# The Origins of Geometry

## 1.1: Introduction

In the beginning geometry was a collection of rules for computing lengths, areas, and volumes. Many were crude approximations derived by trial and error. This body of knowledge, developed and used in construction, navigation, and surveying by the Babylonians and Egyptians, was passed to the Greeks. The Greek historian Herodotus ($5^{th}$ century BC) credits the Egyptians with having originated the subject, but there is much evidence that the Babylonians, the Hindu civilization, and the Chinese knew much of what was passed along to the Egyptians.

The Babylonians of 2,000 to 1,600 BC knew much about navigation and astronomy, which required knowledge of geometry. Clay tablets from the Sumerian (2100 BC) and the Babylonian cultures (1600 BC) include tables for computing products, reciprocals, squares, square roots, and other mathematical functions useful in financial calculations. Babylonians were able to compute areas of rectangles, right and isosceles triangles, trapezoids and circles. They computed the area of a circle as the square of the circumference divided by twelve. The Babylonians were also responsible for dividing the circumference of a circle into 360 equal parts. They also used the Pythagorean Theorem (long before Pythagoras), performed calculations involving ratio and proportion, and studied the relationships between the elements of various triangles. See Appendices A and B for more about the mathematics of the Babylonians.

## 1.2: A History of the Value of $\pi$

The Babylonians also considered the circumference of the circle to be three times the diameter. Of course, this would make $\pi = 3$ — a small problem. This value for $\pi$ carried along to later times. The Roman architect Vitruvius took $\pi = 3$. Prior to this it seems that the Chinese mathematicians had taken the same value for $\pi$. This value for $\pi$ was sanctified by the ancient Jewish civilization and sanctioned in the scriptures. In I Kings 7:23 we find:

> *He then made the sea of cast metal: it was round in shape, the diameter rim to rim being ten cubits: it stood five cubits high, and it took a line thirty cubits long to go round it.*
>
> – The New English Bible

The same verse can be found in II Chronicles 4:2. It occurs in a list of specifications for the great temple of Solomon, built around 950 BC and its interest here is that it gives $\pi = 3$. Not a very accurate value of course and not even very accurate in its day, for the Egyptian and Mesopotamian values of $\frac{25}{8} = 3.125$ and $\sqrt{10} = 3.162$ have been traced to much earlier dates. Now in defense of Solomon's craftsmen it should be noted that the item being described seems to have been a very large brass casting, where a high degree of geometrical precision is neither possible nor necessary. Rabbi Nehemiah attempted to change the value of $\pi$ to 22/7 but was rejected.

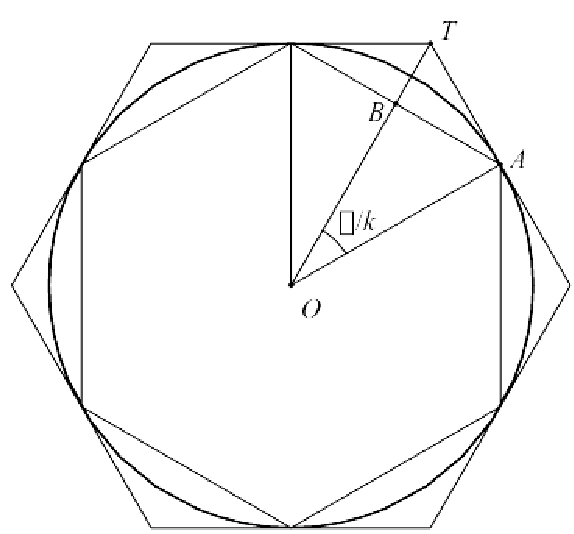
The fact that the ratio of the circumference to the diameter of a circle is constant has been known for so long that it is quite untraceable. The earliest values of $\pi$ including the *Biblical* value of 3, were almost certainly found by measurement. In the Egyptian Rhind Papyrus, which is dated about 1650 BC, there is good evidence for $4\left(\frac{8}{9}\right)^2 = 3.16$ as a value for $\pi$.

The first theoretical calculation of $\pi$ seems to have been carried out by Archimedes of Syracuse (287–212 BC). He obtained the approximation

$$\frac{223}{71} < \pi < \frac{22}{7}.$$

Before giving an indication of his proof, notice that very considerable sophistication involved in the use of inequalities here. Archimedes knew what so many people to this day do not that $\pi$ does not equal 22/7, and made no claim to have discovered the exact value. If we take his best estimate as the average of his two bounds we obtain 3.1418, an error of about $0.0002\pi$.

The following is Archimedes' argument. Consider a circle of radius 1, in which we inscribe a regular polygon of $3\times 2^{n-1}$ sides, with semiperimeter $b_n$, and superscribe a regular polygon of $3\times 2^{n-1}$ sides, with semiperimeter $a_n$. The diagram for the case $n$ = 2 is on the right.




The effect of this procedure is to define an increasing sequence $\{b_1, b_2, b_3, \ldots\}$ and a decreasing sequence $\{a_1, a_2, a_3, \ldots\}$ so that both sequences have limit $\pi$.

