Similarity

MA 341 – Topics in Geometry
Lecture 10

Similarity

- Definition
- AAA Theorem
- Proportionality
- Similarity and parallelism
- Similarity theorems
- Ceva's Theorem

Definition

1. Definition: Two triangles are similar if their corresponding angles are equal.
2. Definition: Triangles are similar if they have the same shape, but can be different sizes.
3. Definition: Two geometric shapes are similar if there is a rigid motion of the plane that maps one onto the other.

Definition

Working Definition: Two triangles are similar if their corresponding angles are equal.

$\Delta ABC \sim \Delta DEF$

means that

$\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$

AA Similarity Theorem

Theorem: Given two triangles ΔABC and ΔDEF, suppose that $\angle A = \angle D$ and $\angle B = \angle E$. Then $\angle C = \angle F$, and so $\Delta ABC \sim \Delta DEF$.

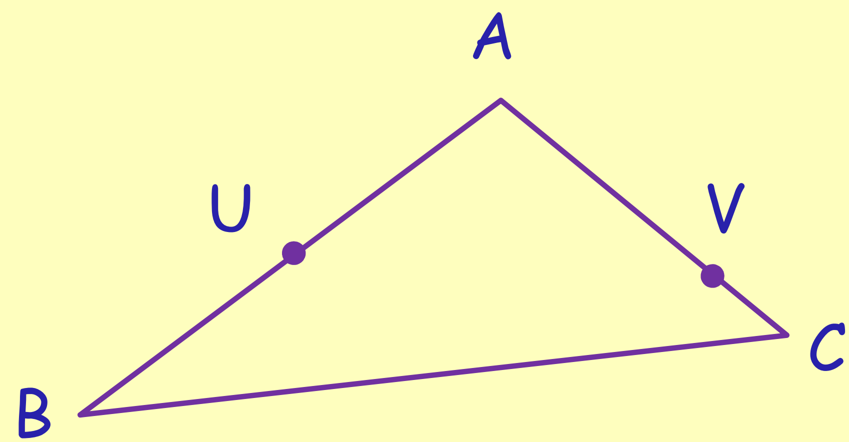
Proportionality Lemma

If U and V are points on sides AB and AC of triangle ΔABC, UV||BC if and only if

$$\frac{AU}{AV} = \frac{AB}{AC}$$

or

$$\frac{AU}{AB} = \frac{AV}{AC}$$



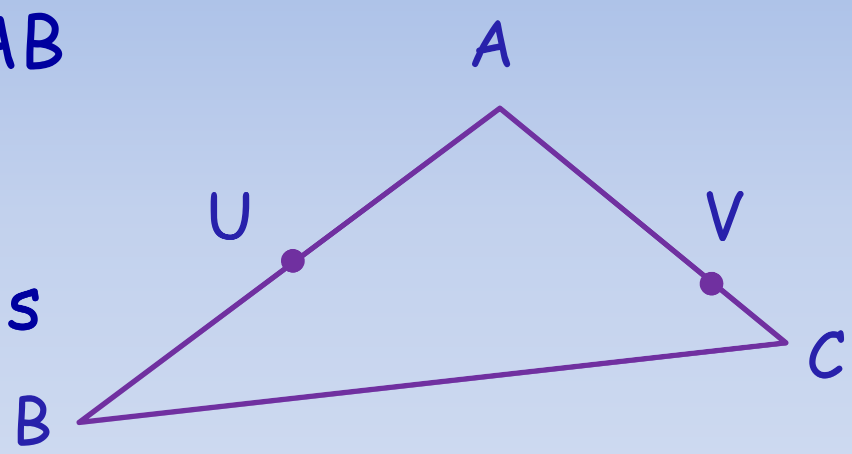
Proof

Let $r = \frac{UB}{AB}$ and $s = \frac{VC}{AC}$

$$AU = AB - UB = (1 - r)AB$$

$$AV = (1 - s)AC$$

$$\frac{AU}{AB} = 1 - r \qquad \frac{AV}{AC} = 1 - s$$



So AU/AB=AV/AC if and only if r=s.

