



MATH 6101
Fall 2008

Continuity

Continuity

Euler (1748) defined continuous, discontinuous and mixed functions.

Continuous = expressible by a single analytic expression.

Mixed = expressible in two or more analytic expressions.

Discontinuous = defined by different expression at different places

19-Nov-2008 MATH 6101 2

Continuity

Fourier (1807) - *On the Propagation of Heat in Solid Bodies*


Committee of Lagrange, Laplace, Monge and Lacroix was set up to report on the work.

Memoir highly regarded but caused controversy.

I. First objection (Lagrange and Laplace, 1808): Fourier's expansions of functions as trigonometrical series

II. Second objection (Biot): Fourier's derivation of equations of transfer of heat.

Théorie analytique de la chaleur (1822)



19-Nov-2008 MATH 6101 3

Continuity

Bolzano (1817) *Rein analytischer Beweis*
(Pure Analytical Proof)
Attempt to free calculus from concept of infinitesimal
Bolzano achieved what he set out to achieve
His ideas only became well known after his death – almost 100 years.
Bolzano purged the concepts of limit, convergence, and derivative of geometrical components and replace them by purely arithmetical concepts

19-Nov-2008 MATH 6101 4

Continuity

Bolzano (1817) *Rein analytischer Beweis*
(Pure Analytical Proof)
Defined a function f to be **continuous** on an interval if for any value of x in this interval the difference $f(x+\Delta x) - f(x)$ becomes and remains less than any given quantity for Δx sufficiently small, whether positive or negative.

19-Nov-2008 MATH 6101 5

Continuity

Cauchy (1821) – *Cours d'analyse*
Let f be a function that maps a set of real numbers to another set of real numbers, and suppose c is an element of the domain of f . The function f is said to be **continuous** at the point c if the following holds: For any number $\epsilon > 0$, however small, there exists some number $\delta > 0$ such that for all x in the domain with $c - \delta < x < c + \delta$, the value of $f(x)$ satisfies

$$f(c) - \epsilon < f(x) < f(c) + \epsilon$$

19-Nov-2008 MATH 6101 6

Continuity

Cauchy (1821) – *Cours d'analyse*
 Pointed out Euler's definition of continuity was imprecise

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases} = \sqrt{x^2} = \frac{2}{\pi} \int_0^{\infty} \frac{x^2}{t^2 + x^2} dt$$

The first is discontinuous by Euler, but last two are clearly analytic expressions.

19-Nov-2008 MATH 6101 7

Continuity

Heine – 1860's

A real function f is **continuous** if for any sequence $\{x_n\}$ such that $\lim_{n \rightarrow \infty} x_n = L$ it holds that $\lim_{n \rightarrow \infty} f(x_n) = f(L)$.
 (Assume $\{x_n\}$ & L are in domain of f .)

A function is continuous if and only if it preserves limits. (Cauchy's and Heine's definitions of continuity are equivalent on the reals.)

19-Nov-2008 MATH 6101 8

Limits

Let $f: D \rightarrow \mathbf{R}$ be a function. Let $a \in D$.
 Definition 1: $\lim_{x \rightarrow a} f(x) = L$ provided

(1) There is a sequence $\{x_n\} \subset D - \{a\}$ such that $\lim_{n \rightarrow \infty} x_n = a$, and

(2) for every sequence $\{x_n\} \subset D - \{a\}$ such that $\lim_{n \rightarrow \infty} x_n = a$, $\lim_{n \rightarrow \infty} f(x_n) = L$.

19-Nov-2008 MATH 6101 9

Simple Proposition

Proposition:

(1) $\lim_{x \rightarrow a} r = r$; (2) $\lim_{x \rightarrow a} x = a$; (3) $\lim_{x \rightarrow a} |x| = |a|$.

Big Theorem:

Suppose $f, g: D \rightarrow \mathbf{R}$ so that $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$, then

(1) $\lim_{x \rightarrow a} (f(x) + g(x)) = L + M$

(2) $\lim_{x \rightarrow a} f(x)g(x) = LM$

(3) $\lim_{x \rightarrow a} f(x) - g(x) = L - M$

(4) $\lim_{x \rightarrow a} f(x)/g(x) = L/M$ if $M \neq 0$.

19-Nov-2008

MATH 6101

10

Problems

$$\lim_{x \rightarrow 5} \frac{x-5}{x^2-25}$$

19-Nov-2008

MATH 6101

11

Problems

$$\lim_{x \rightarrow 4} \frac{4}{x-4}$$

19-Nov-2008

MATH 6101

12

Problems

$$\lim_{x \rightarrow 0} \frac{x}{|x|}$$

19-Nov-2008 MATH 6101 13

The Dirichlet Function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

$\lim_{x \rightarrow a} f(x)$ does not exist for all $a \in \mathbb{R}$

Let $g(x) = x \cdot f(x)$

Claim: $\lim_{x \rightarrow 0} g(x) = 0$

but $\lim_{x \rightarrow a} g(x)$ does not exist for all $a \neq 0$.