We are going to use some trigonometric notation which was not available to Archimedes. We can see that the two semiperimeters are given by

$$a_n = K\tan(\pi / K), b_n = K\sin(\pi / K),$$

where $K = 3\times 2^{n-1}$. Likewise, we have

$$a_{n+1} = 2K\tan(\pi / 2K), b_{n+1} = 2K\sin(\pi / 2K).$$

Now, you can use a couple of trigonometric identities to show that

$$\frac{1}{a_n}+\frac{1}{b_n}=\frac{2}{a_{n+1}} \qquad (0.1)$$
$$a_{n+1}b_n=(b_{n+1})^2$$

Archimedes started from $a_1=3\tan(\pi/3)=3\sqrt{3}$ and $b_1=3\sin(\pi/3)=3\sqrt{3}/2$ and calculated $a_2$ using Equation (2.1), then $b_2$ using (2.1), then $a_3$ using (2.1), then $b_3$ using (2.1), and so forth. He continued until he had calculated $a_6$ and $b_6$. His conclusion was that $b_6<\pi<a_6$.

Archimedes did not have the advantage of an algebraic and trigonometric notation and had to derive (2.1) purely by geometry. Moreover he did not even have the advantage of our decimal notation for numbers, so that the calculation of $a_6$ and $b_6$ from (2.1) was not an easy task. So this was a pretty stupendous feat both of imagination and of calculation. Our real wonder is not that he stopped with polygons of 96 sides, but that he went so far.

Now, if we can compute $\pi$ to this accuracy, we should be able to compute it to greater accuracy. Various people did, including:

| | | |
|---|---|---|
| Ptolemy | (c. 150 AD) | 3.1416 |
| Zu Chongzhi | (430–501 AD) | 355/113 |
| al-Khwarizmi | (c. 800 ) | 3.1416 |
| al-Kashi | (c. 1430) | 14 places |
| Viète | (1540–1603) | 9 places |
| Roomen | (1561–1615) | 17 places |
| Van Ceulen | (c. 1600) | 35 places |

Table 1: Early Calculations of $\pi$

Except for Zu Chongzhi, about whom little to nothing is known and who is very unlikely to have known about Archimedes' work, there was no theoretical progress involved in these improvements, only greater stamina in calculation. Notice how the all of this work, as in all scientific matters, passed from Europe to the East for the millennium 400 to 1400 AD.

Al-Khwarizmi lived in Baghdad, and incidentally gave his name to *algorithm*, while the words *al jabr* in the title of one of his books gave us the word *algebra*. Al-Kashi lived still further east, in Samarkand, while Zu Chongzhi, of course, lived in China.

The European Renaissance brought about in due course a whole new mathematical world. It was also a move from the geometry calculation of $\pi$ seen earlier to an analytic computation. Among the first effects of this reawakening was the emergence of mathematical formulæ for $\pi$. One of the earliest was the work of Wallis (1616–1703)

$$\frac{2}{\pi}=\frac{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot \ldots}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot \ldots}.$$

One of the best-known formulæ is

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots.$$

This formula is sometimes attributed to Leibniz (1646–1716) but it seems to have been first discovered by James Gregory (1638–1675). These are both dramatic and astonishing formulæ, because the expressions on the righthand side are completely arithmetical in nature, while $\pi$ arises from geometry.

From the point of view of the calculation of $\pi$, however, neither of these is of any use at all. In Gregory's series, for example, to get 4 decimal places correct we require the error to be less than $0.00005=1/20,000$, and so we need about 10,000 terms of the series to get this type of accuracy. However, Gregory also showed the more general result than

$$\tan^{-1}(x)=x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots\ (-1\le x\le 1) \tag{0.2}$$

We get the first series from this one by substituting $x=1$. Now, use the fact that

$$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}$$

we get

$$\frac{\pi}{6}=\frac{1}{\sqrt{3}}\left(1-\frac{1}{3\cdot 3}+\frac{1}{5\cdot 3\cdot 3}-\frac{1}{7\cdot 3\cdot 3\cdot 3}+\ldots\right)$$

which converges much more quickly. The 10th term is

$$\frac{1}{19\times 3^9\sqrt{3}},$$

which is less than 0.00005, and so we have at least 4 places correct after just 9 terms.

An even better idea is to take the formula

$$\frac{\pi}{4}=\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right) \tag{0.3}$$

and then calculate the two series obtained by putting first 1/2 and the 1/3 into (2.2).

We could get even faster convergence if we could find some large integers $a$ and $b$ so that

$$\frac{\pi}{4} = \tan^{-1}\left(\frac{1}{a}\right) + \tan^{-1}\left(\frac{1}{b}\right)$$

In 1706 Machin found such a formula:

$$\frac{\pi}{4} = 4\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{239}\right) \qquad (0.4)$$

This is not too hard to prove. If you can prove (2.3) then there is no real extra difficulty about (2.4), except that the arithmetic is worse. The amazing fact is that someone came up with it in the first place.

