

# Chapter 3

## The Real Numbers, $\mathbb{R}$

### 3.1 Notation and Definitions

We will NOT define set, but will accept the common understanding for sets. Let  $A$  and  $B$  be sets.

**Definition 3.1** The **union** of  $A$  and  $B$  is the collection of all elements that belong to  $A$  or to  $B$ ;  $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$ .

The union of a finite number of sets is defined as:

$$\bigcup_{i=1}^n A_i = \{x \mid x \in A_i \text{ for some } i = 1, 2, 3, \dots, n\}.$$

An arbitrary union of sets indexed by some set  $\Lambda$  is defined similarly:

$$\bigcup_{\lambda \in \Lambda} A_\lambda = \{x \mid x \in A_\lambda \text{ for some } \lambda \in \Lambda\}.$$

**Definition 3.2** The **intersection** of  $A$  and  $B$  is the collection of all elements that belong to both  $A$  and to  $B$ ;  $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$ .

The intersection of a finite number of sets is defined as:

$$\bigcap_{i=1}^n A_i = \{x \mid x \in A_i \text{ for all } i = 1, 2, 3, \dots, n\}.$$

An arbitrary intersection of sets indexed by some set  $\Lambda$  is defined similarly:

$$\bigcap_{\lambda \in \Lambda} A_\lambda = \{x \mid x \in A_\lambda \text{ for all } \lambda \in \Lambda\}.$$

**Definition 3.3** The **product**, or *Cartesian product*, of  $A$  and  $B$  is the collection of all ordered pairs,  $(a, b)$ , so that the first coordinate belongs to  $A$  and the second coordinate belongs to  $B$ ;  $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$ .

The product of a finite number of sets is defined as:

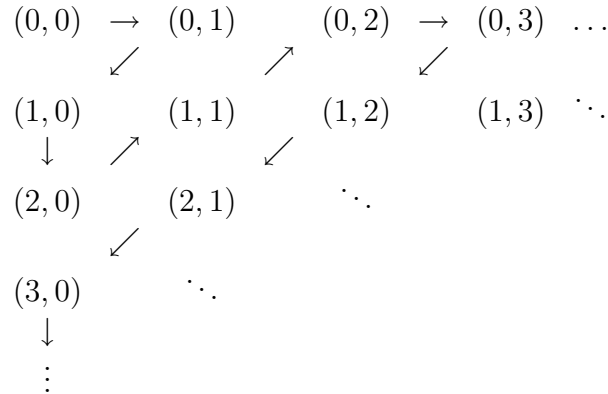
$$\prod_{i=1}^n A_i = \{(x_1, x_2, \dots, x_n) \mid x_i \in A_i \text{ for all } i = 1, 2, 3, \dots, n\}.$$

## 3.2 Infinities

It was not until the 19th Century that mathematicians discovered that infinity comes in different sizes. Georg Cantor (1845 to 1918) defined the following.

**Definition 3.4** Any set which can be put into one-one correspondence with  $\mathbb{N}$  is called *denumerable*. A set is *countable* if it is finite or denumerable.

**Example 3.1** The set of all ordered pairs,  $(a_i, b_i)$  with  $a_i, b_i \in \mathbb{N}$  is countable. The proof of this is the usual Cantor diagonalization argument. List all ordered pairs and follow a path like the following:



The resulting mapping is like this:  $0 \leftrightarrow (0, 0)$ ,  $1 \leftrightarrow (0, 1)$ ,  $2 \leftrightarrow (1, 0)$ ,  $3 \leftrightarrow (2, 0)$ ,  $4 \leftrightarrow (1, 1)$ ,  $5 \leftrightarrow (0, 2)$ ,  $\dots$ . Clearly this mapping will cover all such ordered pairs.

**Lemma 3.1** Any subset of a countable set is countable.

You can use the above countable set and the following mapping to show that the set of all integers are countable:

$$\begin{array}{ccc}
 0 & \leftrightarrow & (0, 0) \\
 1 & \leftrightarrow & (1, 0) \\
 -1 & \leftrightarrow & (1, 1) \\
 2 & \leftrightarrow & (2, 0) \\
 -2 & \leftrightarrow & (2, 1) \\
 3 & \leftrightarrow & (3, 0) \\
 \vdots & \vdots & \vdots
 \end{array}$$

**Lemma 3.2** *The product of two countable sets is countable.*

The proof of this is exactly what we did in the example above.