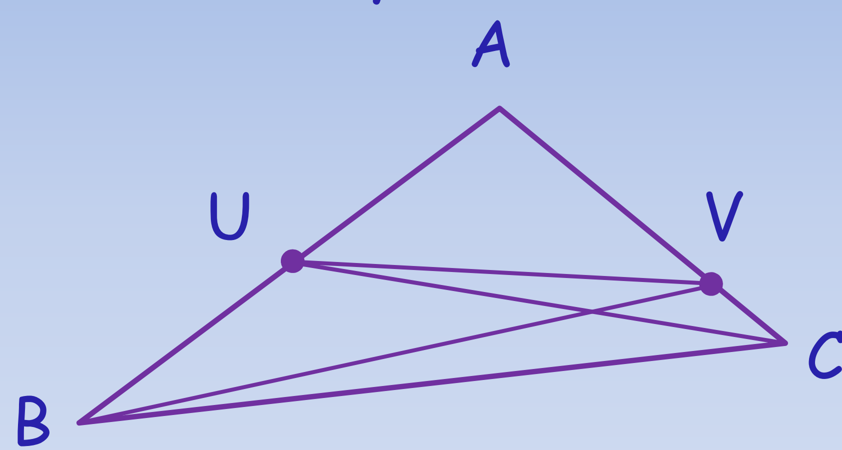
Proof

Now we need to show that UV||BC iff r=s.

Consider ΔACB and ΔUCB. They have same height. Thus:

$$\frac{K_{BUC}}{K_{ABC}} = \frac{UB}{AB} = r$$

$$K_{BUC} = rK_{ABC}$$



Similarly, $K_{BVC}=sK_{ABC}$.

Thus, r=s iff $K_{BUC}=K_{BVC}$.

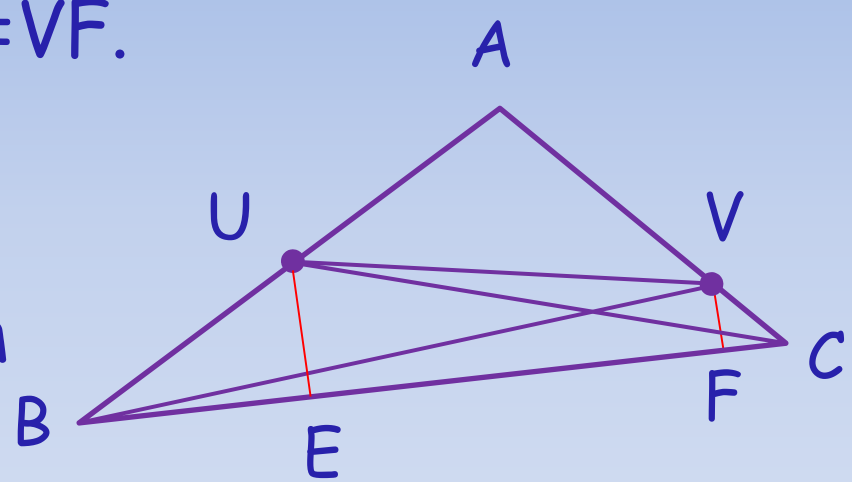
Thus, UV||BC iff $K_{BUC}=K_{BVC}$.

Proof

ΔBUC and ΔBVC have same base. Their areas are equal iff they have same height, or UE=VF. Thus, UV||BC iff UE=VF.

Note: UE||VF.

Thus, if UE=VF then UVEF = parallelogram and UV||BC.



If UV||BC, then UVEF is a parallelogram hence UE=VF.

Proportionality Theorem

Theorem: If $\Delta ABC \sim \Delta DEF$, then the lengths of the corresponding sides are proportional.

Proof: We need to find k so that:

$$AB = k\,DE$$

$$AC = k\,DF$$

$$BC = k\,EF$$

Proportionality Theorem

We want to show that DE/AB=DF/AC.

If DE=AB, then $\Delta ABC \cong \Delta DEF$ by ASA. In this case, $DF \cong AC$ and DF/AC=1.

So assume DE<AB.

Choose $U \in AB$ so that AU=DE and construct UV parallel to BC with $V \in AC$.

