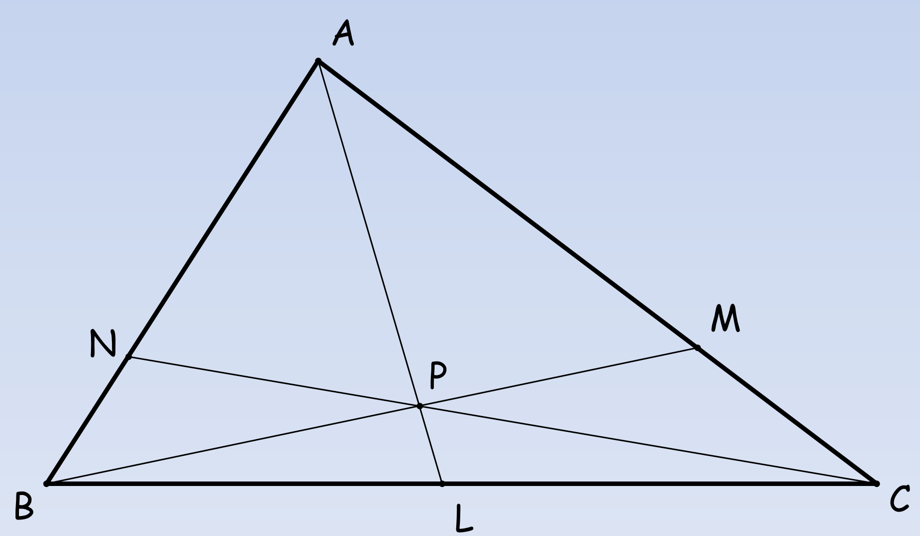
Ceva's Theorem

MA 341 – Topics in Geometry
Lecture 11

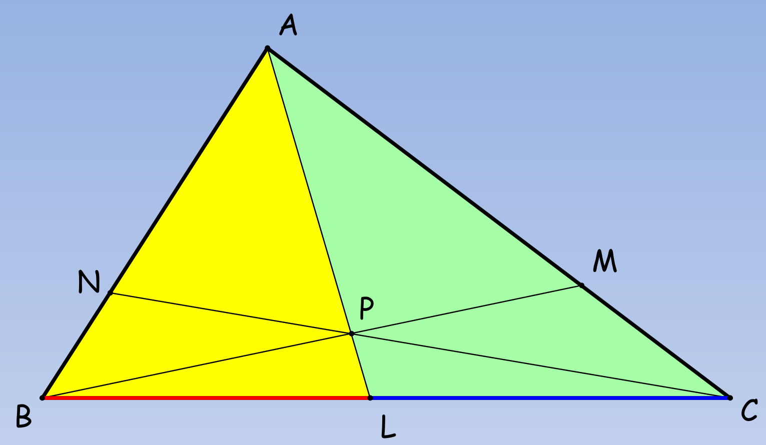
Ceva's Theorem

The three lines containing the vertices A, B, and C of $\triangle ABC$ and intersecting opposite sides at points L, M, and N, respectively, are concurrent if and only if

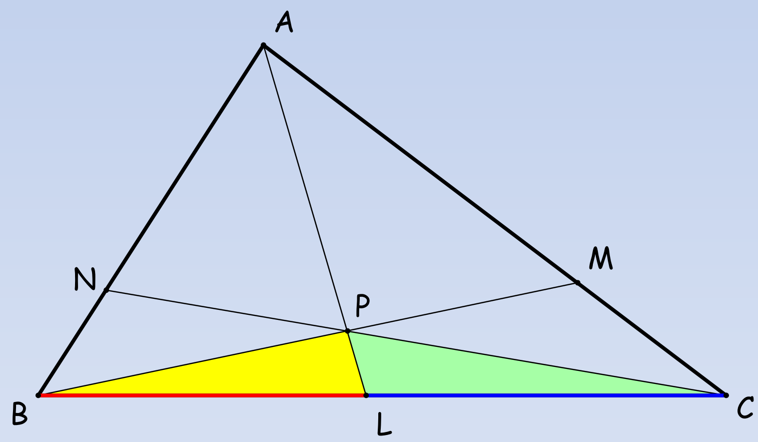
$$\frac{AN}{NB} \bullet \frac{BL}{LC} \bullet \frac{CM}{MA} = 1$$