**Lemma 3.3** *The product of a finite number of countable sets is countable.*

Using this we can show that the set of rationals is countable. The rationals are the set of all fractions  $a/b$  where  $a, b \in \mathbb{Z}$  and  $b > 0$ . This can be mapped onto the subset of ordered triples of natural numbers  $(a, b, c)$  such that  $b > 0$ ,  $a$  and  $b$  are coprime, and  $c \in \{0, 1\}$  so that  $c = 0$  if  $a/b \geq 0$  and  $c = 1$  otherwise.

$$\begin{array}{rcl}
 0 & \mapsto & (0, 1, 0) \\
 1 & \mapsto & (1, 1, 0) \\
 -1 & \mapsto & (1, 1, 1) \\
 \frac{1}{2} & \mapsto & (1, 2, 0) \\
 -\frac{1}{2} & \mapsto & (1, 2, 1) \\
 2 & \mapsto & (2, 1, 0) \\
 -2 & \mapsto & (2, 1, 1) \\
 \frac{1}{3} & \mapsto & (1, 3, 0) \\
 -\frac{1}{3} & \mapsto & (1, 3, 1) \\
 3 & \mapsto & (3, 1, 0) \\
 -3 & \mapsto & (3, 1, 1) \\
 \frac{1}{4} & \mapsto & (1, 4, 0) \\
 -\frac{1}{4} & \mapsto & (1, 4, 1) \\
 \frac{1}{2} & \mapsto & (2, 3, 0) \\
 -\frac{1}{2} & \mapsto & (2, 3, 1) \\
 \frac{1}{3} & \mapsto & (3, 2, 0) \\
 -\frac{1}{3} & \mapsto & (3, 2, 1) \\
 4 & \mapsto & (4, 1, 0) \\
 -4 & \mapsto & (4, 1, 1) \\
 \dots & & 
 \end{array}$$

The amazing insight achieved by Cantor is the following result.

**Theorem 3.1 (Cantor, 1874)** *The set of real numbers  $\mathbb{R}$  is not countable.*

PROOF: We will show that the set of reals in the interval  $(0, 1)$  is not countable. From our lemma above that will make the reals uncountable, since if they were countable, then  $(0, 1)$  would also be countable.

Assume that  $(0, 1)$  is countable. Then we could write down all the decimal expansions of the reals in  $(0, 1)$  in a list:

$$\begin{array}{l}
 0.a_{11}a_{12}a_{13}a_{14}a_{15} \dots \\
 0.a_{21}a_{22}a_{23}a_{24}a_{25} \dots \\
 0.a_{31}a_{32}a_{33}a_{34}a_{35} \dots
 \end{array}$$

$0.a_{41}a_{42}a_{43}a_{44}a_{45}\dots$   
 $\vdots$

Now define a decimal  $x = x_1x_2x_3x_4\dots$  by

$$\begin{aligned}
 x_1 &\neq a_{11} \text{ (or 9)}, \\
 x_2 &\neq a_{22} \text{ (or 9)}, \\
 x_3 &\neq a_{33} \text{ (or 9)}, \\
 x_4 &\neq a_{44} \text{ (or 9)},
 \end{aligned}$$

and so forth. Then the decimal expansion of  $x$  does not end in recurring 9's and it differs from the  $n$ th element of the list in the  $n$ th decimal place. Hence it represents an element of the interval  $(0, 1)$  which is not in the list and so we do not have a list of the reals in  $(0, 1)$ . ■

**Lemma 3.4** *A countable union of countable sets is countable.*

One of the amazing consequences of Cantor's work is that it proves the existence of a class of real numbers which previously had been very difficult to investigate.

Recall that a real number is called algebraic if it is a root of a polynomial with rational (or integer) coefficients. Other real numbers are called transcendental.

**Theorem 3.2 (Cantor)** *The set of algebraic numbers is countable. Hence there are uncountably many transcendental numbers.*

**PROOF:** Since a polynomial of degree  $n$  with rational coefficients has  $n+1$  coefficients, such polynomials can be put into one-to-one correspondence with  $\mathbb{Q} \times \mathbb{Q} \times \dots \times \mathbb{Q}$  ( $n+1$  times). This is countable by Lemma 3.3 and so there are only countably many such polynomials. Such a polynomial can have at most  $n$  roots and so there are only countably many such roots. Finally, the set of all such roots is the union over  $n$  of the roots of polynomials of degree  $n$  and so there are countably many of these. ■

Many mathematicians found this proof of the existence of transcendentals unsatisfactory. Notice that it does not construct a transcendental number, it only shows that they must exist. In 1851, Liouville was the first to provide the existence of a transcendental. He prove that the number  $0.110001000\dots$  (with a 1 in the  $n!$  place and 0 elsewhere) is transcendental. In 1873, Hermite proved that  $e$  is transcendental. In 1882, Lindemann proved that  $\pi$  is transcendental. In 1900, Hilbert published a famous set of problems to challenge mathematicians in the new century. The 23rd problem was to prove that if  $a$  is algebraic ( $a \neq 0, 1$ ) and  $x$  is irrational then  $a^x$  is transcendental. This was proved by Gelfond in 1934.