$UV || BC \Rightarrow \angle AUV = \angle B = \angle E$ and $\angle AVU = \angle C = \angle F$

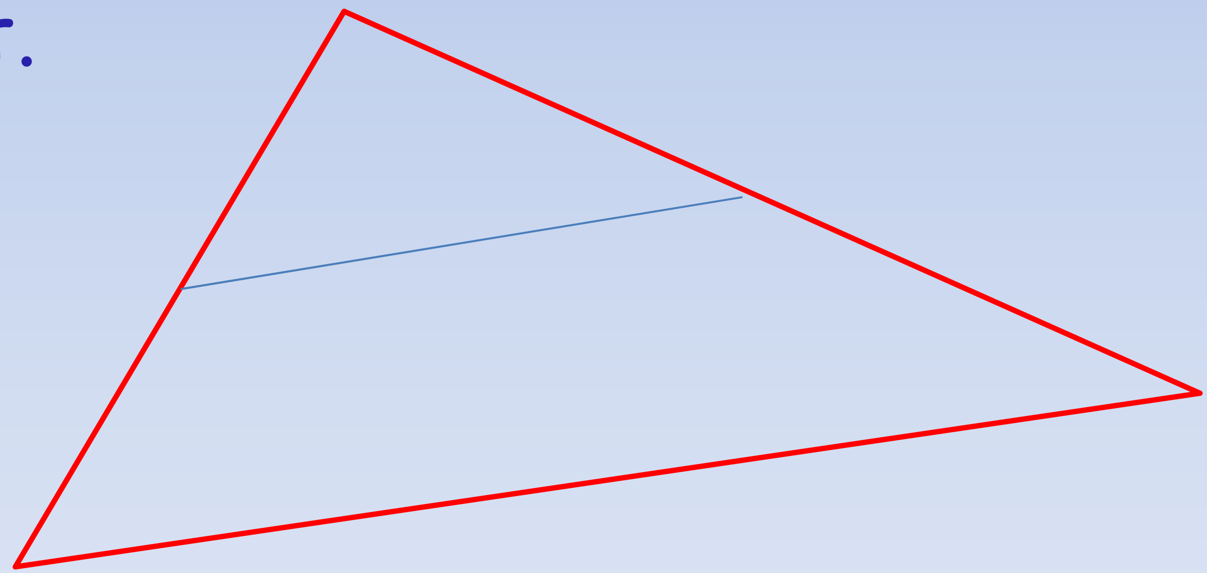
$AU=DF \Rightarrow \Delta AUV \cong \Delta DEF$ by AAS $\Rightarrow$ AV=DF.

$UV || BC \Rightarrow AU/AB=AV/AC$, but AU=DE & AV=DF

So DE/AB=DF/AC.

Midline of a Triangle

A line joining two midpoints of a triangle is called a midline of the triangle. A segment joining two midpoints is called a midsegment.



Midline Theorem

Theorem: In ΔABC if U and V are the midpoints of AB and AC, respectively, then $UV \| BC$ and $UV = \frac{1}{2}BC$.