Ceva's Theorem

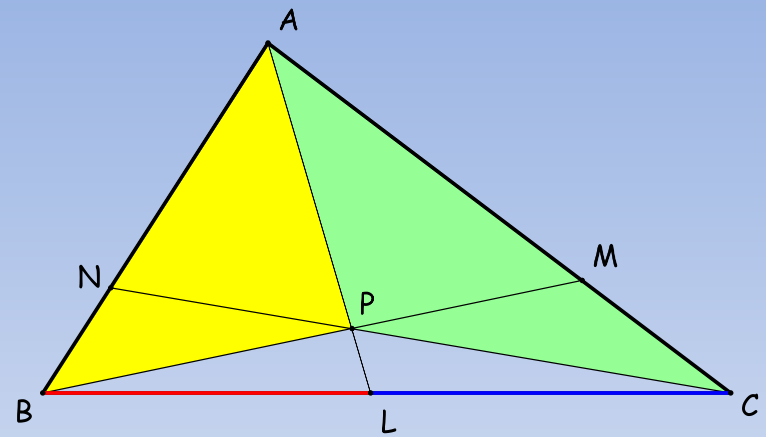


$$\frac{K(\triangle ABL)}{K(\triangle ACL)} = \frac{BL}{LC}$$



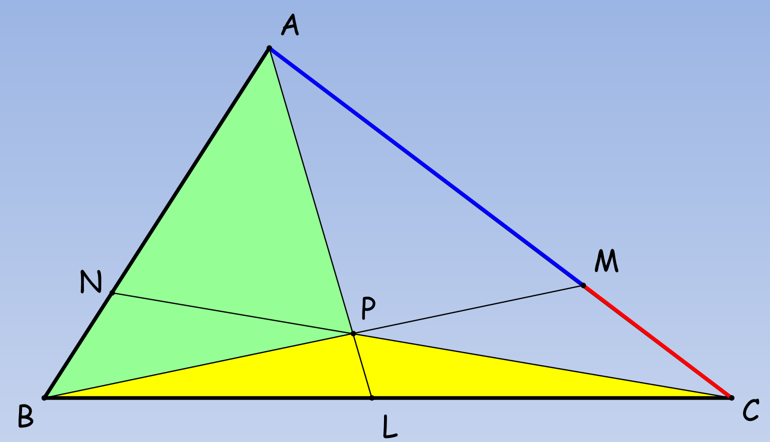
$$\frac{K(\triangle PBL)}{K(\triangle PCL)} = \frac{BL}{LC}$$

Ceva's Theorem



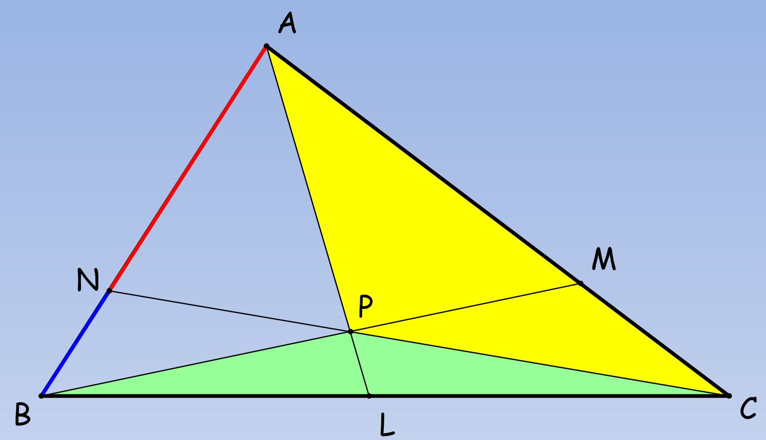
$$\frac{BL}{LC}=\frac{K(\triangle ABL)-K(\triangle PBL)}{K(\triangle ACL)-K(\triangle PCL)}=\frac{K(\triangle ABP)}{K(\triangle ACP)}$$

Ceva's Theorem



$$\frac{CM}{MA} = \frac{K(\triangle BMC) - K(\triangle PMC)}{K(\triangle BMA) - K(\triangle PMA)} = \frac{K(\triangle BCP)}{K(\triangle BAP)}$$

Ceva's Theorem



$$\frac{AN}{NB}=\frac{K(\triangle ACN)-K(\triangle APN)}{K(\triangle BCN)-K(\triangle BPN)}=\frac{K(\triangle ACP)}{K(\triangle BCP)}$$

Ceva's Theorem

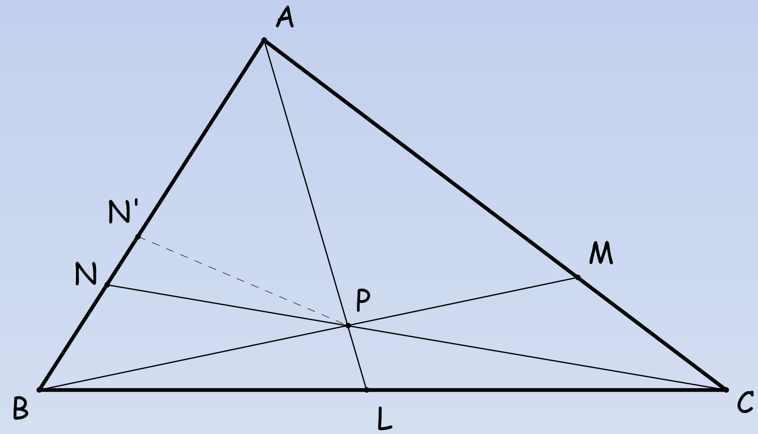
$$\frac{AN}{NB} \bullet \frac{BL}{LC} \bullet \frac{CM}{MA} = \frac{K(\triangle ACP)}{K(\triangle BCP)} \bullet \frac{K(\triangle ABP)}{K(\triangle ACP)} \bullet \frac{K(\triangle BCP)}{K(\triangle ABP)} = 1$$

Ceva's Theorem

Now assume that

$$\frac{AN}{NB} \bullet \frac{BL}{LC} \bullet \frac{CM}{MA} = 1$$

Let BM and AL intersect at P and construct CP intersecting AB at N', N' different from N.