### 3.3 Proofs by Induction

We will need this method of proof later, so now is a good time to introduce this process. One of the most basic properties of the natural numbers,  $\mathbb{N}$ , is the principle of *mathematical induction*. The first known proof by mathematical induction appears in Francesco Maurolico's *Arithmeticonum libri fuo* (1575). Maurolico used the technique to prove that the sum of the first  $n$  odd integers is  $n^2$ .

Suppose  $P(x)$  means that property  $P$  holds for the number  $x$ .

**Principle of Mathematical Induction** A property  $P$  holds true for all natural numbers  $n$  provided that

1. The *basis*:  $P(1)$  is true, *i.e.*, the statement holds for  $n = 1$ <sup>1</sup>.
2. The *inductive step*: If  $P(k)$  is true for  $k \geq 1$ , then  $P(k + 1)$  is true.

One common variant on the inductive step is if the property holds for all  $k < n$ , then it also holds for  $n$ . Another version allows us to start at any finite number  $m$  and prove that it holds for all integers greater than  $m$ , *i.e.*, you do not have to start at 1.

**Example 3.2** Show that  $1 + \cdots + n = \frac{n(n+1)}{2}$ .

First, we need to check this for  $n = 1$ . The formula reads:  $1 = \frac{1 \cdot 2}{2} = 1$ , which is true. Thus, our basis is true. Now we will assume that the statement is true for some  $k > 1$

$$1 + \cdots + k = \sum_{i=1}^k i = \frac{k(k+1)}{2},$$

and we need to prove, then, that the statement is true for  $k + 1$ .

$$\begin{aligned} \sum_{i=1}^{k+1} i &= 1 + 2 + 3 + \cdots + k + (k + 1) \\ &= [1 + 2 + 3 + \cdots + k] + (k + 1) \\ &= \frac{k(k+1)}{2} + (k + 1) \\ &= \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2} \end{aligned}$$

This completes the proof.

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<sup>1</sup>Some texts will start with  $n = 0$ .

### 3.4 Axioms for the Real Numbers

The real numbers  $\mathbb{R}$  have some rather unexpected properties. In fact, there are many things that it is difficult to prove rigorously. For example, how do we know that  $\sqrt{2}$  exists? In other words how can we be sure that there is some real number whose square is 2? Also, it is easy to convince yourself that  $2 + 3 = 3 + 2$ . Can you be so sure about  $\sqrt{2} + \sqrt{3} = \sqrt{3} + \sqrt{2}$  or  $e + \pi = \pi + e$ , if you can really write down what those numbers are?

Our intuition works pretty well about what should be true for  $\mathbb{N}$  or  $\mathbb{Z}$  or even for  $\mathbb{Q}$ . Things don't get hard until we are forced to admit the existence of irrationals. There are constructive methods for making the full set  $\mathbb{R}$  from  $\mathbb{Q}$ . The first rigorous construction was given by Richard Dedekind in 1872.

He developed the idea first in 1858 though he did not publish it until 1872. This is what he wrote at the beginning of the article.

*As professor in the Polytechnic School in Zürich I found myself for the first time obliged to lecture upon the ideas of the differential calculus and felt more keenly than ever before the lack of a really scientific foundation for arithmetic. In discussing the notion of the approach of a variable magnitude to a fixed limiting value, and especially in proving the theorem that every magnitude which grows continuously but not beyond all limits, must certainly approach a limiting value, I had recourse to geometric evidences. . . . This feeling of dissatisfaction was so overpowering that I made the fixed resolve to keep mediating on the question till I should find a purely arithmetic and perfectly rigorous foundation for the principles of infinitesimal analysis.*

He defined a real number to be a pair  $(L, R)$  of sets of rationals which have the following properties.

- Every rational is in exactly one of the sets.
- Every rational in  $L$  is less than every rational in  $R$ .