When you have a formula like this, the only difficulty in computing $\pi$ is the monotony of completing the calculation. A few people did devote vast amounts of time and effort to this tedious effort. One of them, an Englishman named Shanks, used Machin's formula to calculate $\pi$ to 707 places, publishing the results of his many years of labor in 1873. Here is a summary of how the improvement went:

| | | |
|---|---|---|
| Sharp | 1699 | 71 digits using Gregory's result |
| Machin | 1701 | used an improvement to get 100 digits |
| and the following used his methods: | | |
| de Lagny | 1719 | 112 correct digits |
| Vega | 1789 | 126 places and in 1794 got 136 |
| Rutherford | 1841 | calculated 152 digits and in 1853 got 440 |
| Shanks | 1873 | calculated 707 places of which 527 were correct |

Table 2: Calculation of decimal value of $\pi$

Shanks knew that $\pi$ was irrational since this had been proved in 1761 by Lambert. Shortly after Shanks' calculation it was shown by Lindemann in 1882 that $\pi$ is transcendental, that is, $\pi$ is not the solution of any polynomial equation with integer coefficients. In fact this result of Lindemann showed that *squaring the circle* is impossible. The transcendentality of $\pi$ implies that there is no ruler and compass construction to construct a square equal in area to a given circle. Note that this takes us back to the geometry from the analytic and answers a geometry question *via* algebraic and analytic means.

Very soon after Shanks' calculation a curious statistical anomaly was noticed by De Morgan, who found that in the last of 707 digits there was a ***suspicious*** shortage of 7's. He mentions this in his *Budget of Paradoxes* of 1872 and it remained a curiosity until 1945 when Ferguson discovered that Shanks had made an error in the 528th place, after which all his digits were wrong. In 1949 a computer was used to calculate $\pi$ to 2000 places. In this and all subsequent computer expansions the number of 7's does not differ significantly from its expectation, and indeed the sequence of digits has so far passed all statistical tests for randomness.

We do know a lot more about, and all of it is analytic information. There are very interesting algorithms for computing $\pi$ with great accuracy and speed. Since the beginning of the *computer age* we have $\pi$ computed to greater and greater numbers of digits. The current record is over 1.2 trillion digits. See Table 3.

| Mathematician | Date | Places | Type of computer |
|---|---|---|---|
| Ferguson | 01/1947 | 710 | Desk calculator |
| Ferguson & Wrench | 09/1947 | 808 | Desk calculator |
| Smith & Wrench | 1949 | 1,120 | Desk calculator |
| Reitwiesner *et al* | 1949 | 2,037 | ENIAC |
| Nicholson & Jeenel | 1954 | 3,092 | NORAC |
| Felton | 1957 | 7,480 | PEGASUS |
| Genuys | 01/1958 | 10,000 | IBM 704 |
| Felton | 05/1958 | 10,021 | PEGASUS |
| Guilloud | 1959 | 16,167 | IBM 704 |
| Shanks & Wrench | 1961 | 10,0265 | IBM 7090 |
| Guilloud & Filliatre | 1966 | 250,000 | IBM 7030 |
| Guilloud & Dichampt | 1967 | 500,000 | CDC 6600 |
| Guilloud & Bouyer | 1973 | 1,001,250 | CDC 7600 |
| Miyoshi & Kanada | 1981 | 2,000,036 | FACOM M-200 |
| Guilloud | 1982 | 2,000,050 | unknown |
| Tamura | 1982 | 2,097,144 | MELCOM 900II |
| Tamura & Kanada | 1982 | 4,194,288 | HITACHI M-280H |
| Tamura & Kanada | 1982 | 8,388,576 | HITACHI M-280H |
| Kanada, Yoshino & Tamura | 1982 | 16,777,206 | HITACHI M-280H |
| Ushiro & Kanada | 10/1983 | 10,013,395 | HITACHI S-810/20 |
| Gosper | 10/1985 | 17,526,200 | SYMBOLICS 3670 |
| Bailey | 01/1986 | 29,360,111 | CRAY-2 |
| Kanada & Tamura | 09/1986 | 33,554,414 | HITACHI S-810/20 |
| Kanada & Tamura | 10/1986 | 67,108,839 | HITACHI S-810/20 |
| Kanada, Tamura & Kubo | 01/1987 | 134,217,700 | NEC SX-2 |
| Kanada & Tamura | 01/1988 | 201,326,551 | HITACHI S-820/80 |
| Chudnovskys | 05/1989 | 480,000,000 | CRAY-2 |
| Chudnovskys | 06/1989 | 525,229,270 | IBM 3090 |
| Kanada & Tamura | 07/1989 | 53,6870,898 | Hitachi S-820/80 |
| Chudnovskys | 08/1989 | 1,011,196,691 | m-zero |
| Kanada & Tamura | 11/1989 | 1,073,741,799 | m-zero |
| Chudnovskys | 08/1991 | 2,260,000,000 | Hitachi S-3800/480 |
| Chudnovskys | 05/1994 | 4,044,000,000 | Hitachi S-3800/480 |
| Kanada & Takahashi | 06/1995 | 3,221,220,000 | Hitachi S-3800/480 |
| Kanada & Takahashi | 08/1995 | 4,294,967,286 | Hitachi S-3800/480 |
| Kanada & Takahashi | 10/1995 | 6,442,450,000 | Hitachi S-3800/480 |

| | | | |
|---|---|---|---|
| Chudnovskys | 03/1996 | 8,000,000,000 | m-zero |
| Kanada & Takahashi | 04/1997 | 17,179,869,142 | Hitachi SR2201 |
| Kanada & Takahashi | 06/1997 | 51,539,600,000 | Hitachi SR2201 |
| Kanada & Takahashi | 04/1999 | 68,719,470,000 | Hitachi SR8000 |
| Kanada & Takahashi | 09/1999 | 206,158,430,000 | Hitachi SR8000 |
| Kanada *et al.* | 12/2002 | 1,241,100,000,000 | Hitachi SR8000/MP |

