Euclid's Axioms of Geometry

Let the following be postulated

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any center and distance.
4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles. (Euclid's Parallel Postulate)

Hilbert's Axioms of Geometry

Undefined Terms: *point*, *line*, *incidence*, *betweenness*, and *congruence*.

Incidence Geometry

AXIOM I-1: *For every point P and for every point Q not equal to P there exists a unique line ℓ that passes through P and Q.*

AXIOM I-2: *For every line ℓ there exist at least two distinct points incident with ℓ.*

AXIOM I-3: *There exist three distinct points with the property that no line is incident with all three of them.*

Betweenness Axioms

AXIOM B-1: If $A*B*C$, *then A, B, and C are three distinct points all lying on the same line and* $C*B*A$.

AXIOM B-2: *Given any two distinct points B and D, there exists points A, C, and E lying on* $\overleftrightarrow{BD}$ *such that* $A*B*D$, $B*C*D$, and $B*D*E$.

Axiom B-3: *If A, B, and C are three distinct point lying on the same line, then one and only one of the points is between the other two.*

Axiom B-4: (Plane Separation Axiom) *For every line ℓ and for any three points A, B, and C not lying on ℓ:*

(i) If A and B are on the same side of ℓ and B and C are on the same side of ℓ, then A and C are on the same side of ℓ.

(ii) If A and B are on opposite sides of ℓ and B and C are on opposite sides of ℓ, then A and C are on the same side of ℓ.

Congruence Theorems

Axiom C-1: *If A and B are distinct points and if A' is any point, then for each ray* r *emanating from A' there is a* unique *point B' on* r *such that* $B' \neq A'$ *and* $AB \cong A'B'$.

Axiom C-2: *If* $AB \cong CD$ *and* $AB \cong EF$, *then* $CD \cong EF$. *Moreover, every segment is congruent to itself.*

Axiom C-3: *If* $A*B*C$, $A'*B'*C'$, $AB \cong A'B'$, *and* $BC \cong B'C'$, *then* $AC \cong A'C'$.

Axiom C-4: *Given any* $\angle BAC$ *and given any ray* $\overrightarrow{A'B'}$ *emanating from a point A', then there is a* unique *ray* $\overrightarrow{A'C'}$ *on a given side of line* $\overleftrightarrow{A'B'}$ *such that* $\angle BAC \cong \angle B'A'C'$.

Axiom C-5: *If* $\angle A \cong \angle B$ *and* $\angle B \cong \angle C$, *then* $\angle A \cong \angle C$. *Moreover, every angle is congruent to itself.*

Axiom C-6: (**SAS**) *If two sides and the included angle of one triangle are congruent respectively to two sides and the included angle of another triangle, then the two triangles are congruent.*

Axioms of Continuity

Archimedes' Axiom: *If AB and CD are any segments, then there is a number n such that if segment CD is laid off n times on the ray* $\overrightarrow{AB}$ *emanating from A, then a point E is reached where* $n \cdot CD \cong AE$ *and B is between A and E.*

Dedekind's Axiom: *Suppose that the set of all points on a line* ℓ *is the union* $\Sigma_1 \cup \Sigma_2$ *of two nonempty subsets such that no point of* Σ_1 *is between two points of* Σ_2 *and* vice versa. *Then there is a unique point, O, lying on* ℓ *such that* $P_1 * O * P_2$ *if and only if* $P_1 \in \Sigma_1$ *and* $P_2 \in \Sigma_2$ *and* $O \neq P_1, P_2$.

Elementary Continuity Principle: *If one endpoint of a segment is inside a circle and the other outside, then the segment intersects the circle.*

Circular Continuity Principle: *If a circle* γ *has one point inside and one point outside another circle* γ', *then the two circles intersect in two points.*

Axiom of Parallelism

Euclidean Parallel Postulate: *Through a given external point there is at most one line parallel to a given line.*

Birkhoff's Axioms of Geometry

B1. There exist nonempty subsets of the plane called lines, with the property that each two points belong to exactly one line.

B2. Corresponding to any two points A and B in the plane there exists a unique real number $d(AB) = d(BA)$, the distance from A to B, which is 0 if and only if $A = B$.

B3. (*Birkhoff Ruler Axiom*) If k is a line and $\mathbb{R}$ denotes the set of real numbers, there exists a one-to-one correspondence ($X \to x$) between the points X in k and the numbers $x \in \mathbb{R}$ such that $d(A,B) = |a - b|$ where $A \to a$ and $B \to b$.

B4. For each line k there are exactly two nonempty convex sets R' and R'' satisfying

a) $R' \cup k \cup R''$ is the entire plane,

b) $R' \cap R'' = \varnothing$, $R' \cap k = \varnothing$, and $R'' \cap k = \varnothing$,

c) if $X \in R'$ and $Y \in R''$ then $\overline{XY} \cap k \neq \varnothing$.

B5. For each angle $\angle ABC$ there exists a unique real number x with $0 \leq x \leq 180$ which is the (degree) measure of the angle $x = \angle ABC^\circ$.