Such a pair is called a Dedekind cut (*Schnitt* in German). You can think of it as defining a real number which is the least upper bound of the “left-hand set”  $L$  and also the greatest lower bound of the “right-hand set”  $R$ . If the cut defines a rational number then this may be in either of the two sets. It is a long task to define the arithmetic operations and order relation on such cuts and to verify that they do then satisfy the axioms for the reals — including even the Completeness Axiom. Richard Dedekind was one of the last research students of Gauss. His arithmetization of analysis was his most important contribution to mathematics. It was not enthusiastically received by leading mathematicians of his day.

For the moment we will simply give a set of axioms for the reals and leave it to intuition that there is something that satisfies these axioms.

We start with a set, which we will call  $\mathbb{R}$  and a pair  $+, \cdot$  of binary operations.

### 3.4.1 The Axioms

These are divided into three groups.

#### I The algebraic axioms

$\mathbb{R}$  is a field under  $+$  and  $\cdot$ . This means that  $(\mathbb{R}, +)$  and  $(\mathbb{R}, \cdot)$  are both abelian groups and the distributive law  $(a + b)c = ab + ac$  holds.

- (a) For all  $a, b \in \mathbb{R}$ ,  $a + b \in \mathbb{R}$ .
- (b) For all  $a, b \in \mathbb{R}$ ,  $a + b = b + a$
- (c) For all  $a, b, c \in \mathbb{R}$ ,  $(a + b) + c = a + (b + c)$ .
- (d) For all  $a \in \mathbb{R}$ ,  $a + 0 = 0 + a = a$ .
- (e) For all  $a \in \mathbb{R}$  there is an element  $-a \in \mathbb{R}$  so that  $a + (-a) = (-a) + a = 0$ .
- (f) For all  $a, b \in \mathbb{R}$ ,  $a \cdot b \in \mathbb{R}$ .
- (g) For all  $a, b \in \mathbb{R}$ ,  $a \cdot b = b \cdot a$
- (h) For all  $a, b, c \in \mathbb{R}$ ,  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ .
- (i) For all  $a \in \mathbb{R}$ ,  $a \cdot 1 = 1 \cdot a = a$ .
- (j) For all  $a \in \mathbb{R}$ ,  $a \neq 0$  there is an element  $b \in \mathbb{R}$  so that  $a \cdot b = b \cdot a = 1$ .
- (k) For all  $a, b, c \in \mathbb{R}$ ,  $a \cdot (b + c) = a \cdot b + a \cdot c$ .

#### II The order axioms<sup>2</sup>

There is a relation  $>$  on  $\mathbb{R}$ . (That is, given any pair  $a, b \in \mathbb{R}$  then  $a > b$  is either true or false). It satisfies:

- (a) Trichotomy: For any  $a \in \mathbb{R}$  exactly one of  $a > 0$ ,  $a = 0$ ,  $0 > a$  is true.
- (b) If  $a, b > 0$  then  $a + b > 0$  and  $a \cdot b > 0$ .
- (c) If  $a > b$  then  $a + c > b + c$  for any  $c \in \mathbb{R}$ .

#### III The Completeness Axiom<sup>3</sup>

If a non-empty set  $A$  has an upper bound, it has a least upper bound.

**Examples:** The field  $\mathbb{Q}$  of rationals is an ordered field. While the field  $\mathbb{C}$  of complex numbers is not an ordered field under any ordering. To see this suppose  $i > 0$ . Then  $-1 = i^2 > 0$  and adding 1 to both sides gives  $0 > 1$ . But if we square both sides

<sup>2</sup>Something satisfying axioms I and II is called an ordered field.

<sup>3</sup>Something which satisfies Axioms I, II and III is called a complete ordered field.

instead we have  $(-1)^2 = 1 > 0$  and so we get a contradiction. A similar argument starting with  $i < 0$  also gives a contradiction.

These axioms are used to show more properties of the reals. For example, the ordering  $>$  on  $\mathbb{R}$  is transitive. That is, if  $a > b$  and  $b > c$  then  $a > c$ . The proof is straightforward:

$$\begin{aligned} a > b &\Leftrightarrow a - b > b - b = 0 \text{ by Axiom II c)} \\ a > c &\Leftrightarrow a - c > c - c = 0 \end{aligned}$$

Hence  $(a - b) + (a - c) > 0$  and so  $a - c > 0$  and we have  $a > c$ .