SS Similarity Theorem

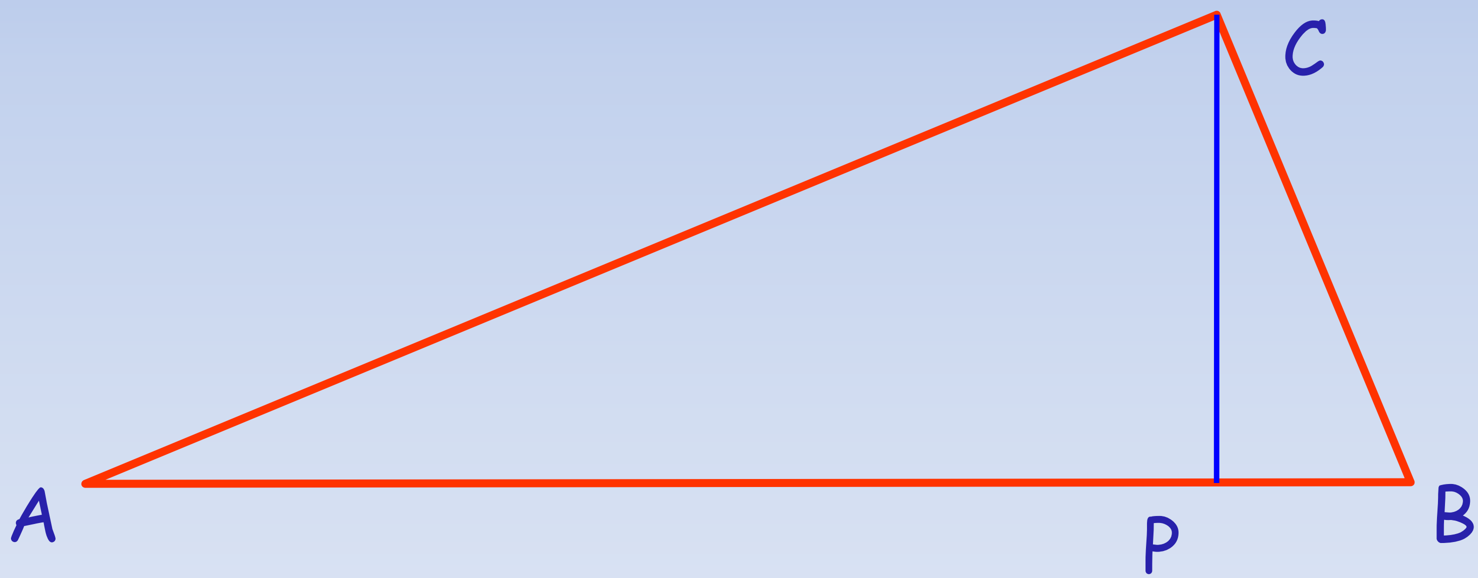
Theorem: Suppose that the sides of ΔABC are proportional to the corresponding sides of ΔDEF. Then $\Delta ABC \sim \Delta DEF$.

SAS Similarity Theorem

Theorem: Suppose that in ΔABC and ΔDEF we have that $\angle A = \angle D$ and $DE/AB=DF/AC$. Then $\Delta ABC \sim \Delta DEF$.

Right Triangle Theorem

Theorem: Suppose ΔABC is a right triangle with hypotenuse AB and CP is the altitude from C. $\Delta ACP \sim \Delta ABC \sim \Delta CBP$.



Ceva's Theorem

The three lines containing the vertices A, B, and C of $\triangle ABC$ and intersecting opposite sides at points L, M, and N, respectively, are concurrent if and only if