B6. If ray $\overrightarrow{BD}$ lies in $\angle ABC$, then $\angle ABD^\circ + \angle DBC^\circ = \angle ABC^\circ$.

B7. If $\overrightarrow{AB}$ is a ray in the edge, k, of an open half plane $H(k;P)$ then there exist a one-to-one correspondence between the open rays in $H(k;P)$ emanating from A and the set of real numbers between 0 and 180 so that if $\overrightarrow{AX} \to x$ then $\angle BAX^\circ = x$.

B8. (SAS) If a correspondence of two triangles, or a triangle with itself, is such that two sides and the angle between them are respectively congruent to the corresponding two sides and the angle between them, the correspondence is a congruence of triangles.

B9. (Euclidean Parallel Postulate) Through a given external point there is at most one line parallel to a given line.

SMSG Axioms for Euclidean Geometry

Introductory Note. *The School Mathematics Study Group (SMSG) developed an axiomatic system designed for use in high school geometry courses. The axioms are not independent of each other, but the system does satisfy all the requirements for Euclidean geometry; that is, all the theorems in Euclidean geometry can be derived from the system. The lack of independence of the axiomatic system allows high school students to more quickly study a broader range of topics without becoming trapped in detailed study of obvious concepts or difficult proofs. You should compare the similarity and differences between the SMSG axioms and those by Hilbert and Birkhoff.*

Undefined Terms: *point*, *line*, *and plane*

Postulate 1. (*Line Uniqueness*) Given any two distinct points there is exactly one line that contains them.

Postulate 2. (*Distance Postulate*) To every pair of distinct points there corresponds a unique positive number. This number is called the distance between the two points.

Postulate 3. (*Ruler Postulate*) The points of a line can be placed in a correspondence with the real numbers such that:

a) To every point of the line there corresponds exactly one real number.
b) To every real number there corresponds exactly one point of the line.
c) The distance between two distinct points is the absolute value of the difference of the corresponding real numbers.

Postulate 4. (*Ruler Placement Postulate*) Given two points P and Q of a line, the coordinate system can be chosen in such a way that the coordinate of P is zero and the coordinate of Q is positive.

Postulate 5. (*Existence of Points*)

a) Every plane contains at least three non-collinear points.
b) Space contains at least four non-coplanar points.

Postulate 6. (*Points on a Line Lie in a Plane*) If two points lie in a plane, then the line containing these points lies in the same plane.

Postulate 7. (*Plane Uniqueness*) Any three points lie in at least one plane, and any three non-collinear points lie in exactly one plane.

Postulate 8. (*Plane Intersection*) If two planes intersect, then that intersection is a line.

Postulate 9. (*Plane Separation Postulate*) Given a line and a plane containing it, the points of the plane that do not lie on the line form two sets such that:

a) each of the sets is convex;
b) if P is in one set and Q is in the other, then segment PQ intersects the line.

Postulate 10. (*Space Separation Postulate*) The points of space that do not lie in a given plane form two sets such that:

a) Each of the sets is convex.

b) If P is in one set and Q is in the other, then segment PQ intersects the plane.

Postulate 11. (*Angle Measurement Postulate*) To every angle there corresponds a real number between 0° and 180°.

Postulate 12. (*Angle Construction Postulate*) Let $\overrightarrow{AB}$ be a ray on the edge of the half-plane H. For every r between 0 and 180, there is exactly one ray $\overrightarrow{AP}$ with P in H such that $m\measuredangle PAB = r$.

Postulate 13. (*Angle Addition Postulate*) If D is a point in the interior of $\measuredangle BAC$, then $m\measuredangle BAC = m\measuredangle BAD + m\measuredangle DAC$.

Postulate 14. (*Supplement Postulate*) If two angles form a linear pair, then they are supplementary.

Postulate 15. (*SAS Postulate*) Given a one-to-one correspondence between two triangles (or between a triangle and itself). If two sides and the included angle of the first triangle are congruent to the corresponding parts of the second triangle, then the correspondence is a congruence.

Postulate 16. (*Parallel Postulate*) Through a given external point there is at most one line parallel to a given line.

Postulate 17. (Area of Polygonal Region) To every polygonal region there corresponds a unique positive real number called the area.

Postulate 18. (Area of Congruent Triangles) If two triangles are congruent, then the triangular regions have the same area.

Postulate 19. (Summation of Areas of Regions) Suppose that the region R is the union of two regions R_1 and R_2. If R_1 and R_2 intersect at most in a finite number of segments and points, then the area of R is the sum of the areas of R_1 and R_2.

Postulate 20. (Area of a Rectangle) The area of a rectangle is the product of the length of its base and the length of its altitude.

Postulate 21. (Volume of Rectangular Parallelpiped) The volume of a rectangular parallelpiped is equal to the product of the length of its altitude and the area of its base.

Postulate 22. (*Cavalieri's Principle*) Given two solids and a plane. If for every plane that intersects the solids and is parallel to the given plane the two intersections determine regions that have the same area, then the two solids have the same volume.

School Mathematics Study Group, *Geometry*. New Haven: Yale University Press, 1961.