19-Nov-2008 MATH 6101 14

Continuity

Definition:
The function $f: D \rightarrow \mathbf{R}$ is **continuous** at $a \in D$ if

$$\lim_{x \rightarrow a} f(x) = f(a)$$

This means:
For a function to be continuous at a point a :

1. $f(a)$ exists,
2. $\lim_{x \rightarrow a} f(x)$ exists, and
3. $\lim_{x \rightarrow a} f(x) = f(a)$.

19-Nov-2008 MATH 6101 15

Simple Proposition & Big Theorem

Proposition: Redux

The constant function $f(x) = r$, the identity function $f(x) = x$ and the absolute value function $f(x) = |x|$ are all continuous for all real numbers.

Big Theorem:

Suppose $f, g: D \rightarrow \mathbf{R}$ are continuous at $x = a$. Then

- (1) $f(x) + g(x)$ is continuous at $x = a$.
- (2) $f(x)g(x)$ is continuous at $x = a$.
- (3) $f(x) - g(x)$ is continuous at $x = a$.
- (4) $f(x)/g(x)$ is continuous at $x = a$ if $g(a) \neq 0$.

19-Nov-2008

MATH 6101

16

Continuity and Composition

Theorem:

Suppose $f: D \rightarrow \mathbf{R}$ and $g: E \rightarrow \mathbf{R}$ are functions such that the composition $f \circ g$ is defined in E . If g continuous at $x = a \in E$ and f continuous at $g(a)$, then $f \circ g$ is continuous at $x = a$.

19-Nov-2008

MATH 6101

17

Continuity and Trig

Theorem:

1. $|\sin \phi| \leq |\phi|$ for all ϕ

2. $\lim_{\phi \rightarrow 0} \frac{\sin \phi}{\phi} = 1$

19-Nov-2008

MATH 6101

18

Continuity and Trig

19-Nov-2008 MATH 6101 19

Continuity and Trig

Theorem:
The functions $\sin(x)$ and $\cos(x)$ are continuous.

$$\sin \alpha - \sin \beta = 2 \sin \frac{\alpha - \beta}{2} \cos \frac{\alpha + \beta}{2}$$

$$|\sin x_n - \sin a| \leq 2 \left| \sin \frac{x_n - a}{2} \right| \leq 2 \frac{|x_n - a|}{2} = |x_n - a|$$

19-Nov-2008 MATH 6101 20

Questions

Let $\{f_n\}$ be a sequence of continuous functions. By induction we know that

$$\sum_{k=1}^n f_k(x)$$

is continuous. Is it true that the following sum is continuous?

$$\sum_{n=1}^{\infty} f_n(x)$$

Is it true that if $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ exists, then it will be continuous?

19-Nov-2008 MATH 6101 21

Continuity Pathologies

Let

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Let $g(x) = x \cdot f(x)$
 Then g is continuous at $x = 0$ and discontinuous at every other real number.

19-Nov-2008 MATH 6101 22

Continuity II

Determine where the following function is continuous.

$$f(x) = x + \lceil x^2 \rceil - \lfloor x \rfloor$$

19-Nov-2008 MATH 6101 23

Continuity II

The function, $f(x)$, is a combination of three simpler functions.

$f_1(x) = x$ is continuous at each point;

$f_2(x) = \lfloor x \rfloor$ is continuous $\Leftrightarrow x$ is not an integer;

$f_3(x) = \lceil x^2 \rceil$ is continuous $\Leftrightarrow x^2$ is not an integer;

19-Nov-2008 MATH 6101 24

More Pathologies

1. A function that is continuous at only one point ($x=0$)

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ -x & \text{if } x \text{ is irrational} \end{cases}$$

19-Nov-2008

MATH 6101

25

More Pathologies

2. A function with a derivative defined for all x , but whose derivative is discontinuous

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x=0 \end{cases}$$

19-Nov-2008

MATH 6101

26

More Pathologies

3. A function continuous at all irrationals and discontinuous at all rationals.

$f(x)=1/q$ if $x = p/q$ is rational and in lowest terms, otherwise $f(x)=0$.

