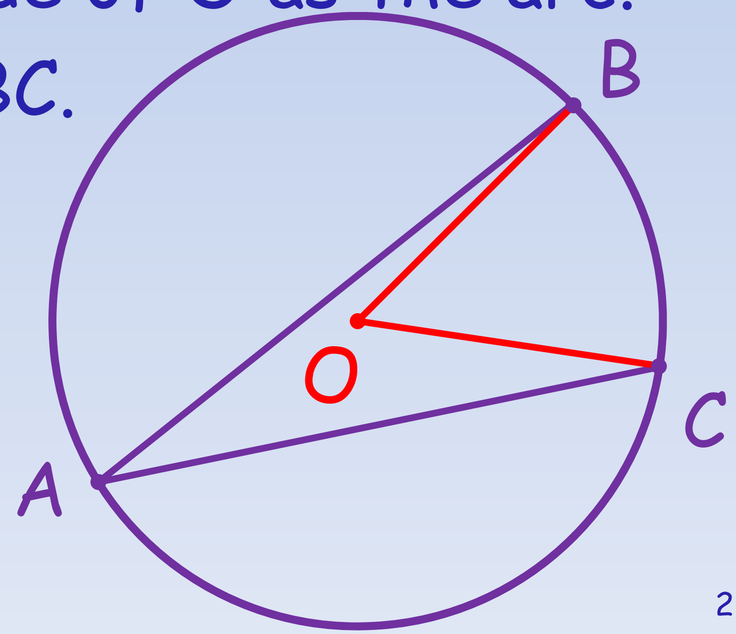
Circles, Arcs and Angles

MA 341 – Topics in Geometry
Lecture 09

Star Trek Lemma

- A, B, C are points on a circle centered at O.
- Angle $\angle BAC$ an *inscribed* angle
- Angular measure of arc BC is the measure of the central angle $\angle BOC$, where the angle is measured on the same side of O as the arc.
- $\angle BAC$ subtends the arc BC.



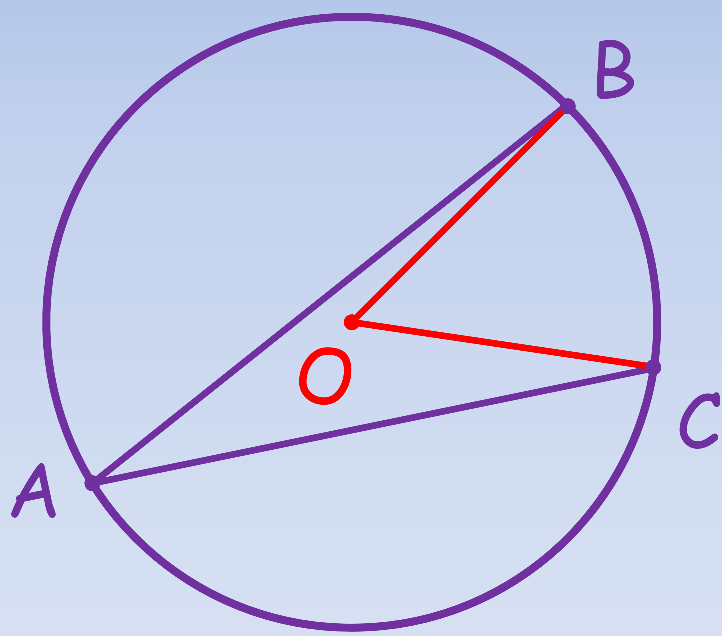
Star Trek Lemma

The measure of the inscribed angle is half of the angular measure of the arc it subtends.

There are several cases to the proof of the lemma. We will look only at the case where $\angle BAC$ is an acute angle and the center, O, lies in the interior of the angle, as in our figure.

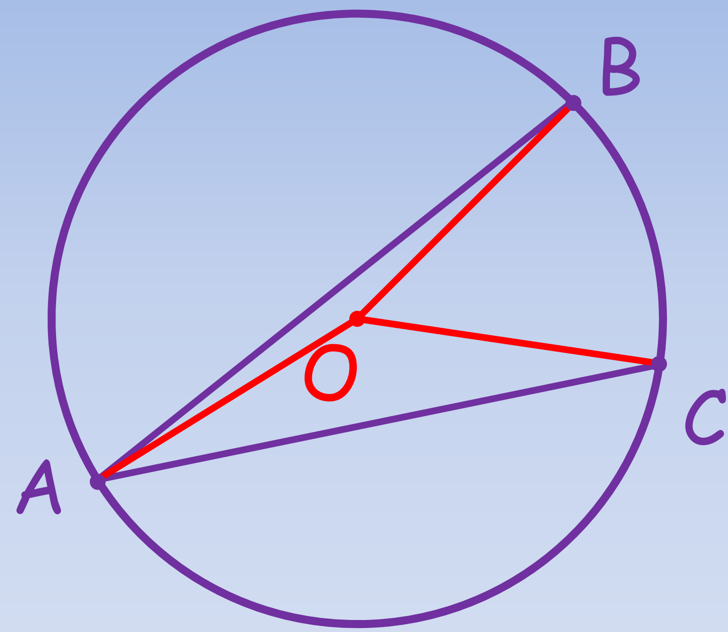
Star Trek Lemma

The measure of the inscribed angle is half of the angular measure of the arc it subtends.