Ceva's Theorem

Then AL, BM, and CN' are concurrent and

$$\frac{AN'}{N'B} \bullet \frac{BL}{LC} \bullet \frac{CM}{MA} = 1$$

From our hypothesis it follows that

$$\frac{AN'}{N'B} = \frac{AN}{NB}$$

So N and N' must coincide.

Medians

In ΔABC, let M, N, and P be midpoints of AB, BC, AC.

Medians: CM, AN, BP

Theorem: In any triangle the three medians meet in a single point, called the centroid.

M – midpoint $\Rightarrow$ AM=BM, N - midpoint $\Rightarrow$ BN=CN

P - midpoint $\Rightarrow$ AP=CP

$$\frac{AM}{MB} \bullet \frac{BN}{NC} \bullet \frac{CP}{PA} = 1$$

By Ceva's Theorem they are concurrent.

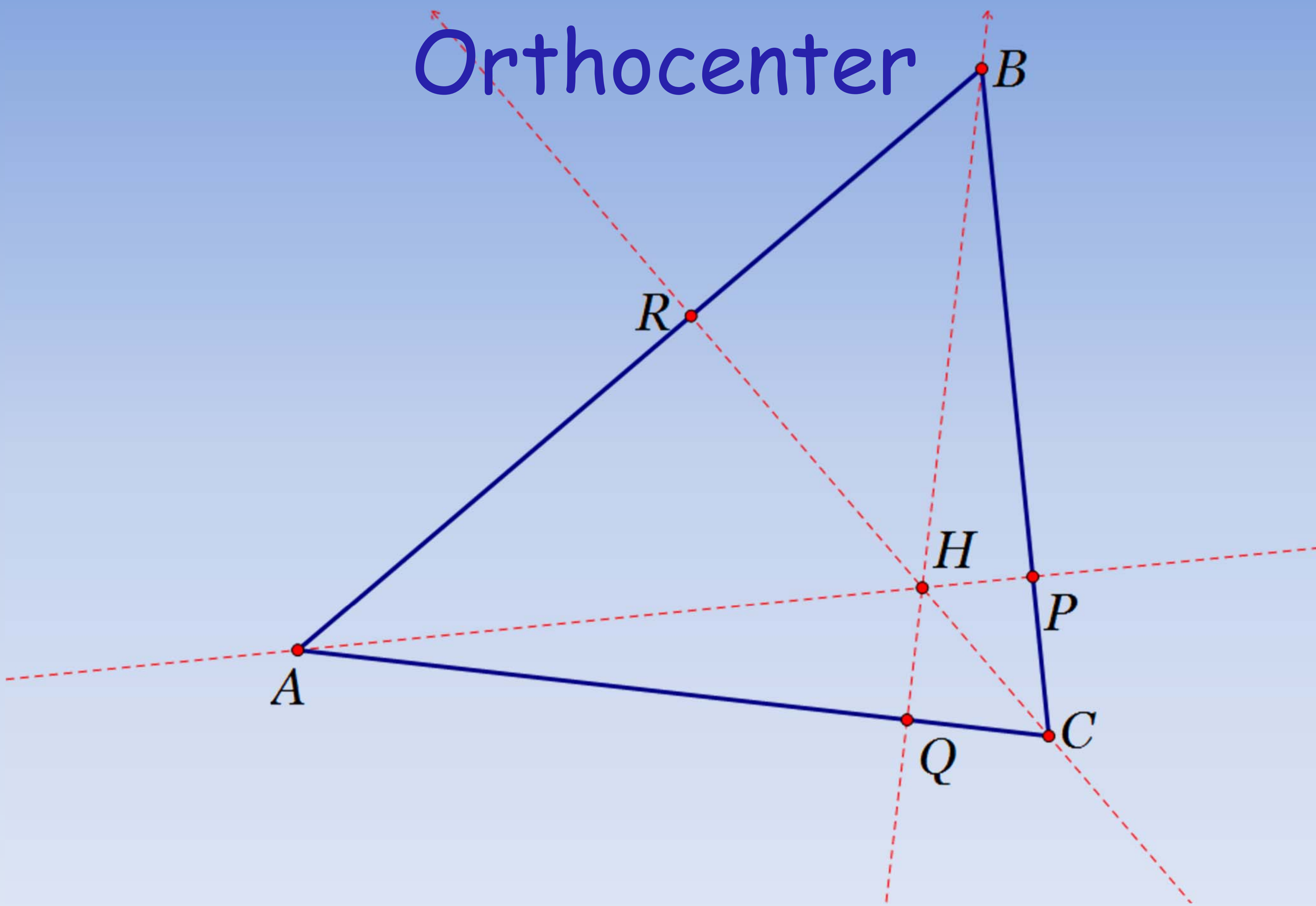
Orthocenter

Let ΔABC be a triangle and let P, Q, and R be the feet of A, B, and C on the opposite sides.

AP, BQ, and CR are the altitudes of ΔABC.

Theorem: The altitudes of a triangle ΔABC meet in a single point, called the orthocenter, H.