**Definition 3.5** An **upper bound** of a non-empty subset  $A$  of  $\mathbb{R}$  is an element  $b \in \mathbb{R}$  with  $b \geq a$  for all  $a \in A$ . An element  $M \in \mathbb{R}$  is a **least upper bound** or **supremum** of  $A$  if  $M$  is an upper bound of  $A$  and if  $b$  is an upper bound of  $A$  then  $b \geq M$ . That is, if  $M$  is a least upper bound of  $A$  then there is a  $b \in \mathbb{R}$  so that for all  $x \in A$  if  $b \geq x$  then  $b \geq M$ .  $M$  is sometimes written as  $\text{lub}\{A\}$  or  $\text{inf}\{A\}$ .

A lower bound of a non-empty subset  $A$  of  $\mathbb{R}$  is an element  $d \in \mathbb{R}$  with  $d \leq a$  for all  $a \in A$ . An element  $m \in \mathbb{R}$  is a **greatest lower bound** or **infimum** of  $A$  if  $m$  is a lower bound of  $A$  and if  $d$  is a lower bound of  $A$  then  $m \geq d$ .

**Lemma 3.5** A subset  $A$  which has a lower bound has a greatest lower bound.

PROOF: Let  $B = \{x \in \mathbb{R} \mid -x \in A\}$ . Then  $B$  is bounded above by  $-\text{lub}\{A\}$  and so has a least upper bound. Call it  $b$ . It is then easy to check that  $-b$  is a greatest lower bound of  $A$ . ■

**Lemma 3.6 (The Archimedean property of  $\mathbb{R}$ )** If  $a > 0$  in  $\mathbb{R}$ , then for some  $n \in \mathbb{N}$  it is true that  $1/n < a$ .

PROOF: This is equivalent to proving that for any  $x \in \mathbb{R}$  there is an  $n \in \mathbb{N}$  so that  $n > x$ . This statement is equivalent to saying that  $\mathbb{N}$  is not bounded above. This seems like a very obvious fact, but we will prove it rigorously from the axioms. Suppose  $\mathbb{N}$  were bounded above. Then it would have a least upper bound,  $M$ . So then  $M - 1$  is not an upper bound and so there is an integer  $n > M - 1$ . But then  $n + 1 > M$  contradicting the fact that  $M$  is an upper bound. ■

**Lemma 3.7** Between any two real numbers is an rational number.

PROOF: Let  $a, b$  be real numbers and assume  $a < b$ . Choose  $n$  so that  $1/n < b - a$ . Then look at multiples of  $1/n$ . Since these are unbounded, we may choose the first such multiple with  $m/n > a$ . We claim that  $m/n < b$ . If not, then since  $(m-1)/n < a$  and  $m/n > b$  we would have  $1/n > b - a$ . ■



**Definition 3.6** A set  $A$  with the property that an element of  $A$  lies in every interval  $(a, b)$  of  $\mathbb{R}$  is called **dense** in  $\mathbb{R}$ .

We have just proved that the rationals are dense in the reals.

We can now answer the question we stated earlier.

**Lemma 3.8** The real number  $\sqrt{2}$  exists.

PROOF: We will get  $\sqrt{2}$  as the least upper bound of the set  $A = \{r \in \mathbb{Q} \mid r^2 < 2\}$ . We know that  $A$  is bounded above at least by 2. Thus its least upper bound  $b$  exists by Axiom III. Now we have to prove that  $b^2 < 2$  and  $b^2 > 2$  both lead to contradictions, leaving only  $b^2 = 2$  (by the Trichotomy rule).

So suppose that  $b^2 > 2$ . Look at  $(b - 1/n)^2 = b^2 - 2b/n + 1/n^2 > b^2 - 2b/n$ . When is this greater than 2? Now  $b^2 - 2b/n > 2$  if and only if  $b^2 - 2 > 2b/n$  or  $1/n < (b^2 - 2)/2b$  and we can find such an  $n$  by the Archimedean property. Thus  $b - 1/n$  is an upper bound, contradicting the assumption that  $b$  was the least upper bound.