Table 3: Computer calculations of $\pi$

Just a note on how the notation $\pi$ arose. Oughtred in 1647 used the symbol $d/\pi$ for the ratio of the diameter of a circle to its circumference. David Gregory (1697) used $\pi/r$ for the ratio of the circumference of a circle to its radius. The first to use $\pi$ with its present meaning was an Welsh mathematician, William Jones, in 1706 when he states 3.14159 and $c=\pi$. Euler adopted the symbol in 1737 and it quickly became a standard notation.

### 1.2.3: Indiana and $\pi$

There are many false claims about the value of $\pi$, most of them coming from attempts to *square the circle* — one of the great unsolved problems of Greek mathematics. One of the best stories is that of the Indiana legislature in 1897. In the State of Indiana in 1897 the House of Representatives unanimously passed a Bill introducing *a new mathematical truth*.

> *Be it enacted by the General Assembly of the State of Indiana: It has been found that a circular area is to the square on a line equal to the quadrant of the circumference, as the area of an equilateral rectangle is to the square of one side.*
>
> (Section I, House Bill No. 246, 1897)}

The author of the bill was Edwin J. Goodwin, an M.D., of Solitude, Indiana. It seems that he was a crank (or amateur) mathematician who had been working on a procedure to *square the circle*. He contacted his Representative, one Taylor I. Record of Posey County on January 18, 1897, with his epoch-making suggestion: if the State would pass an Act recognizing his discovery, he would allow all Indiana textbooks to use it without paying him a royalty.

What a novel idea! How could the legislature pass up such an offer? Nobody in the Indiana Legislature knew enough mathematics to know that the *discovery* was nonsense, as it had been proved in the 1700's that it is impossible to *square the circle*. In due course the bill had its third House reading, and passed 67–0. At this point the text of the bill was published *and, of course, became the target for ridicule*}, in this and other states.

The next day, the following article appeared in the *Indianapolis Sentinel*:

> To SQUARE THE CIRCLE
>
> *Claims Made That This Old Problem Has Been Solved.*
>
> *The bill telling how to square a circle, introduced in the House by Mr. Record, is not intended to be a hoax. Mr. Record knows nothing of the bill with the exception that he*

*introduced it by request of Dr.Edwin Goodwin of Posey County, who is the author of the demonstration. The latter and State Superintendent of Public Instruction Geeting believe that it is the long-sought solution of the problem, and they are seeking to have it adopted by the legislature. Dr. Goodwin, the author, is a mathematician of note. He has it copyrighted and his proposition is that if the legislature will endorse the solution, he will allow the state to use the demonstration in its textbooks free of charge. The author is lobbying for the bill.'*

On *February 2, 1897, ...Representative S.E. Nicholson, of Howard County, chairman of the Committee on Education, reported to the House.*

*Your Committee on Education, to which was referred House Bill No. 246, entitled a bill for an act entitled an act introducing a new mathematical truth, has had same under consideration, and begs leave to report the same back to the House with the recommendation that said bill do pass.*

*The report was concurred in, and on February 8,1897, it was brought up for the second reading, following which it was considered engrossed. Then 'Mr. Nicholson moved that the constitutional rule requiring bills to be read on three days be suspended, that the bill may be read a third time now.' The constitutional rule was suspended by a vote of 72 to 0 and the bill was then read a third time. It was passed by a vote of 67 to 0, and the Clerk of the House was directed to inform the Senate of the passage of the bill.*

The newspapers reported the suspension of the constitutional rules and the unanimous passage of the bill matter-of-factly, except for one line in the *Indianapolis Journal* to the effect that *this is the strangest bill that has ever passed an Indiana Assembly*.

The bill was referred to the Senate on Feb. 10,1897 and was read for the first time on Feb. 11 and referred to the Committee on Temperance. *On Feb.12 Senator Harry S. New, of Marion County, Chairman of the Committee on Temperance, made the following report to the Senate:*

*Your committee on Temperance, to which was referred House Bill No.246, introduced by Mr.Record, has had the same under consideration and begs leave to report the same back to the Senate with the recommendation that said bill do pass.*

The Senate Journal mentions only that the bill was read a second time on Feb. 12, 1897, that there was an unsuccessful attempt to amend the bill by strike out the enacting clause, and finally it was postponed indefinitely. That the bill was killed appears to be a matter of dumb luck rather than the superior education or wisdom of the Senate. It is true that the bill was widely ridiculed in Indiana and other states, but what actually brought about the defeat of the bill is recorded by Professor C. A. Waldo in an article he wrote for the *Proceedings of the Indiana Academy of Science* in 1916. The reason he knows is that he happened to be at the State Capitol lobbying for the appropriation of the Indiana Academy of Science, on the day the Housed passed House Bill 246. When he walked in he found the debate on House Bill 246 already in progress. In his article, he writes (according to Edington):