Star Trek Lemma

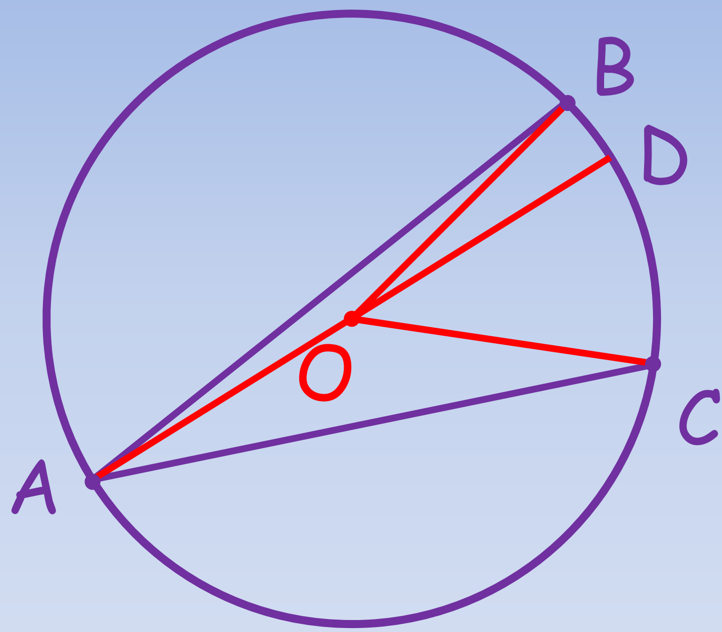
Note that OA, OB and OC are all radii



Thus, ΔAOB, ΔAOC and ΔBOC are isosceles

Star Trek Lemma

Extend the segment OA until it meets the circle at a point D.

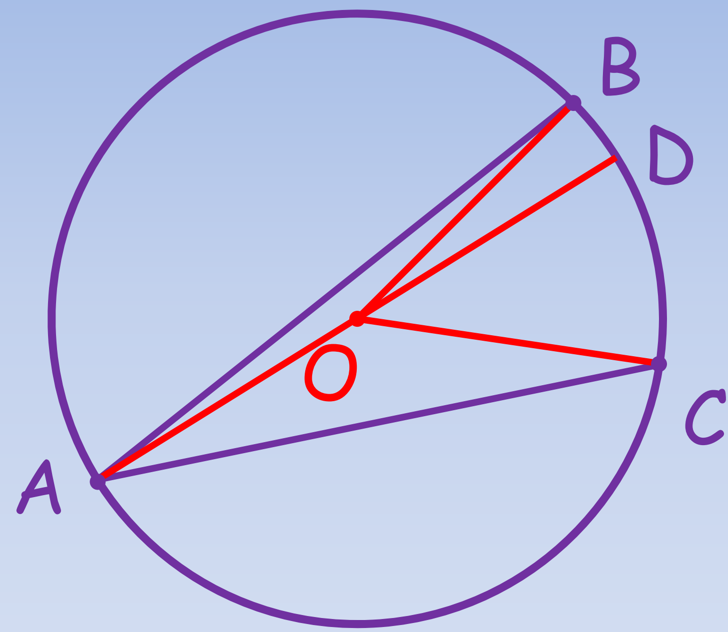


ΔAOB isosceles => $\angle BAO = \angle OBA$

$$\angle BOD = \angle 0BA + \angle BAO = 2\angle BAO$$

Star Trek Lemma

$\angle BOD = \angle OBA + \angle BAO = 2\angle BAO$



ΔAOC isosceles => $\angle CAO = \angle OCA$

$\angle COD = \angle OCA + \angle CAO = 2\angle CAO$

Star Trek Lemma

$$\angle BOD = \angle OBA + \angle BAO = 2\angle BAO$$

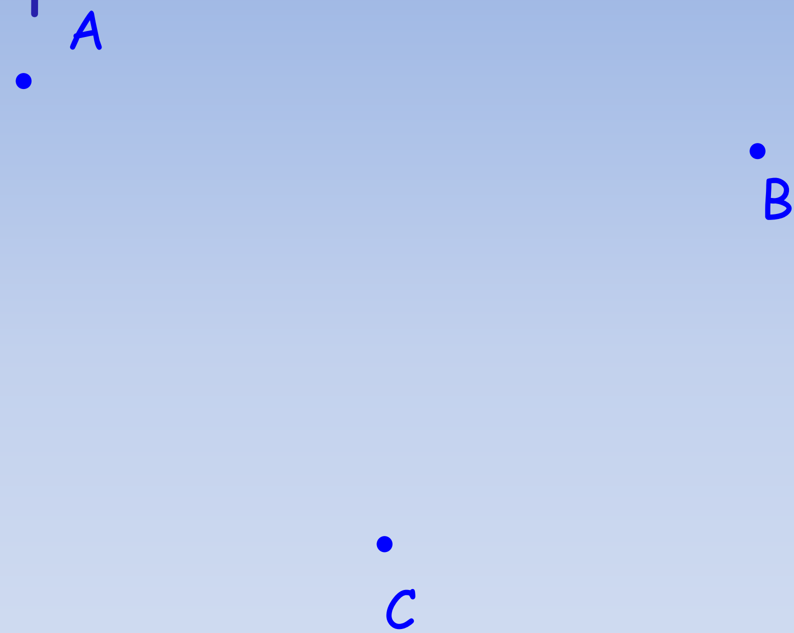
$$\angle COD = \angle OCA + \angle CAO = 2\angle CAO$$

$$\begin{aligned} \angle BOC &= \angle BOD + \angle DOC \\ &= 2\angle BAO + 2\angle CAO \\ &= 2\angle BAC \end{aligned}$$