Orthocenter



Orthocenter

By AA

$\Delta BRC \sim \Delta BPA$ (a right angle and $\angle B$)

$\Rightarrow BR/BP = BC/BA$

$\Delta AQB \sim \Delta ARC$ (a right angle and $\angle A$)

$\Rightarrow AQ/AR = AB/AC$

$\Delta CPA \sim \Delta CQB$ (a right angle and $\angle C$)

$\Rightarrow CP/CQ = AC/BC$

$$\frac{BR}{BP} \bullet \frac{AQ}{AR} \bullet \frac{CP}{CQ} = \frac{BC}{AB} \bullet \frac{AB}{AC} \bullet \frac{AC}{BC} = 1$$

Orthocenter

By Ceva's Theorem, the altitudes meet at a single point.

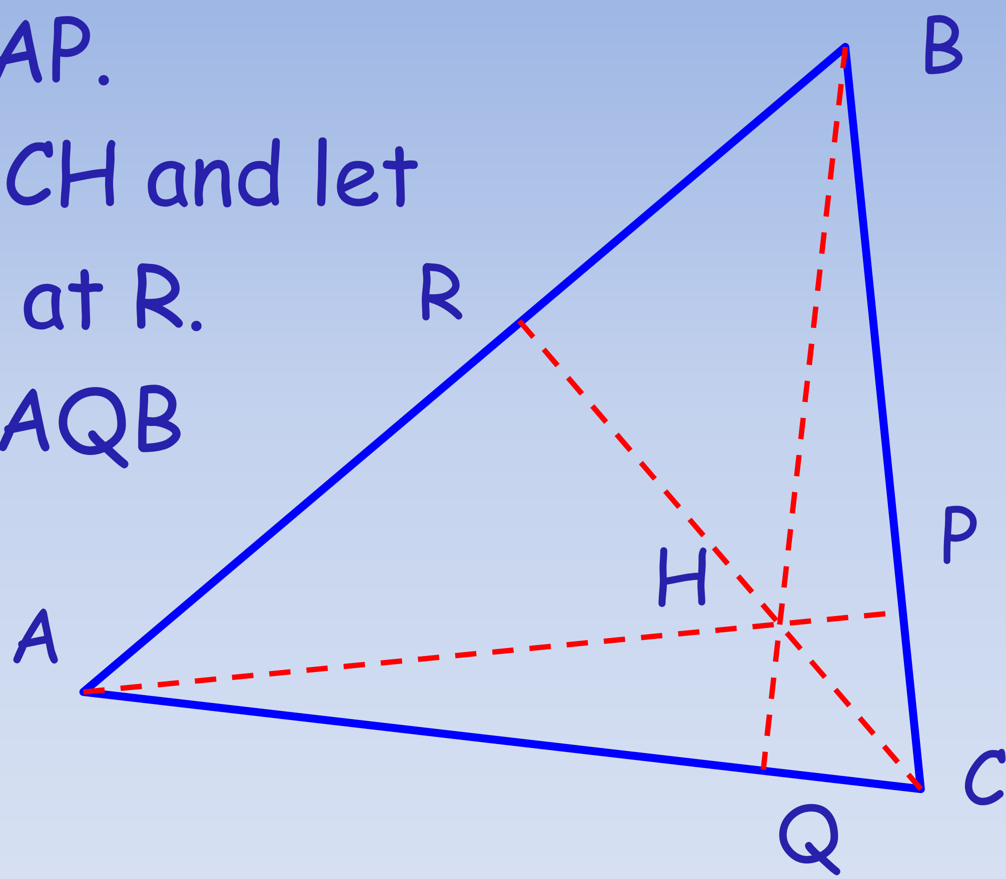
Orthocenter

Traditional route:

BQ intersects AP.

Now construct CH and let it intersect AB at R.

Prove ΔARC~ΔAQB making ∠R=90.



Incenter

Let ΔABC be a triangle and let AP, BQ, and CR be the angle bisectors of $\angle A$, $\angle B$, and $\angle C$.

Angle Bisector Theorem: If AD is the angle bisector of $\angle A$ with D on BC, then

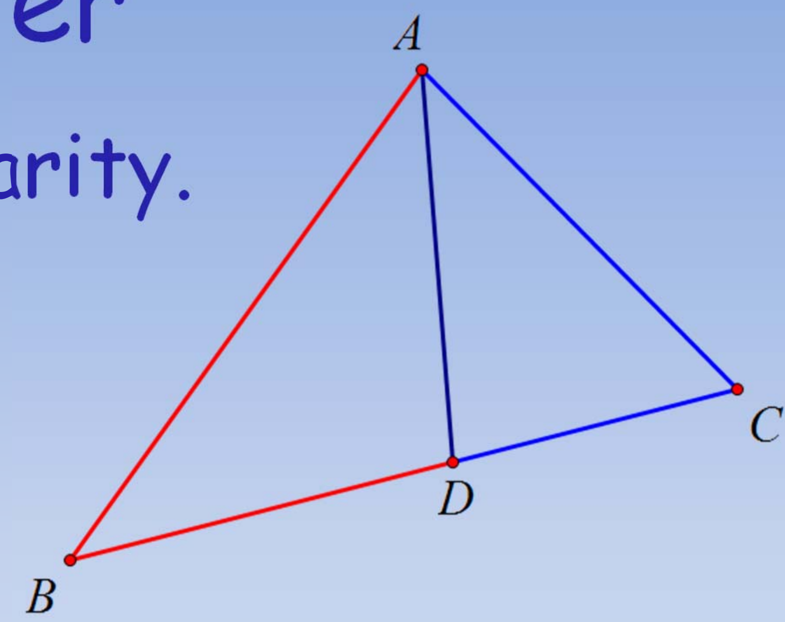
$$\frac{AB}{AC} = \frac{BD}{CD}$$

Incenter

Proof: Want to use similarity.