Similarly, if  $b^2 < 2$  then  $(b + 1/n)^2 = b^2 + 2b/n + 1/n^2 > b^2 + 2b/n$ . Can this be less than 2? Yes, when  $b^2 + 2b/n < 2$  which happens if and only if  $2 - b^2 > 2b/n$  or  $1/n < (2 - b^2)/2b$  and we can find an  $n$  satisfying this, leading to the conclusion that  $b$  would not be an upper bound. ■

## 3.5 Intervals and Decimal Expansion

In school algebra we usually define real numbers as those numbers that can be represented by finite or infinite decimals. When the student gets to geometry he is told that the real numbers are those that are in a one-to-one correspondence with the points on a line. What we have seen is that neither one of these descriptions is really complete. There is a lot more to real numbers than we would probably want to tell our students. We need to know more about these numbers, however, in order not to mislead the students when they ask questions about numbers, especially real numbers. We have just seen that we can define the real numbers in terms of least upper bounds and we hinted at a way in which Dedekind cuts can be used to define real numbers. Another method — and one more appropriate to high school mathematics — is describe the rational numbers in a particular way and then use the Nested Interval Property of the reals to obtain real numbers as decimals.

The way that we usually introduce the *number line* or *real line* in school is to start with a straight line and select two points on that line that we will let represent the integers 0 and 1.<sup>4</sup>

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<sup>4</sup>We usually take 0 to be the left hand point and 1 the right hand point. Do we have to make this choice?

Now, you can represent the counting numbers 2, 3, 4, ... by laying off the length from 0 to 1 successively to the right of 1. Likewise, we can represent the negative integers as lengths to the left of 0.

We know how to represent fractions (rational numbers) by a geometric construction, so we can represent any rational number on the number line as a length. In fact if we write the rational number  $x = \frac{a}{b}$  as an integral part and a fractional part:  $x = \frac{a}{b} = q + \frac{r}{b}$  where  $q$  is the greatest integer in  $x$  ( $q = \lfloor x \rfloor$ ) and  $0 < r < b$ , then  $\frac{r}{b}$  is between 0 and 1, and we can think of each integer interval as just a translation of the interval between 0 and 1.

We also know how to construct a number of irrational numbers, as well, such as  $\sqrt{2}$  among others. Each of these numbers that we know how to represent as a length then we know where to put that irrational number on the number line. That doesn't represent all of the real numbers though. How can we represent each real number on the number line?

### 3.5.1 Intervals

**Definition 3.7** *An interval of numbers is a set containing all numbers between two given numbers together with one, both, or neither of the two numbers.*

As we will see later an interval is called *open* if it contains neither of its endpoints and *closed* if it contains both of its endpoints. The *length* of an interval with endpoints  $a$  and  $b$  is  $|b - a|$ .

### 3.5.2 Decimals

**Definition 3.8** *If a nonnegative real number  $x$  can be expressed as a (finite) sum of the form*

$$x = D + \frac{d_1}{10^1} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \cdots + \frac{d_k}{10^k} = D + d_1 \cdot 10^{-1} + d_2 \cdot 10^{-2} + d_3 \cdot 10^{-3} + \cdots + d_k \cdot 10^{-k},$$

where  $D$  and  $d_i$ ,  $i = 1, \dots, k$  are nonnegative integers and  $0 \leq d_i \leq 9$  for  $i = 1, 2, \dots, k$ , then  $D.d_1d_2d_3 \dots d_k$  is the **finite decimal** representation of  $x$ .

We say that  $D$  is the *integer part*,  $.d_1d_2 \dots d_k$  is the *decimal part* and  $d_i$  is the  $i$ th *decimal place* of  $x$ . The integer part is the greatest integer less than or equal to  $x$ ,  $D = \lfloor x \rfloor$ .

If  $x$  is a negative number and there is a finite decimal  $D.d_1d_2 \dots d_k$  representing  $-x$ , then we write  $-D.d_1d_2 \dots d_k$  for the finite decimal representing  $x$ . In this case,  $\lfloor x \rfloor = -D - 1$ .

Of course, there are rational numbers that do not have a finite decimal representation, such as  $1/3$ . To deal with these we need another definition.

**Definition 3.9** An *infinite decimal* representation of a real number  $x$  is an infinite sequence  $d = \{D, d_1, d_2, \dots, d_n, \dots\}$  of integers such that  $0 \leq d_k \leq 9$  for all  $k \in \mathbb{N}$ . Every finite decimal can be regarded as an infinite decimal by identifying it with the infinite sequence  $d = \{D, d_1, d_2, \dots, d_k, 0, 0, \dots\}$ .