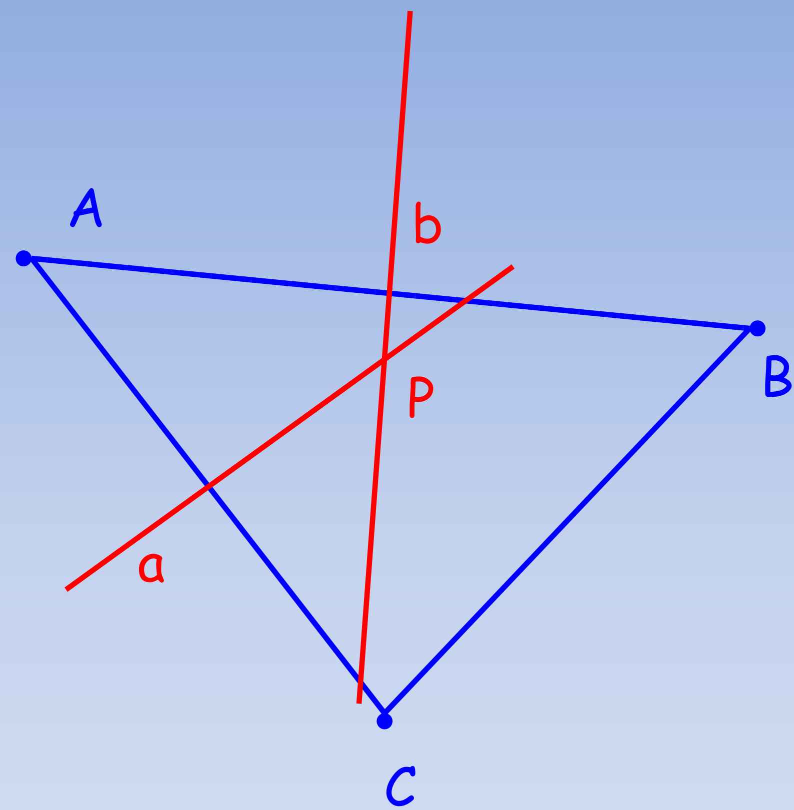
Why the Star Trek Lemma?

Circle Theorem

There is exactly one circle through any three non-collinear points.



Circle Theorem

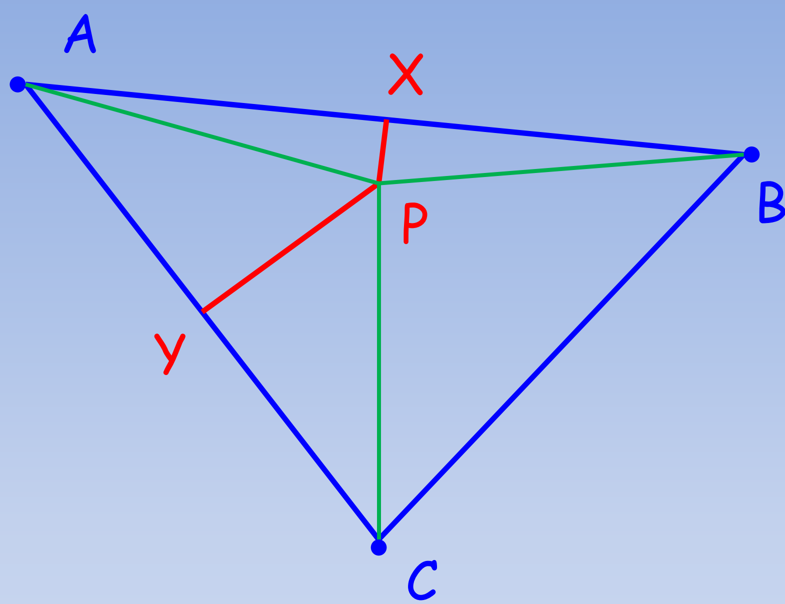


Let a be the perpendicular bisector of *AC* and b the perpendicular bisector of *AB*.

Why do they intersect?

Let the intersection be P.

Circle Theorem



Construct PA, PB, PC

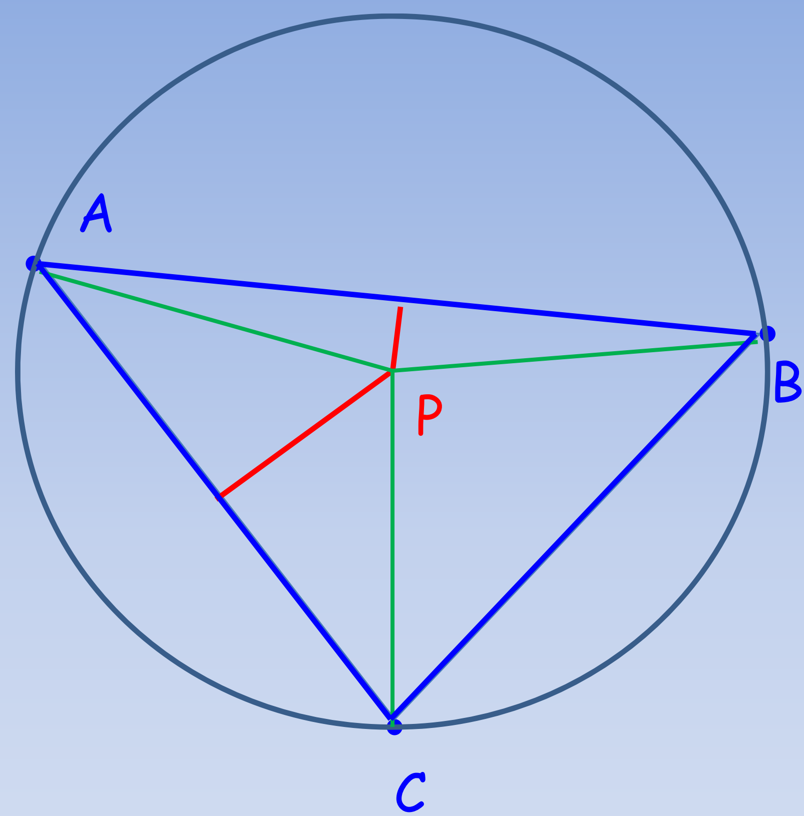
$\Delta PBX \cong \Delta PAX$ – why? $\Rightarrow$ $PB \cong PA$