Where is similarity?

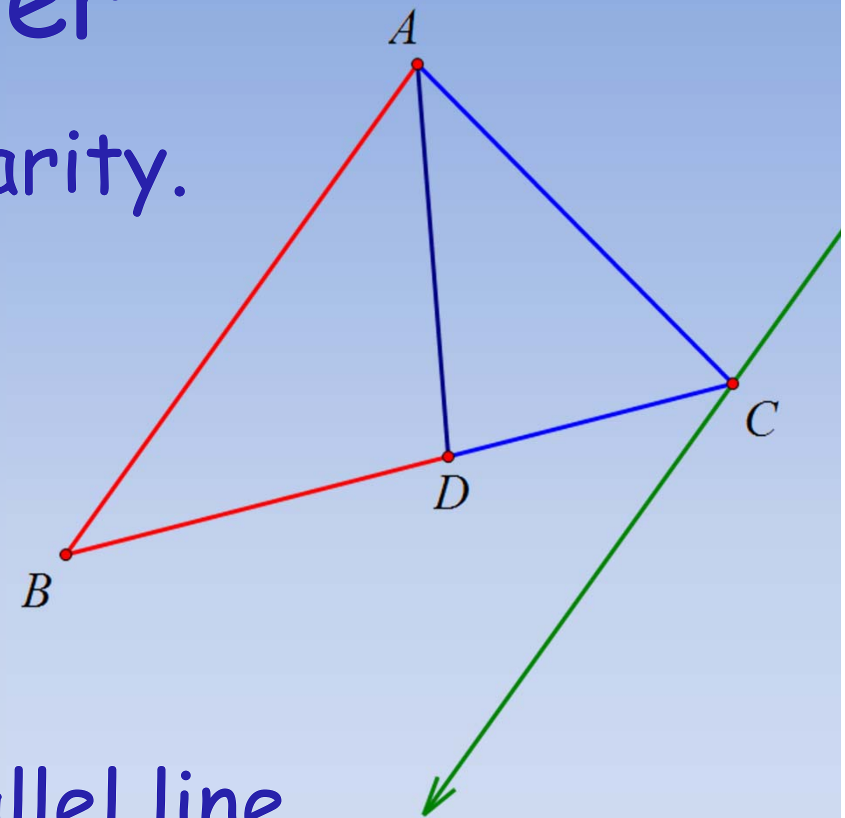
Construct line through
C parallel to AB



Incenter

Proof: Want to use similarity.

Where is similarity?



Construct line through C parallel to AB

Extend AD to meet parallel line through C at point E.

Incenter

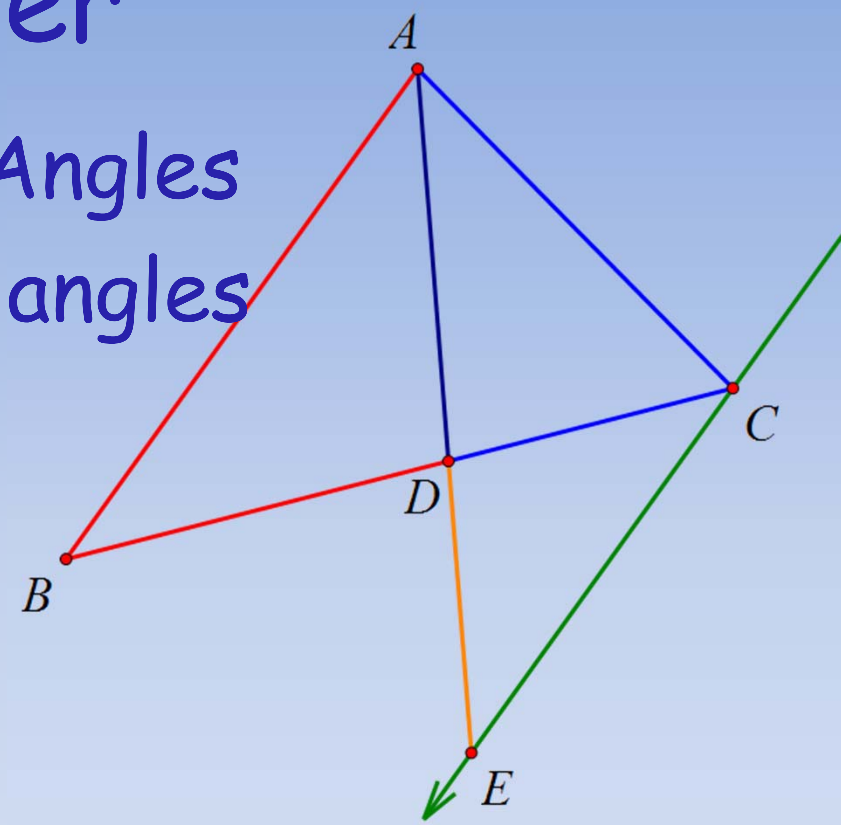
$\angle BAE \cong \angle CEA$ - Alt Int Angles