$$\frac{AN}{NB}\bullet\frac{BL}{LC}\bullet\frac{CM}{MA}=1$$

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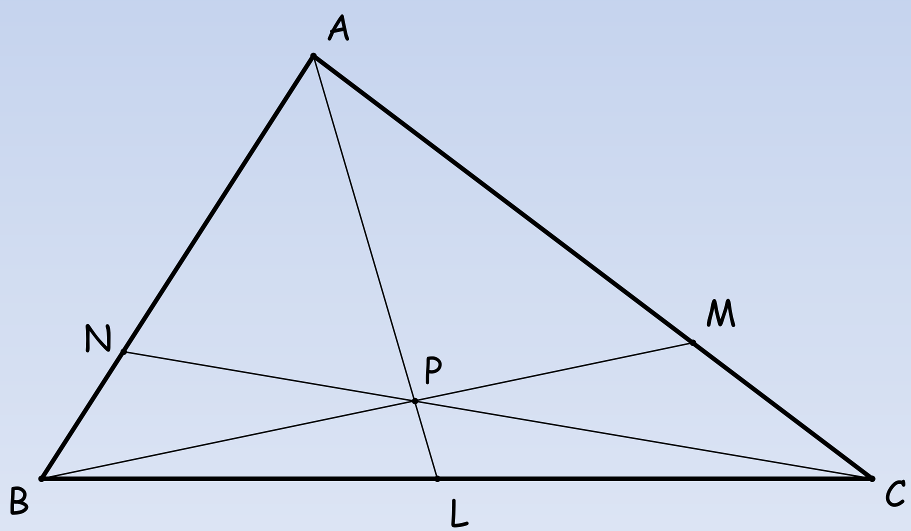
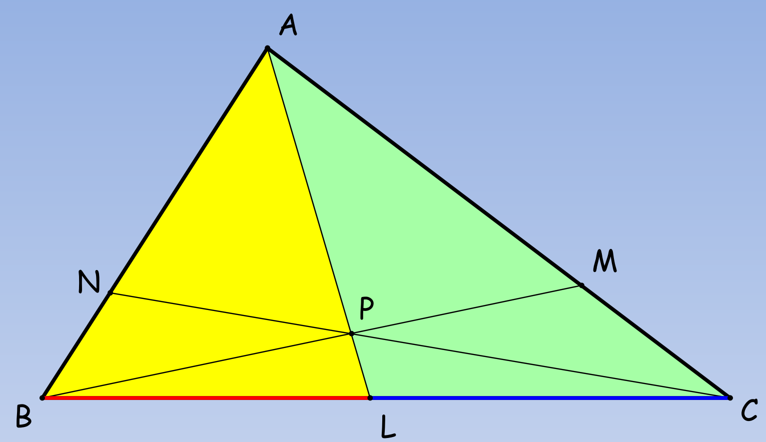
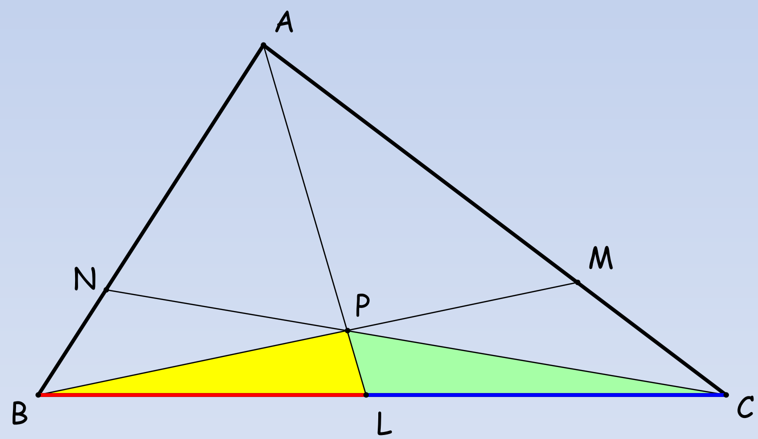


Ceva's Theorem

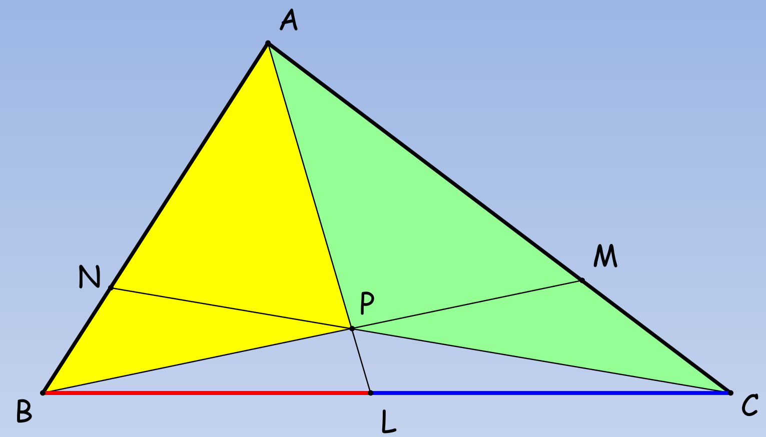


$$\frac{K(\triangle ABL)}{K(\triangle ACL)} = \frac{BL}{LC}$$



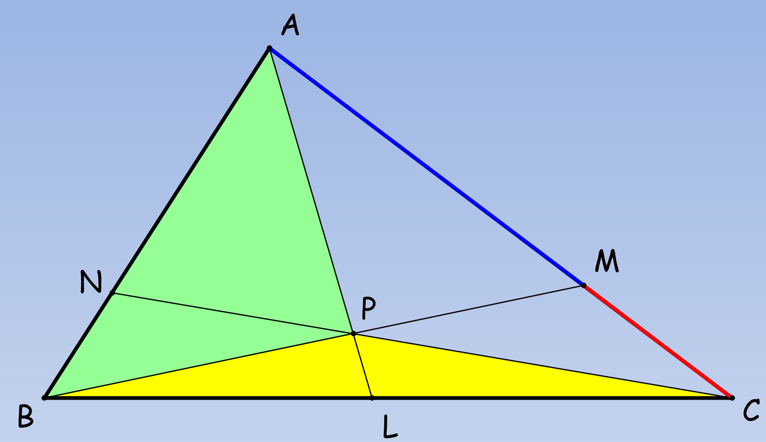
$$\frac{K(\triangle PBL)}{K(\triangle PCL)} = \frac{BL}{LC}$$