$\Delta PAY \cong \Delta PCY$ – why? $\Rightarrow$ $PA \cong PC$

$\Rightarrow$ $PA \cong PB \cong PC$

A, B, C equidistant from P. Text shows other direction

Circle Theorem



Inscribed Parallelograms

A polygon is inscribed in a circle if all of its vertices lie on the circle and its interior is interior to the circle.

Theorem: Opposite angles of an inscribed quadrilateral are supplementary.

Inscribed Parallelograms

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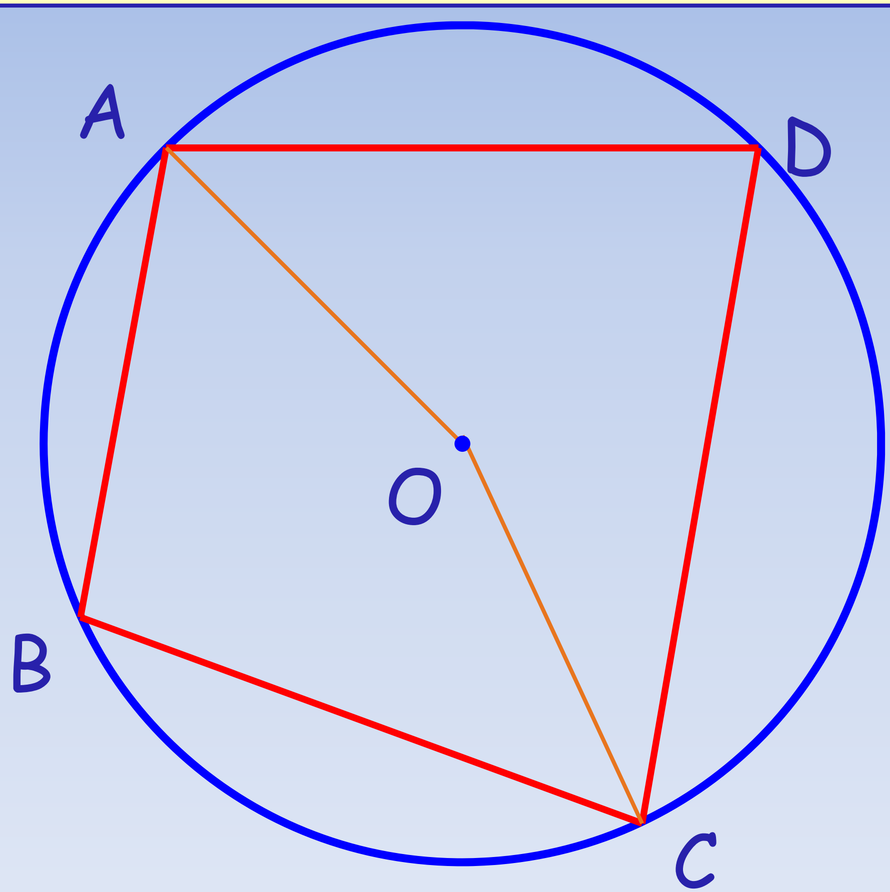
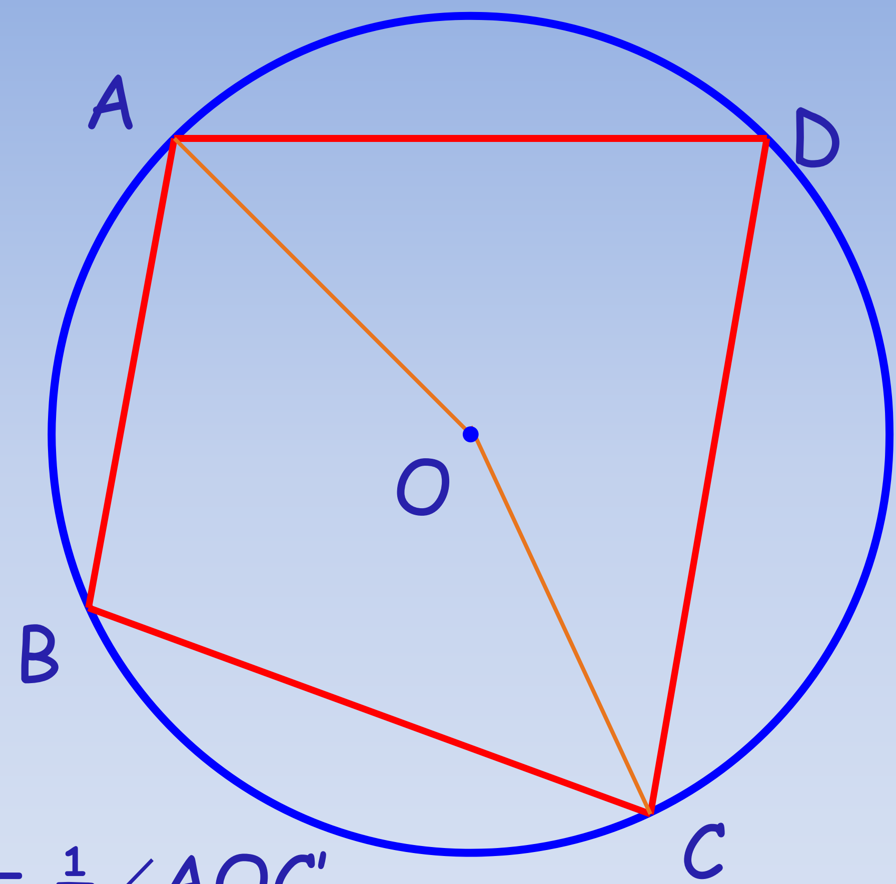