$\angle BDA \cong \angle CDE$ - vertical angles

$\Delta BAD \sim \Delta CDE$ - AA

Therefore

$$\frac{AB}{CE} = \frac{BD}{CD}$$



Note that $\angle CEA \cong \angle BAE \cong \angle CAE$

$\Rightarrow \Delta ACE$ isosceles $\Rightarrow CE = AC$ and $\frac{AB}{AC} = \frac{BD}{CD}$

Incenter

Let ΔABC be a triangle and let AP, BQ, and CR be the angle bisectors of $\angle A$, $\angle B$, and $\angle C$.

Theorem: The angle bisectors of a triangle ΔABC meet in a single point, called the incenter, I.

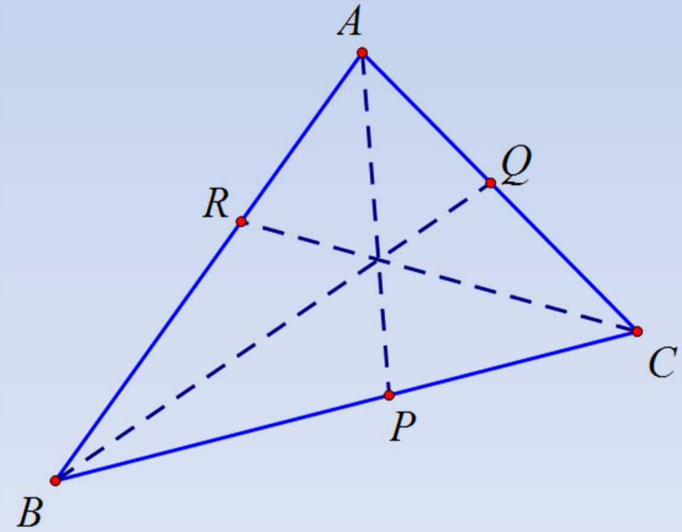
Incenter

Proof: Angle bisector means:

$$\frac{AB}{AC} = \frac{BP}{PC} \qquad \frac{BA}{BC} = \frac{AQ}{QC} \qquad \frac{CA}{CB} = \frac{AR}{RB}$$

By Ceva's Theorem we need to find the product:

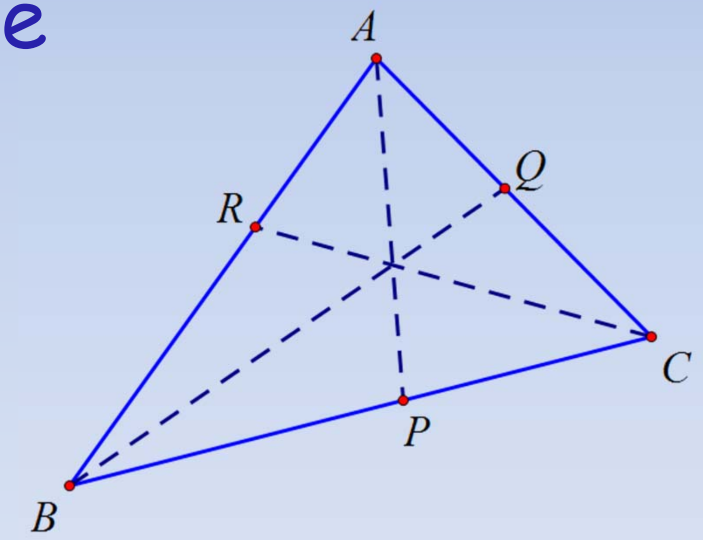
$$\frac{AR}{RB} \bullet \frac{BP}{PC} \bullet \frac{CQ}{QA}$$