*An ex-teacher from the eastern part of the state was saying: 'The case is perfectly simple. If we pass this bill which establishes a new and correct value for $\pi$, the author offers to our state without cost the use of his discovery and its free publication in our school text books, while everyone else must pay him a royalty.'*

The roll was then called and the bill passed its third and final reading in the lower house. A member then showed the writer [i.e. Waldo -AA] a copy of the bill just passed and asked him if he would like an introduction to the learned doctor, its author. He declined the courtesy with thanks remarking that he was acquainted with as many crazy people as he cared to know.

*That evening the senators were properly coached and shortly thereafter as it came to its final reading in the upper house they threw out with much merriment the epoch making discovery of the Wise Man from the Pocket.*

Note that this value of $\pi$ is mentioned in the *Guiness Book of Records* — as the ***most inaccurate value for π***.

It appears that Dr. Goodwin was not satisfied with just one value of $\pi$, as the following values appear in the bill:

*Since the rule in present use [presumably $\pi$ equals 3.14159...] fails to work ..., it should be discarded as wholly wanting and misleading in the practical applications," the bill declared. Instead, mathematically inclined Hoosiers could take their pick among the following formulae:*

i) *The ratio of the diameter of a circle to its circumference is 5/4 to 4. In other words, $\pi = 16/5 = 3.2$.*
ii) *The area of a circle equals the area of a square whose side is 1/4 the circumference of the circle. Working this out algebraically, we see that $\pi$ must be equal to 4.*
iii) *The ratio of the length of a $90^\circ$ arc to the length of a segment connecting the arc's two endpoints is 8 to 7. This gives us $\pi = \sqrt{2} \times 16/7$, or about 3.23.*

There may have been other values for $\pi$ as well; the bill was so confusingly written that it's impossible to tell exactly what Dr. Goodwin was getting at. Mathematician David Singmaster says he found six different values in the bill, plus three more in Goodwin's other writings and comments, for a total of nine.

## 1.3: Egyptian Geometry

Now, the Egyptians were not nearly as inventive as the Babylonians but they were extensive users of mathematics, especially geometry. They were extremely accurate in their construction, making the right angles in the Great Pyramid of Giza accurate to one part in 27,000. From the above approximation they computed the area of a circle to be the square of 8/9 of the diameter.

$$A = \left(\frac{8}{9}d\right)^2 = \left(\frac{16}{9}\right)^2 r^2$$

They also knew the Pythagorean Theorem and were able to compute volumes and dihedral angles of pyramids and cylinders.

## 1.4: Early Greek Geometry

The ancient knowledge of geometry was passed on to the Greeks. Maybe we should say that they gathered all that they could find about geometry. They seemed to be blessed with an inclination toward speculative thinking and the leisure to pursue this inclination. They insisted that geometric statements be established by deductive reasoning rather than trial and error. This began with Thales of Miletus (624–547 BC). He was familiar with the computations, right or wrong, handed down from Egyptian and Babylonian mathematics. In determining which of the computations were correct, he developed the first logical geometry.

This orderly development of theorems by proof was the distinctive characteristic of Greek mathematics and was new. He is credited with proving the following results:

i) A circle is bisected by any diameter.
ii) The base angles of an isosceles triangle are equal.
iii) The angles between two intersecting straight lines are equal.
iv) Two triangles are congruent if they have two angles and one side equal.
v) An angle inscribed in a semicircle is a right angle.

This new mathematics of Thales was continued over the next two centuries by Pythagoras of Samos (569–475 BC) and his disciples. Pythagoras is regarded as the first pure mathematician to deduce logically geometric facts from basic principles. He is credited with proving many theorems such as the angles of a triangle summing to $180^\circ$, and the infamous *Pythagorean Theorem* for a right-angled triangle (which had been known experimentally in Egypt and Babylon for over 1000 years). The Pythagorean school is considered as the (first documented) source of logic and deductive thought, and may be regarded as the birthplace of reason itself. As philosophers, they speculated about the structure and nature of the universe: matter, music, numbers, and geometry. The Pythagoreans, as a religious sect, believed that the elevation of the soul and union with God were achieved by the study of music and mathematics. They developed a large body of mathematics by using the deductive method.

Their foundation of plane geometry was brought to a conclusion around 440 BC in the *Elements* by the mathematician Hippocrates of Chios (470–410 BC). This treatise has been lost, but many historians agree that it probably covered most of Books I–IV of

Euclid's Elements, which appeared about a century later, circa 300 BC. In this first *Elements* Hippocrates included geometric solutions to quadratic equations and early methods of integration. He studied the classic problem of squaring the circle showing how to square a *lune*. He worked on duplicating the cube which he showed equivalent to constructing two mean proportionals between a number and its double. Hippocrates was also the first to show that the ratio of the areas of two circles was equal to the ratio of the squares of their radii.