Think of the decimal for  $\sqrt{2}$ . If only the first two decimal places are known, then we know that  $\sqrt{2}$  lies in the interval  $[1.41, 1.42]$ , an interval of length  $10^{-2}$ . Each succeeding decimal place places  $\sqrt{2}$  in an interval of length  $\frac{1}{10}$  the preceding interval. To 5 decimal places we know that  $\sqrt{2} = 1.41459$  which places it in the interval  $[1.41459, 1.41460]$ , an interval of length  $10^{-5}$ . Now, this interval is contained in the previous interval.

**Definition 3.10** An interval  $I$  is nested in another interval  $J$  if and only if  $I \subset J$ . A sequence  $\{I_k\}$  of intervals is called a nested sequence if  $I_{k+1} \subset I_k$  for all  $k$ .

**Nested Interval Property:** In the real line for any sequence of finite closed nested intervals there is at least one point that belongs to all of them.

This property is equivalent to the **Least Upper Bound** axiom.

Consider the nested sequence of closed intervals:

$$[1.4, 1.5], [1.41, 1.42], [1.414, 1.415], \dots, [d_k, d_k + 10^{-k}], \dots$$

where  $d_k$  is the rational number whose finite decimal representation consists of the first  $k$  places of  $\sqrt{2}$ . The Nested Interval Property asserts that there is at least one point that belongs to **all** of these intervals. Because the length of the  $k$ th interval is  $10^{-k}$ , at most one point can belong to **all** of these intervals. (If there were two points in the intersection, the distance between them would be more than  $10^{-m}$  for some  $m$ , so they could both be in all the  $[d_k, d_k + 10^{-k}]$ .) This unique point is the real number  $\sqrt{2}$ .

**Lemma 3.9** If  $\{I_k\}$  is a nested sequence of closed intervals with rational endpoints whose length decreases to 0, then there is one and only one point that belongs to all of the intervals in the sequence.

This unique point is the real number determined by the sequence  $\{I_k\}$ .

**Lemma 3.10** Real numbers can be defined by decimal expansions.

PROOF: Given the decimal expansion  $0.a_1a_2a_3\dots$  consider the set of rational numbers

$$\{0.a_1, 0.a_1a_2, 0.a_1a_2a_3, \dots\} = \left\{ \frac{a_1}{10}, \frac{10a_1 + a_2}{100}, \frac{100a_1 + 10a_2 + a_3}{1000}, \dots \right\}.$$

This is bounded above by  $(a_1 + 1)/10$  or by  $(10a_1 + a_2 + 1)/100$  and so it has a least upper bound. This is the real number defined by the decimal expansion. ■

### 3.6 Open and Closed Sets

**Definition 3.11** We say that  $U \subseteq \mathbb{R}$  is open if for all  $x \in U$ , there is  $\epsilon > 0$  such that  $(x - \epsilon, x + \epsilon) \subset U$ . That is, if  $|x - y| < \epsilon$  then  $y \in U$ .

Intuitively, if  $U$  is open and  $x \in U$ , then every  $y$  that is sufficiently close to  $x$  is also in  $U$ .

For example, if  $a < b$ , then  $(a, b)$  is open. To show this, suppose that  $x \in (a, b)$ , and let  $\epsilon = \min(x - a, b - x)$ . So if  $|x - y| < \epsilon$ , then  $y \in (a, b)$ , so  $(x - \epsilon, x + \epsilon) \subset (a, b)$ .

It is also easy to see that  $\mathbb{R}$  is open, and that  $(-\infty, a)$  and  $(a; +\infty)$  are open for any  $a \in \mathbb{R}$ .

The empty set  $\emptyset$  is also open: since  $\emptyset$  has no elements, it is then clearly true that every element of  $\emptyset$  has a neighborhood contained in  $\emptyset$ .

**Lemma 3.11 (An arbitrary union of open sets is open)** If  $U$  and  $V$  are open subsets of  $\mathbb{R}$ , then  $U \cup V$  is open. More generally, if  $A$  is a set (it can even be an uncountable set) and  $U_\lambda \subset \mathbb{R}$  is open for each  $\lambda \in A$ , then  $W = \bigcup_{\lambda \in A} U_\lambda$  is open.

PROOF: We prove the more general case. Suppose  $x \in W = \bigcup_{\lambda \in A} U_\lambda$ . Then there is a  $\lambda \in A$  such that  $x \in U_\lambda$ . Since  $U_\lambda$  is open, there is an  $\epsilon > 0$  such that  $(x - \epsilon, x + \epsilon) \subset U_\lambda \subset W$ . ■

This proof does not use the fact that  $A$  is a finite set. In fact,  $A$  can be infinite, or even uncountable — an arbitrary union of open sets is open.