Incenter

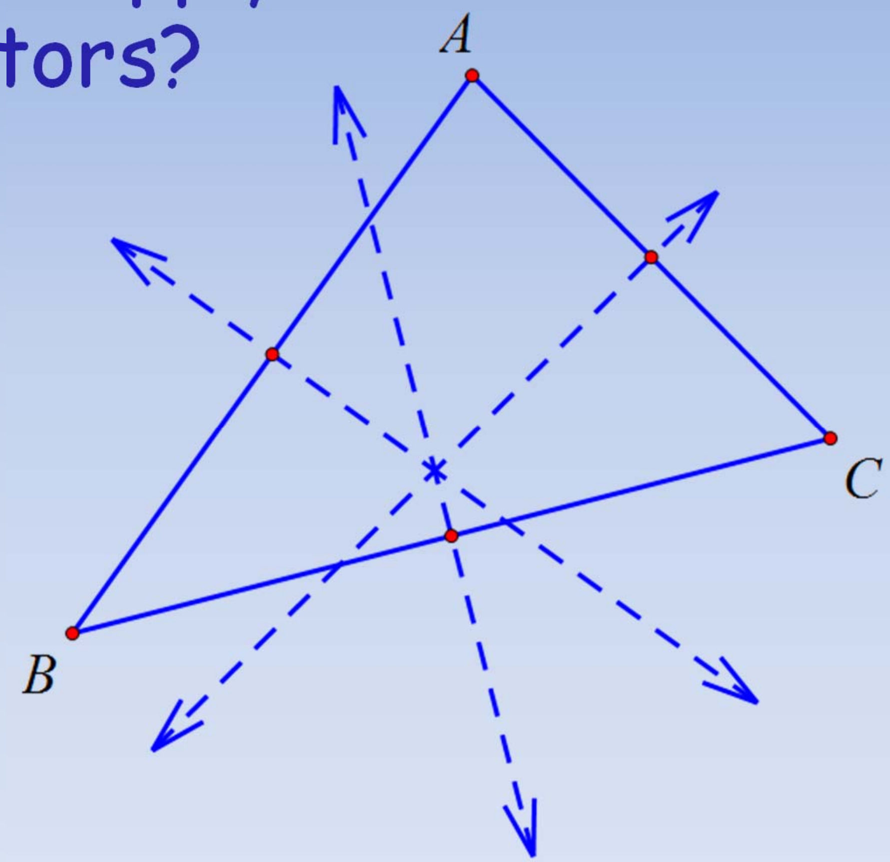
$$\frac{AR}{RB}\bullet\frac{BP}{PC}\bullet\frac{CQ}{QA}=\frac{AC}{BC}\bullet\frac{AB}{AC}\bullet\frac{BC}{AB}=1$$

Thus by Ceva's Theorem the angle bisectors are concurrent.



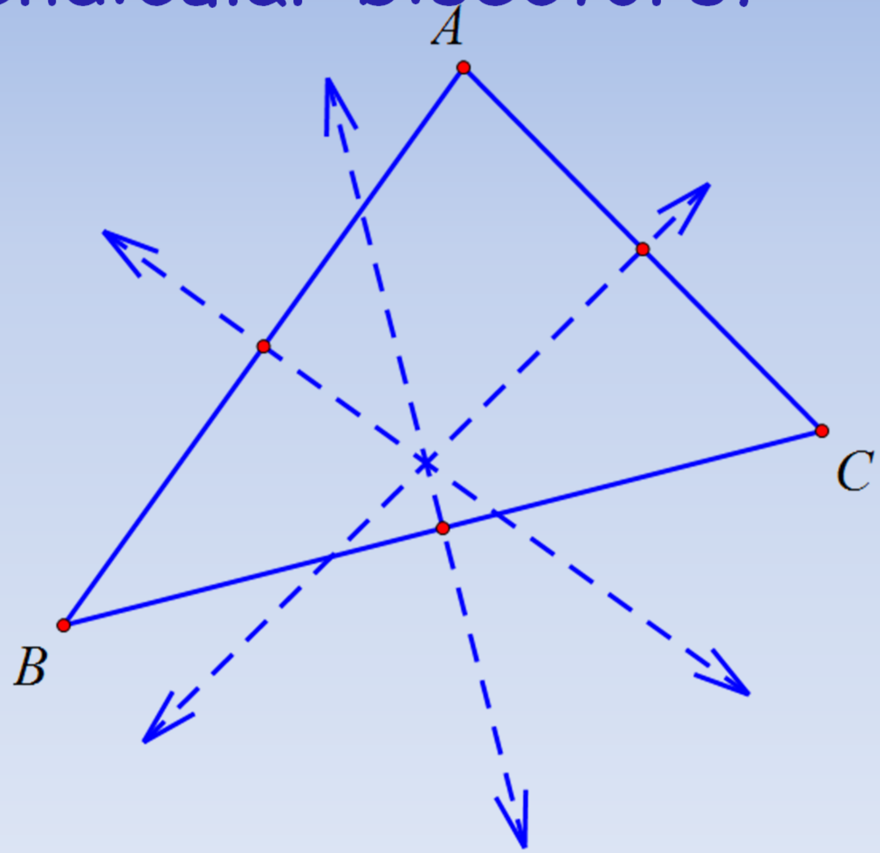
Circumcenter & Perp Bisectors

Does Ceva's Theorem apply to perpendicular bisectors?