Inscribed Parallelograms

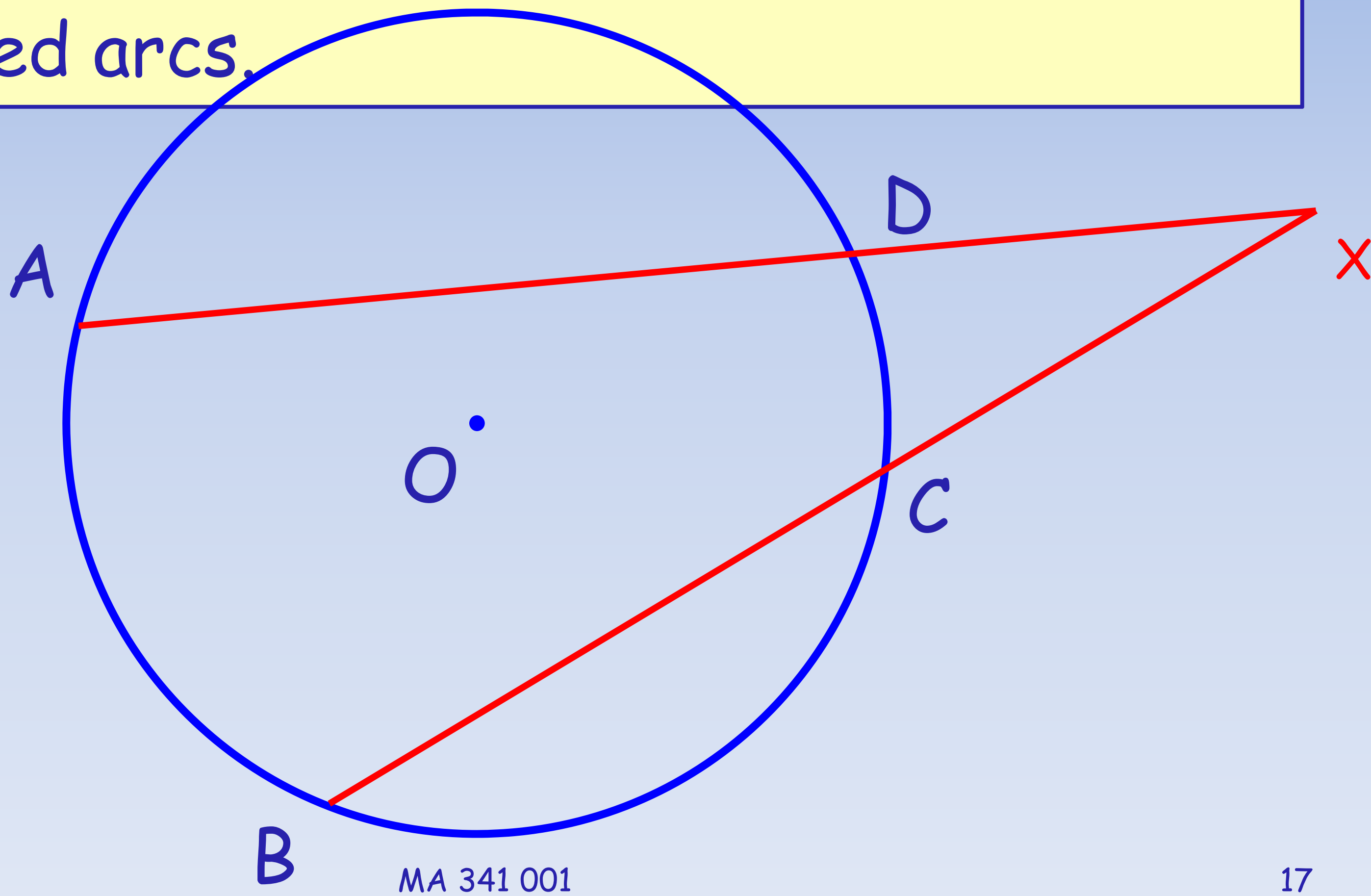
$\angle ABC = \frac{1}{2}\angle AOC$
$\angle ADC = \frac{1}{2}\angle AOC'$