Plato (427–347 BC) founded "The Academy" in 387 BC which flourished until 529 AD. He developed a theory of Forms, in his book *Phaedo*, which considers mathematical objects as perfect forms (such as a line having length but no breadth). He emphasized the idea of *proof* and insisted on accurate definitions and clear hypotheses, paving the way for Euclid, but we cannot attribute any major mathematical discoveries to him.

Theætetus of Athens (417–369 BC) was a student of Plato's, and the creator of solid geometry. He was the first to study the octahedron and the icosahedron, and thus construct all five regular (or *Platonic*) solids. This work of his formed Book XIII of Euclid's Elements. His work about rational and irrational quantities also formed Book X of Euclid.

Eudoxus of Cnidus (408–355 BC) developed a precursor to algebra by developing a theory of proportion which is presented in Book V of Euclid's Elements in which Definitions 4 and 5 establish Eudoxus' landmark concept of proportion. In 1872, Dedekind stated that his work on *cuts* for the real number system was inspired by the ideas of Eudoxus. Eudoxus also did early work on integration using his *method of exhaustion* by which he determined the area of circles and the volumes of pyramids and cones. This was the first seed from which the calculus grew two thousand years later.

Menaechmus (380–320 BC) was a pupil of Eudoxus, and discovered the conic sections. He was the first to show that ellipses, parabolas, and hyperbolas are obtained by cutting a cone in a plane not parallel to the base.

Euclid of Alexandria (325–265 BC) is best known for his 13 Book treatise *The Elements* (~300 BC), collecting the theorems of Pythagoras, Hippocrates, Thætetus, Eudoxus and other predecessors into a logically connected whole.

> *Euclid was a disciple of the Platonic school. Around 300 BC he produced the definitive treatment of Greek geometry and number theory in his thirteen-volume Elements. In compiling this masterpiece Euclid built on the experience and achievements of his predecessors in preceding centuries: on the Pythagoreans for Books I–IV, VII, and IX, on Archytas for Book VIII, on Eudoxus for Books V, VI, and XII, and on Theætetus for Books X and XIII. So completely did Euclid's work supersede earlier attempts at presenting geometry that few traces remain of these efforts. It's a pity that Euclid's heirs have not been able to collect royalties on his work, for he is the most widely read author in*

> *the history of mankind. His approach to geometry has dominated the teaching of the subject for over two thousand years. Moreover, the axiomatic method used by Euclid is the prototype for all of what we now call pure mathematics. It is pure in the sense of pure thought: no physical experiments need be performed to verify that the statements are correct-only the reasoning in the demonstrations need be checked.*

In this treatise, he organized this large body of known mathematics, including discoveries of his own, into the first formal system of mathematics. This formalness was exhibited by the fact that the *Elements* began with an explicit statement of assumptions called axioms or postulates, together with definitions. The other statements — theorems, lemmæ, corollaries — were then shown to follow logically from these axioms and definitions. Books I–IV, VII, and IX of the work dealt primarily with mathematics which we now classify as geometry, and the entire structure is what we now call Euclidean geometry.

## 1.5: Euclidean Geometry

Euclidean geometry was certainly conceived by its creators as an idealization of physical geometry. The entities of the mathematical system are concepts, suggested by, or abstracted from, physical experience but differing from physical entities as an idea of an object differs from the object itself. However, a remarkable correlation existed between the two systems. The angle sum of a mathematical triangle was stated to be $180^\circ$, if one measured the angles of a physical triangle the angle sum did indeed seem to be $180^\circ$, and so it went for a multitude of other relations. Because of this agreement between theory and practice, it is not surprising that many writers came to think of Euclid's axioms as self evident truths. Centuries later, the philosopher Immanuel Kant even took the position that the human mind is essentially Euclidean and can only conceive of space in Euclidean terms. Thus, almost from its inception, Euclidean geometry had something of the character of dogma. Euclid based his geometry on five fundamental assumptions:

**Postulate I**: *For every point $P$ and for every point $Q$ not equal to $P$ there exists a unique line $\ell$ that passes through $P$ and $Q$.*

**Postulate II**: *For every segment AB and for every segment CD there exists a unique point $E$ such that $B$ is between $A$ and $E$ and segment CD is congruent to segment BE.*

**Postulate III**: *For every point O and every point A not equal to O there exists a circle with center O and radius OA.*

**Postulate IV**: *All right angles are congruent to each other.*

Before we study the Fifth Postulate, let me say a few words about his definitions. Euclid's methods are imperfect by modern standards. He attempted to define everything in terms of a more familiar notion, sometimes creating more confusion than he removed. As an example:

*A point is that which has no part.*
*A line is breadthless length. A straight line is a line which lies evenly with the points on itself.*
*A plane angle is the inclination to one another of two lines which meet. When a straight line set upon a straight line makes adjacent angles equal to one another, each of the equal angles is a right angle.*

Euclid did not define *length*, *distance*, *inclination*, or *set upon*. Once having made the above definitions, Euclid never used them. He used instead the rules of interaction between the defined objects as set forth in his five postulates and other postulates that he implicitly assumed but did not state.