Ceva's Theorem



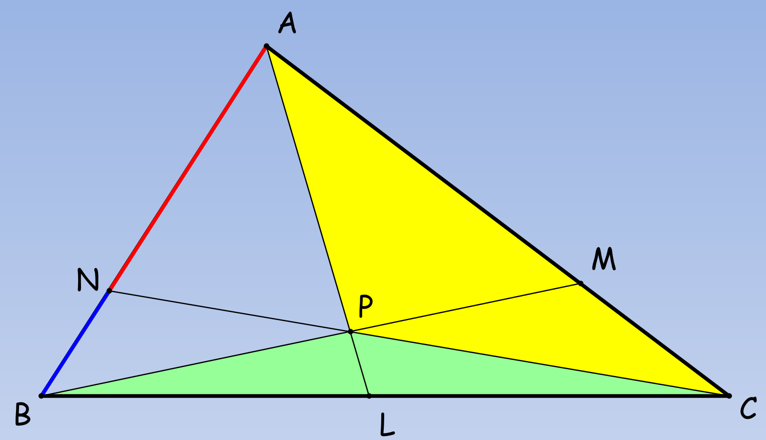
$$\frac{BL}{LC}=\frac{K(\triangle ABL)-K(\triangle PBL)}{K(\triangle ACL)-K(\triangle PCL)}=\frac{K(\triangle ABP)}{K(\triangle ACP)}$$

Ceva's Theorem



$$\frac{CM}{MA}=\frac{K(\triangle BMC)-K(\triangle PMC)}{K(\triangle BMA)-K(\triangle PMA)}=\frac{K(\triangle BCP)}{K(\triangle BAP)}$$

Ceva's Theorem



$$\frac{AN}{NB} = \frac{K(\triangle ACN) - K(\triangle APN)}{K(\triangle BCN) - K(\triangle BPN)} = \frac{K(\triangle ACP)}{K(\triangle BCP)}$$

Ceva's Theorem

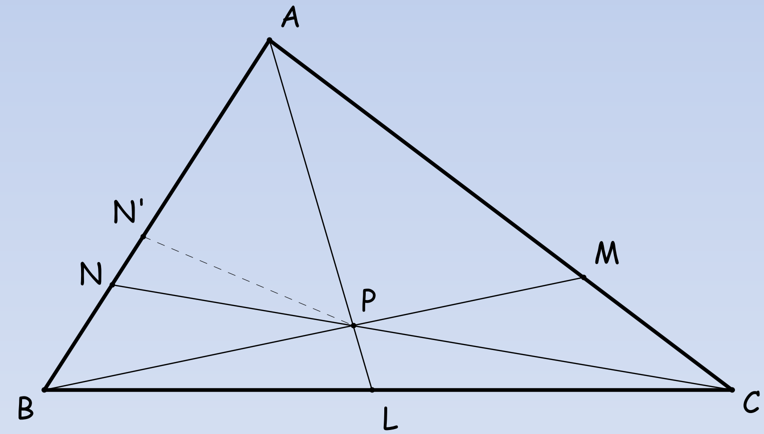
$$\frac{AN}{NB} \bullet \frac{BL}{LC} \bullet \frac{CM}{MA} = \frac{K(\triangle ACP)}{K(\triangle BCP)} \bullet \frac{K(\triangle ABP)}{K(\triangle ACP)} \bullet \frac{K(\triangle BCP)}{K(\triangle ABP)} = 1$$

Ceva's Theorem

Now assume that

$$\frac{AN}{NB} \bullet \frac{BL}{LC} \bullet \frac{CM}{MA} = 1$$

Let BM and AL intersect at P and construct CP intersecting AB at N', N' different from N.



Ceva's Theorem

Then AL, BM, and CN' are concurrent and

$$\frac{AN'}{N'B} \bullet \frac{BL}{LC} \bullet \frac{CM}{MA} = 1$$

From our hypothesis it follows that

$$\frac{AN'}{N'B} = \frac{AN}{NB}$$

So N and N' must coincide.