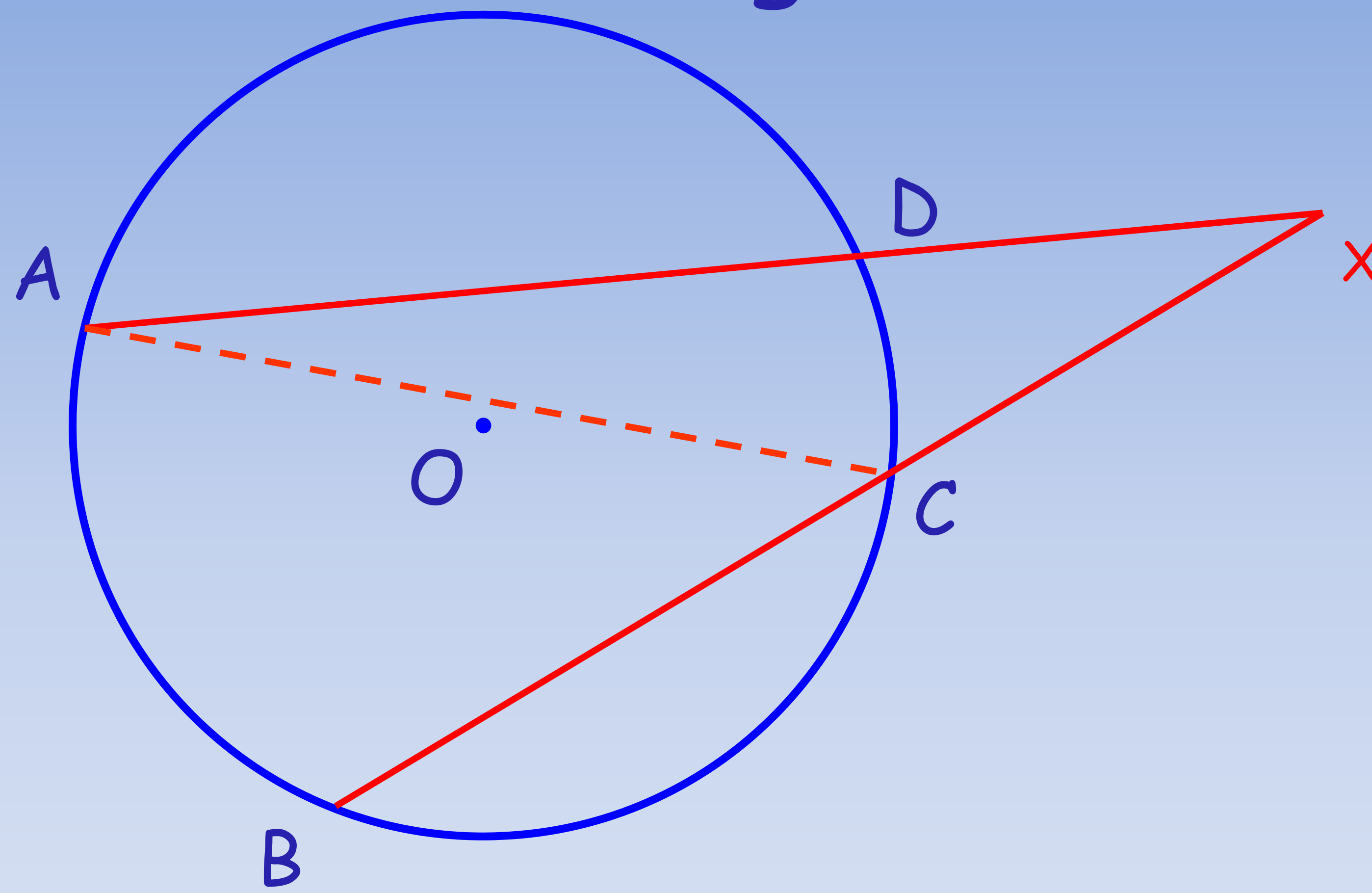
$\angle ABC + \angle ADC = \frac{1}{2}\angle AOC + = \frac{1}{2}\angle AOC'$
$\angle ABC + \angle ADC = \frac{1}{2}(360°) = 180°$

Secants and Angles

Theorem: The angle between two secants drawn to a circle from an exterior point is equal to half the difference of the two subtended arcs.



Secants and Angles



$\angle AXB = 180 - \angle XAC - \angle XCA$

$\angle AXB = 180 - \angle XAC - (180 - \angle ACB) = \angle ACB - \angle XAC$

Secants and Angles

$\angle AXB = 180 - \angle XAC - \angle XCA$

$\angle AXB = 180 - \angle XAC - (180 - \angle ACB) = \angle ACB - \angle DAC$

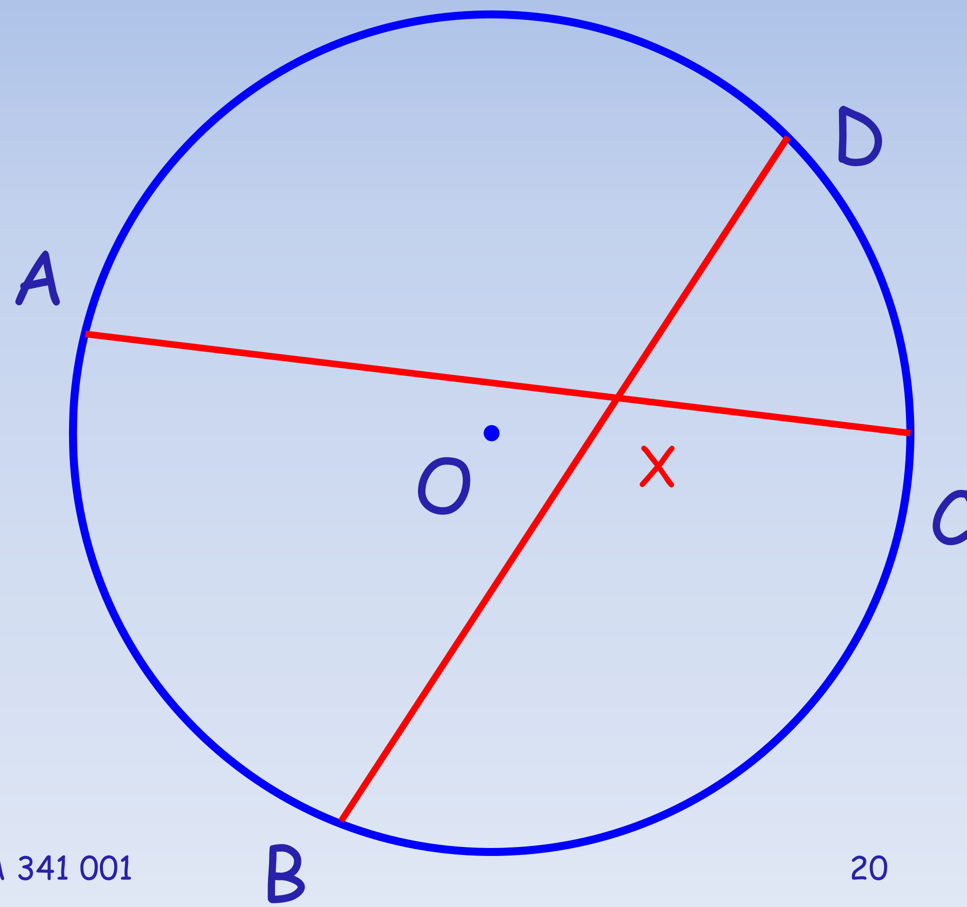
$\angle AXB = \frac{1}{2}\angle AOB - \frac{1}{2}\angle DOA$