**Postulate V**: *If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles.*

## 1.6: Geometry after Euclid

Archimedes of Syracuse (287–212 BC) is regarded as the greatest of Greek mathematicians, and was also an inventor of many mechanical devices (including the screw, the pulley, and the lever). He perfected integration using Eudoxus' method of exhaustion, and found the areas and volumes of many objects. A famous result of his is that the volume of a sphere is two-thirds the volume of its circumscribed cylinder, a picture of which was inscribed on his tomb. He gave accurate approximations to $\pi$ and square roots. In his treatise *On Plane Equilibriums*, he set out the fundamental principles of mechanics, using the methods of geometry, and proved many fundamental theorems concerning the center of gravity of plane figures. In *On Spirals*, he defined and gave the fundamental properties of a spiral connecting radius lengths with angles as well as results about tangents and the area of portions of the curve. He also investigated surfaces of revolution, and discovered the 13 semi-regular (or *Archimedean*) polyhedra whose faces are all regular polygons. He was a great inventor of war machines for the soldiers of Syracuse and is credited with many other inventions. He was killed by a Roman soldier 212 BC when the Romans invaded Syracuse in the Second Punic War.

Apollonius of Perga (262–190 BC) was called "The Great Geometer". His famous work was *Conics* consisting of 8 Books. In Books 5 to 7 he studied normals to conics and determined the center of curvature and the evolute of the ellipse, parabola, and hyperbola. In another work *Tangencies* he showed how to construct the circle which is tangent to three objects (points, lines or circles). He also computed an approximation for $\pi$ better than the one of Archimedes.

Hipparchus of Rhodes (190–120 BC) is the first to systematically use and document the foundations of trigonometry, and may have invented it. He published several books of trigonometric tables and the methods for calculating them. He based his tables on dividing a circle into 360 degrees with each degree divided into 60 minutes. This is the first recorded use of this subdivision. In other work, he applied trigonometry to astronomy making it a practical predictive science.

Heron of Alexandria (10–75 AD) wrote *Metrica* which gives methods for computing areas and volumes. Book I considers areas of plane figures and surfaces of 3D objects, and contains his now-famous formula for the area of a triangle $=\sqrt{s(s-a)(s-b)(s-c)}$ where $s=(a+b+c)/2$ is the semiperimeter. Book II considers volumes of three-dimensional solids. Book III deals with dividing areas and volumes according to a given ratio, and gives a method to find the cube root of a number.

Menelaus of Alexandria (70–130 AD) developed spherical geometry in his only surviving work *Sphaerica*. In Book I, he defines spherical triangles using arcs of great circles. This marked a turning point in the development of spherical trigonometry. Book II applies spherical geometry to astronomy; and Book III deals with spherical trigonometry including *Menelaus's theorem* about how a straight line cuts the three sides of a triangle in proportions whose product is $-1$.

Claudius Ptolemy (85–165 AD) wrote the *Almagest*[1] giving the mathematics for the geocentric theory of planetary motion. The books are believed to have been written in 150 AD. The work is considered to be one of the great masterpieces of early mathematical and scientific works. The *Almagest* remained the major work in astronomy for 1400 years until it was superceded by the heliocentric theory of Copernicus. In Books I and II Ptolemy refined the foundations of trigonometry based on the chords of a circle established by Hipparchus. In this treatise is one of the results for which he is known, known as *Ptolemy's Theorem* it states that for a quadrilateral inscribed in a circle, the product of its diagonals is equal to the sum of the products of its opposite sides. From this, he derived the chord formulæ for $\sin(a+b)$, $\sin(a-b)$, and $\sin(a/2)$, and used these to compute detailed trigonometric tables.

Pappus of Alexandria (290–350 AD) was the last of the great Greek geometers. His major work in geometry is *Synagoge* or *The Collection*, a handbook on a wide variety of topics: arithmetic, mean proportionals, geometrical paradoxes, regular polyhedra, the spiral and quadratrix, trisection, honeycombs, semiregular solids, minimal surfaces, astronomy, and mechanics. In Book VII, he proved *Pappus' Theorem* which forms the basis of modern projective geometry; and also proved *Guldin's Theorem* (rediscovered in 1640 by Guldin) to compute a volume of revolution.

[1] The name *Almagest* is the Latin form of the shortened title "*al mijisti*" of the Arabic title "*al-kitabu-l-mijisti*", meaning "The Great Book".

Hypatia of Alexandria (370–415 AD was the first woman to make a substantial contribution to the development of mathematics. She learned mathematics and philosophy from her father Theon of Alexandria, and assisted him in writing an eleven part commentary on Ptolemy's *Almagest*, and a new version of Euclid's *Elements*. Hypatia also wrote commentaries on Diophantus' *Arithmetica*, Apollonius' *Conics* and Ptolemy's astronomical works. About 400 AD Hypatia became head of the Platonist school at Alexandria and lectured there on mathematics and philosophy. Although she had many prominent Christians as students, she ended up being brutally murdered by a fanatical Christian sect that regarded science and mathematics to be pagan. Nevertheless, she is the first woman in history recognized as a professional geometer and mathematician.