For example  $(n, n + 1)$  is open for all  $n \in \mathbb{Z}$ . Thus

$$\mathbb{R} \setminus \mathbb{Z} = \bigcup_{n \in \mathbb{Z}} (n, n + 1)$$

is open.

**Lemma 3.12 (A finite intersection of open sets is open)** Suppose  $U$  and  $V$  are open subsets of  $\mathbb{R}$ . Then  $U \cap V$  is open. More generally, if  $U_1, U_2, \dots, U_n$  are open sets, then  $U_1 \cap \dots \cap U_n$  is open.

PROOF: Suppose  $x \in U \cap V$ . Since  $U$  and  $V$  are open there are  $\epsilon_1$  and  $\epsilon_2$  such that  $(x - \epsilon_1, x + \epsilon_1) \subset U$  and  $(x - \epsilon_2, x + \epsilon_2) \subset V$ . Let  $\epsilon = \min(\epsilon_1, \epsilon_2)$ . Then  $(x - \epsilon, x + \epsilon) \subset U \cap V$ .

The proof that  $U_1 \cap \dots \cap U_n$  is open follows similarly. ■

It is **not true** that an arbitrary intersection of open sets is open. For example, let  $U_n$  be the interval  $(-\frac{1}{n}, \frac{1}{n})$  for  $n = 1, 2, \dots$ . Then  $\bigcap_{n=1}^{\infty} U_n = \{0\}$  and  $\{0\}$  is not open. The result is that a *finite* intersection of open sets is open.

**Definition 3.12** A subset  $F \subset \mathbb{R}$  is closed if  $\mathbb{R} \setminus F$  is open.

Since  $\mathbb{R}$  and  $\emptyset$  are open, they are also both closed. If  $a < b$ , then  $[a, b]$ ,  $(-\infty, a]$  and  $[b, \infty)$  are closed.

**Lemma 3.13 (An arbitrary intersection of closed sets is closed)** *If  $F$  and  $G$  are open subsets of  $\mathbb{R}$ , then  $F \cap G$  is closed. More generally, if  $A$  is a set — it can even be an uncountable set — and  $F_\lambda \subset \mathbb{R}$  is closed for each  $\lambda \in A$ , then  $W = \bigcap_{\lambda \in A} F_\lambda$  is closed.*

PROOF:  $\mathbb{R} - W = \mathbb{R} - \bigcap_{\lambda \in A} F_\lambda = \bigcup_{\lambda \in A} (\mathbb{R} \setminus F_\lambda)$ . Since each  $\mathbb{R} \setminus F_\lambda$  is open,  $\bigcup_{\lambda \in A} (\mathbb{R} \setminus F_\lambda)$  is open by Lemma 3.11. Thus,  $W$  is closed. ■

**Lemma 3.14 (A finite union of closed sets is closed)** *Suppose  $F$  and  $G$  are closed subsets of  $\mathbb{R}$ . Then  $F \cap G$  is closed. More generally, if  $F_1, F_2, \dots, F_n$  are closed sets, then  $F_1 \cup \dots \cup F_n$  is closed.*

PROOF: Since  $F$  and  $G$  are closed,  $\mathbb{R} \setminus F$  and  $\mathbb{R} \setminus G$  are open. By Lemma 3.12

$$\mathbb{R} \setminus (F \cup G) = (\mathbb{R} \setminus F) \cap (\mathbb{R} \setminus G)$$

is open, hence  $F \cup G$  is closed. The proof that  $F_1 \cup \dots \cup F_n$  is closed is similar. ■

**Lemma 3.15** *Suppose  $F \subset \mathbb{R}$  is closed. If  $F$  is bounded above and  $\beta = \text{lub}\{F\}$ , then  $\beta \in F$ . Similarly, if  $F$  is bounded below and  $\alpha = \text{glb}\{F\}$ , then  $\alpha \in F$ .*

PROOF: Suppose instead that  $\beta \notin F$ . Since  $\mathbb{R} \setminus F$  is open, there is  $\epsilon > 0$  such that  $(\beta - \epsilon, \beta + \epsilon) \subset \mathbb{R} \setminus F$ , or  $|x - \beta| < \epsilon$  which implies that  $x \notin F$ . But then if  $z \in (\beta - \epsilon, \beta)$  then  $z$  is also an upper bound for  $F$ , contradicting the fact that  $\beta$  is the least upper bound.

The proof that the greatest lower bound  $\alpha \in F$  is similar. ■