19-Nov-2008

MATH 6101

27

More Pathologies

4. The function g is non-zero, infinitely differentiable, and any derivative of g at $x = 0$ equals zero.

$$g(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

19-Nov-2008 MATH 6101 28

More Pathologies

5. A function that is everywhere continuous but nowhere differentiable (Bolzano)

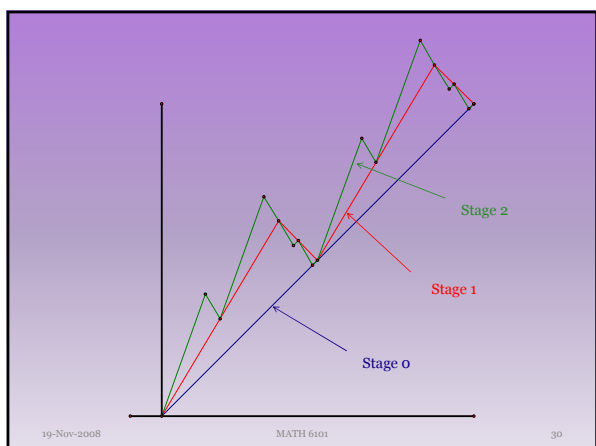
a) The first graph is the line from $(0, 0)$ to $(0, 1)$.

b) Suppose that (a, A) and (b, B) are the endpoints of a segment in some iteration. In the next iteration the segment is replaced by a polygonal line joining the following points:

(a, A) , $(a + 3/8(b-a), A + 5/8(B-A))$,
 $(1/2(a+b), 1/2(A+B))$, $(a + 7/8(b-a), A + 9/8(B-A))$,
 (b, B)


<http://demonstrations.wolfram.com/BolzosFunction/>

19-Nov-2008 MATH 6101 29



More Pathologies

7. A function that is everywhere continuous but nowhere differentiable (Weierstrass, 1872)



$$g(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n |\sin(4^n x)|$$

$$h(x) = \sum_{n=0}^{\infty} (1/2)^n \sin(2^n x)$$

19-Nov-2008
MATH 6101
31

Properties of Continuous Functions

Intermediate Value Theorem (Bolzano, 1817)

Let $f:[a,b] \rightarrow \mathbf{R}$ be a continuous function and let y^ be a real number so that either $f(a) < y^* < f(b)$ or $f(b) < y^* < f(a)$. Then there is x^* , $a < x^* < b$, so that $f(x^*) = y^*$.*

19-Nov-2008
MATH 6101
32

Properties of Continuous Functions

Proof:

Assume $f(a) < y^* < f(b)$.

We are going to set up a binary search for x^* .

Let $a_0 = a$ and $b_0 = b$. Let $c_1 = (a_0 + b_0)/2$

Then $f(c_1) = y^*$, $f(c_1) < y^*$, or $f(c_1) > y^*$.

19-Nov-2008
MATH 6101
33

Properties of Continuous Functions

If $f(c_1) > y^*$ set $a_1 = a_0$ and $b_1 = c_1$.
 If $f(c_1) < y^*$ set $a_1 = c_1$ and $b_1 = b_0$.
 If $f(c_1) = y^*$ set $x^* = c_1$.
 This sets up a recursive algorithm.
 If there is an n so that $f(c_n) = y^*$, we are done.

19-Nov-2008 MATH 6101 34

Properties of Continuous Functions

Assume that this never happens.
 Note that
 $a_0 \leq a_1 \leq a_2 \leq \dots$ and $b_0 \geq b_1 \geq b_2 \geq \dots$ and

$$b_n - a_n = \frac{b_{n-1} - a_{n-1}}{2} = \frac{b_{n-2} - a_{n-2}}{2^2} = \dots = \frac{b_0 - a_0}{2^n}$$
 Therefore $\{a_n\}$ and $\{b_n\}$ converge to the same limit, call it x^* .
 f continuous $\implies \{f(a_n)\}$ and $\{f(b_n)\}$ converge to $f(x^*)$. By the Squeeze Theorem

19-Nov-2008 MATH 6101 35

Properties of Continuous Functions

$$f(x^*) = \lim_{n \rightarrow \infty} f(a_n) \leq y^* \leq \lim_{n \rightarrow \infty} f(b_n) = f(x^*)$$

Therefore $f(x^*) = y^*$.

19-Nov-2008 MATH 6101 36

Properties of Continuous Functions

Corollary: For every positive integer n and positive real number r there is a real number x so that $x^n = r$.

Proof:

Let $f(x) = x^n$, $y^* = r$, $a = 0$, and $b = 1+r$.

$$f(a) = 0 < y^* = r < 1+r < (1+r)^n = f(b)$$

19-Nov-2008

MATH 6101

37

Properties of Continuous Functions

The Maximum Principle (Weierstrauss):

Let $f:[a,b] \rightarrow \mathbf{R}$ be a continuous function. Then f assumes both a maximum and a minimum on $[a,b]$.

19-Nov-2008

MATH 6101

38