One of the themes that ran through Geometry following Euclid was the search for a replacement or a proof of dependence of his Fifth Postulate. No one seemed to like this Fifth Postulate, possibly not even Euclid himself — he did not use it until Proposition 29. The reason that this statement seems out of place is that the first four postulates seem to follow from experience — try to draw more than one line through 2 different points. The Fifth Postulate is non-intuitive. It does come from the study of parallel lines, though. An equivalent statement to this postulate is:

**Playfair's Postulate**: *Given a line and a point not on that line, there exists one and only one line through that point parallel to the given line.*

Euclid's Fifth Postulate seemed to be too burdensome. If it is so complicated, then it should follow from the other axioms. Since it is not intuitive, we should be able to prove it as a theorem. We should be able to prove that it is dependent in this Axiom system. If we have a set of axioms $A_1, A_2, \ldots, A_n$ for our mathematical system and we can prove that Axiom $A_n$ is *derivable*, or *provable*, from the other axioms, then $A_n$ is indeed redundant.

In a sense analogous to linear algebra, we are looking for a *basis* for this mathematical system. Unlike vector spaces and linear algebra, however, there is not a unique number of elements in this basis, for it includes the axioms, definitions, and the rules of logic that you use.

Many people have tried to prove the Fifth Postulate. The first known attempt to prove Euclid V, as it became known, was by Posidonius (1st century BC). He proposed to replace the definition of parallel lines (those that do not intersect) by defining them as coplanar lines that are everywhere equidistant from one another. It turns out that without Euclid V you cannot prove such lines exist. It is true that such a statement that parallel lines are equidistant from one another is equivalent to Euclid V.

Ptolemy followed with a proof that used the following assumption:

> *For every line $\ell$ and every point $P$ not on $\ell$, there exists at most one line $m$ through $P$ such that $m$ is parallel to $l$.*

We will show in Section XX that this statement is equivalent to Euclid V, and therefore this did not constitute a proof of Euclid V.

Proclus (410–485 AD) also attempted to prove Euclid V. His argument used a limiting process. He retained all of Euclid's definitions, all of his assumptions except Euclid V, and hence all of his propositions which did not depend on Euclid V. His plan was (1) to prove on this basis that a line which meets one of two parallels also meets the other, and (2) to deduce Euclid V from this proposition. His handling of step (2) was correct. The argument in step (1) runs substantially as follows. Let $g$ and $h$ be parallel lines and let another line $k$ meet $h$ at $P$. From $Q$, a point of $k$ situated between $g$ and $h$, drop a perpendicular to $h$. As $Q$ recedes indefinitely far from $P$, its distance $QR$ from $h$ increases and exceeds any value, however great. In particular $QR$ will exceed the distance between $g$ and $h$. For some position of $Q$ then $QR$ will equal the distance between $g$ and $h$. When this occurs $k$ will meet $g$.

There are a number of assumptions here which go beyond those found in Euclid. I will mention only the following two:

- the distance from one of two intersecting lines to the other increases beyond all bounds as we recede from their common point,
- the distance between two parallels never exceeds some finite value.

The first of the two assumptions is not a grave error on the part of Proclus, for it can be proved as a theorem on the basis of what he assumed from Euclid. Unfortunately for Proclus, his second assumption is equivalent to Euclid V.

Nasiraddin (1201–1274), John Wallis (1616–1703), Legendre (1752–1833), Wolfgang Bolyai, Girolamo Saccheri (1667–1733), Johann Heinrich Lambert (1728–1777), and many others tried to prove Euclid V, and failed. In these failures there developed a goodly number of substitutes for Euclid V; *i.e.*, statements that were equivalent to the statement of Euclid V. The following is a list of some of these that are more common:

1. *Through a point not on a given line there passes not more than one parallel to the line.*
2. *Two lines that are parallel to the same line are parallel to each other.*
3. *A line that meets one of two parallels also meets the other.*
4. *If two parallels are cut by a transversal, the alternate interior angles are equal.*
5. *There exists a triangle whose angle-sum is two right angles.*
6. *Parallel lines are equidistant from one another.*
7. *There exist two parallel lines whose distance apart never exceeds some finite value.*
8. *Similar triangles exist which are not congruent.*
9. *Through any three non-collinear points there passes a circle.*

10. *Through any point within any angle a line can be drawn which meets both sides of the angle.*
11. *There exists a quadrilateral whose angle-sum is four right angles.*
12. *Any two parallel lines have a common perpendicular*.

It fell to three different mathematicians independently to show that Euclid V is not provable from the other axioms and what is derivable from them. These were Karl Friedrich Gauss, János Bolyai, and Nicolai Ivanovich Lobachevskii. Once these men broke the ice, the pieces of geometry began to fall into place. More was learned about non-Euclidean geometries—hyperbolic and elliptic or doubly elliptic. The elliptic geometry was studied by Riemann, gave rise to Riemannian geometry and manifolds, which gave rise to differential geometry which gave rise to relativity theory, *et al*..

This gives us something to anticipate as we learn more about geometry. We will spend our time studying hyperbolic geometry, for it lends itself to better study — not requiring major changes in the axiom system that we have chosen. We may have an opportunity to see that hyperbolic geometry is now lending itself to considerations in the latest research areas of mathematics.