$\angle AXB = \frac{1}{2}(\overset{\frown}{AB} - \overset{\frown}{CD})$

Chords and Angles

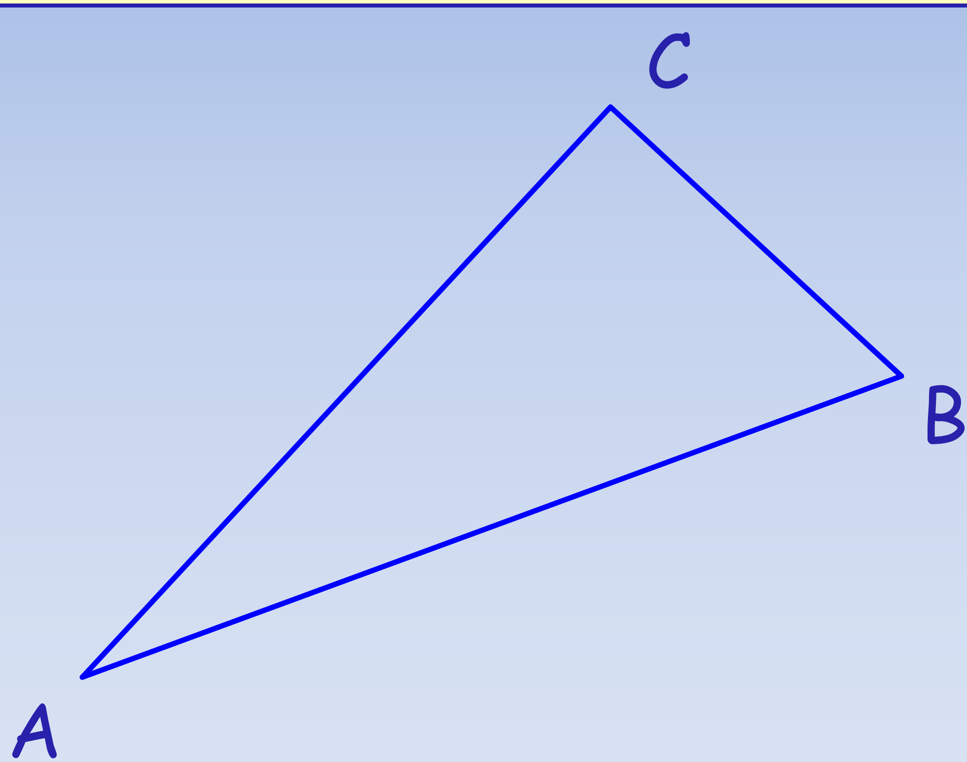
Theorem: The angle between two chords is equal to half the sum of the two subtended arcs.

$$\angle AXB = \tfrac{1}{2}(\overset{\frown}{AB} + \overset{\frown}{CD})$$



Circles and Right Angles

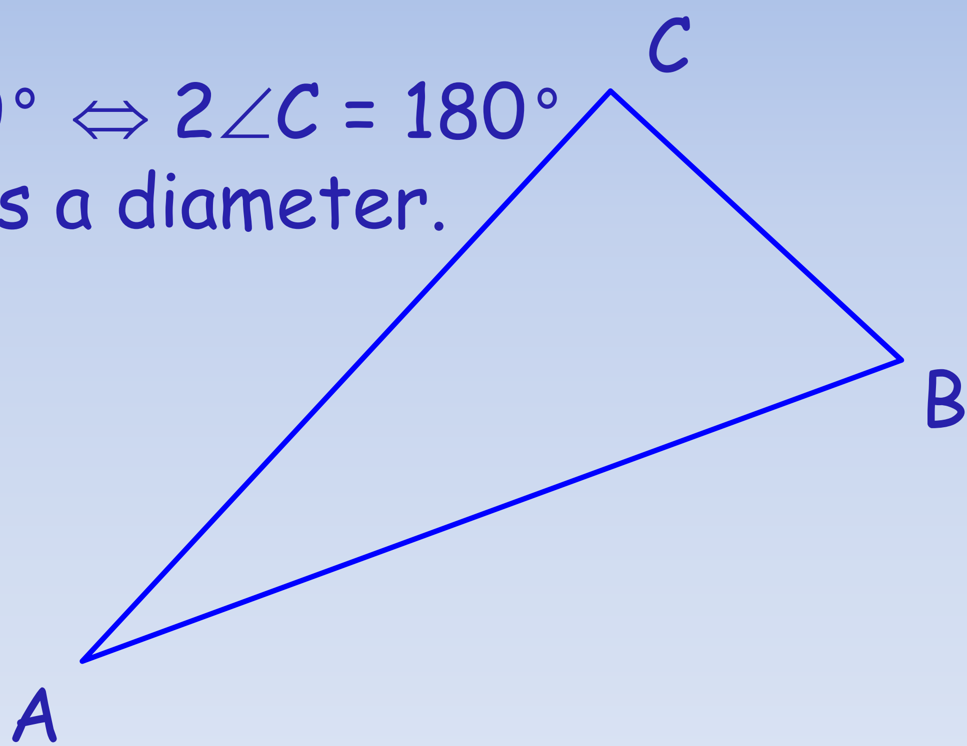
Theorem: Given ΔABC the angle at vertex C is a right angle if and only if AB is a diameter of a circle.



Circles and Right Angles

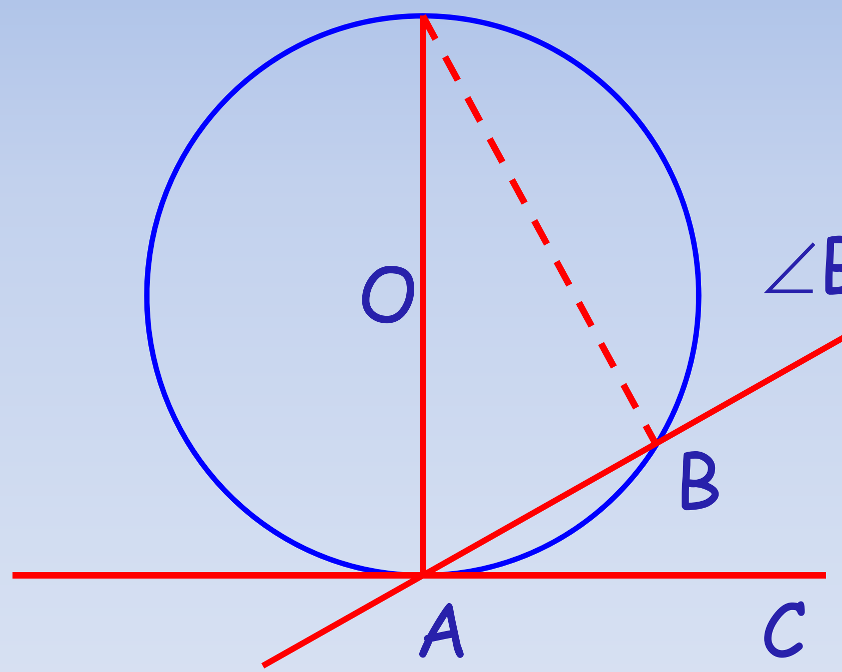
The circumcircle for ΔABC exists. Thus, $\angle ACB$ is inscribed in the circle and the arc AB is measured by $2\angle C$.

$\angle C = 90° \Leftrightarrow 2\angle C = 180°$
$\Leftrightarrow$ AB is a diameter.



Tangents and Chords

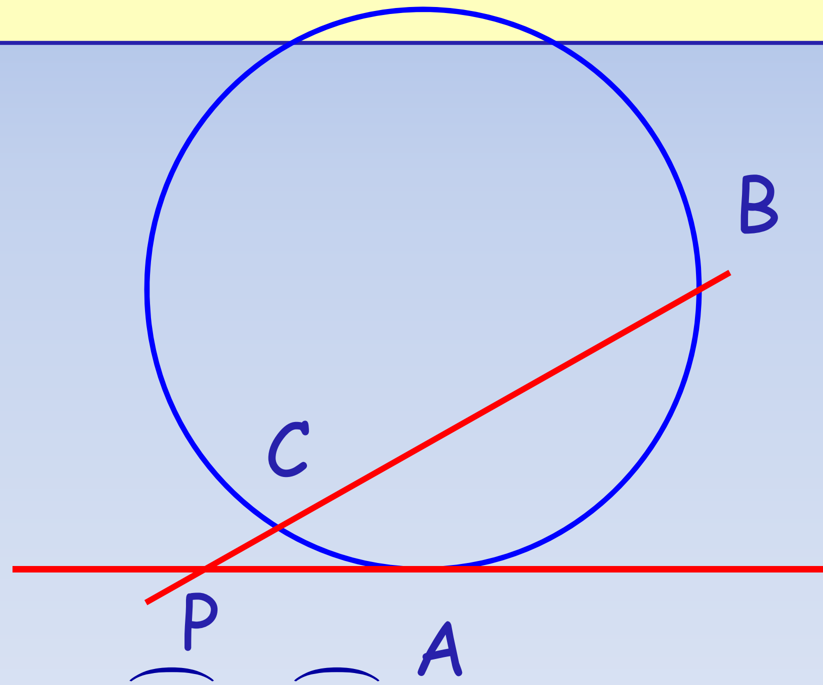
Theorem: The angle between a chord and a tangent at one of its endpoints is equal to half the subtended arc.



$\angle BAC = \frac{1}{2}\angle AOB$

Tangents and Secants

Theorem: The angle between a secant and a tangent at a point outside the circle is equal to half the difference of the two subtended arcs.



$$\angle BPA = \tfrac{1}{2}(\overset{\frown}{AB} - \overset{\frown}{AC})$$