Circumcenter & Perp Bisectors

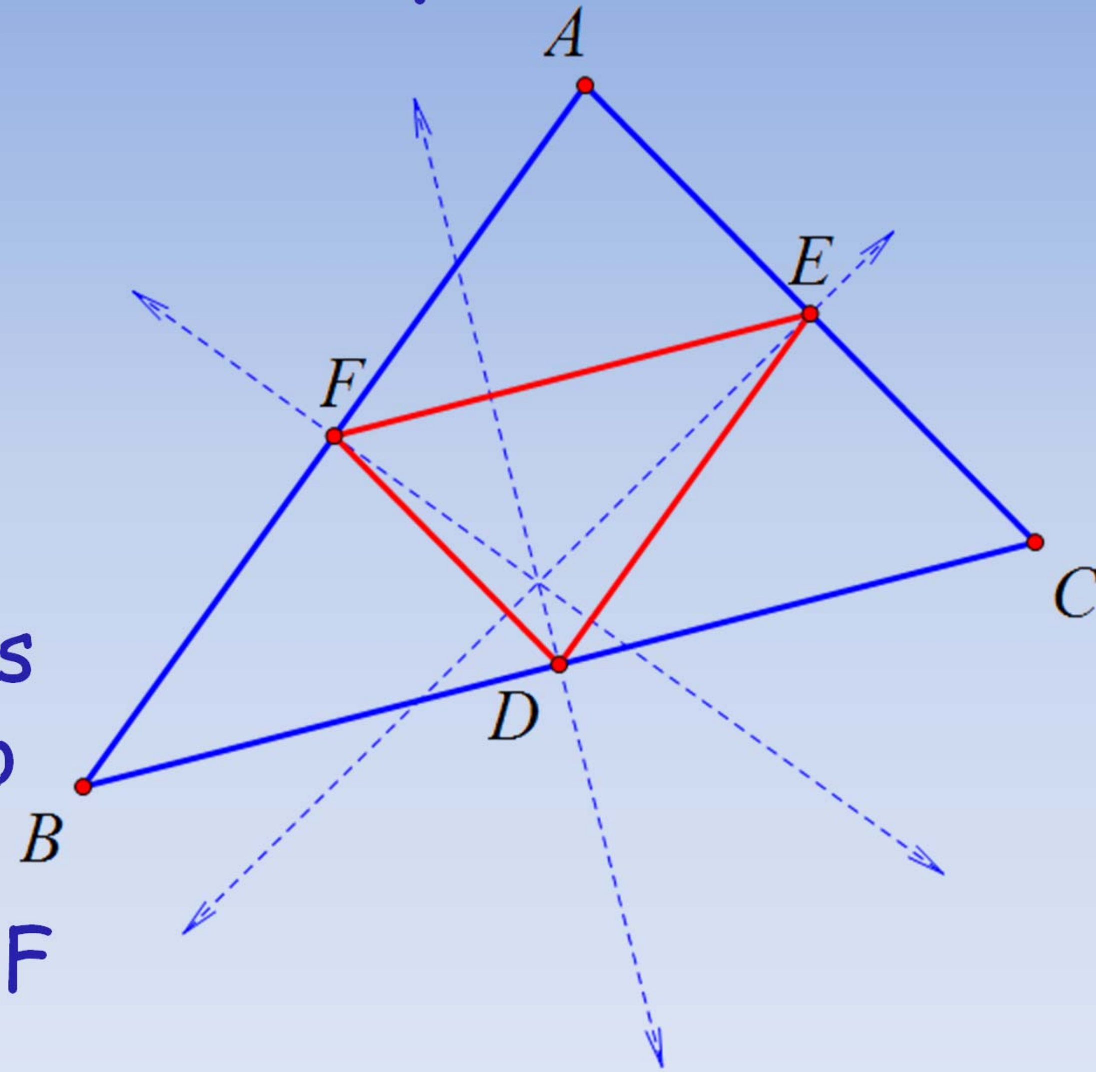
How can we get Ceva's Theorem to apply to perpendicular bisectors?



Circumcenter & Perp Bisectors

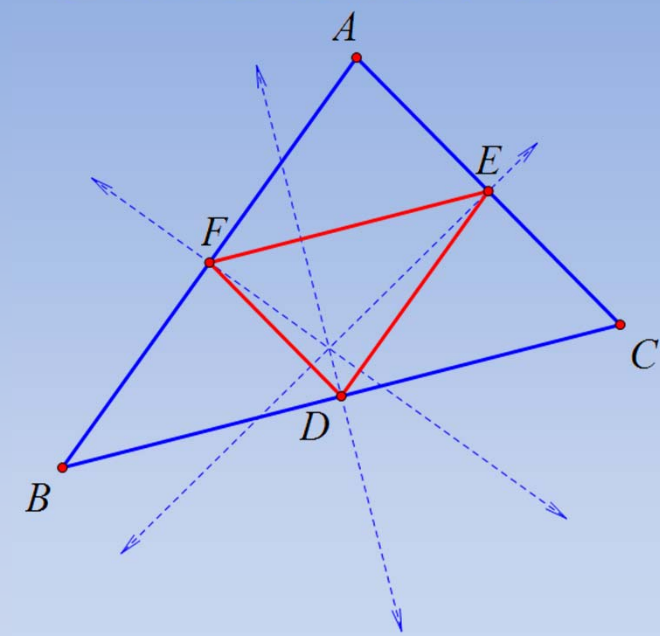
Draw in midsegments

EF||BC $\Rightarrow$ perpendicular bisector of BC is perpendicular to EF $\Rightarrow$ is an altitude of ΔDEF



Circumcenter & Perp Bisectors

Perpendicular bisectors of AB, BC and AC are altitudes of ΔDEF.



Altitudes meet in a single point ⇒ perpendicular bisectors are concurrent.

Circumcircle

Theorem: There is exactly one circle through any three non-collinear points.

The circle = the circumcircle
The center = the circumcenter, O.
The radius = the circumradius, R.

Theorem: The circumcenter is the point of intersection of the three perpendicular bisectors.

Question

Where do the perpendicular bisectors of the sides intersect the circumcircle?

Question

Where do the perpendicular bisectors of the sides intersect the circumcircle?

At one end is point of intersection of angle bisector with circumcircle

The other end is point of intersection of exterior angle bisector with circumcircle.

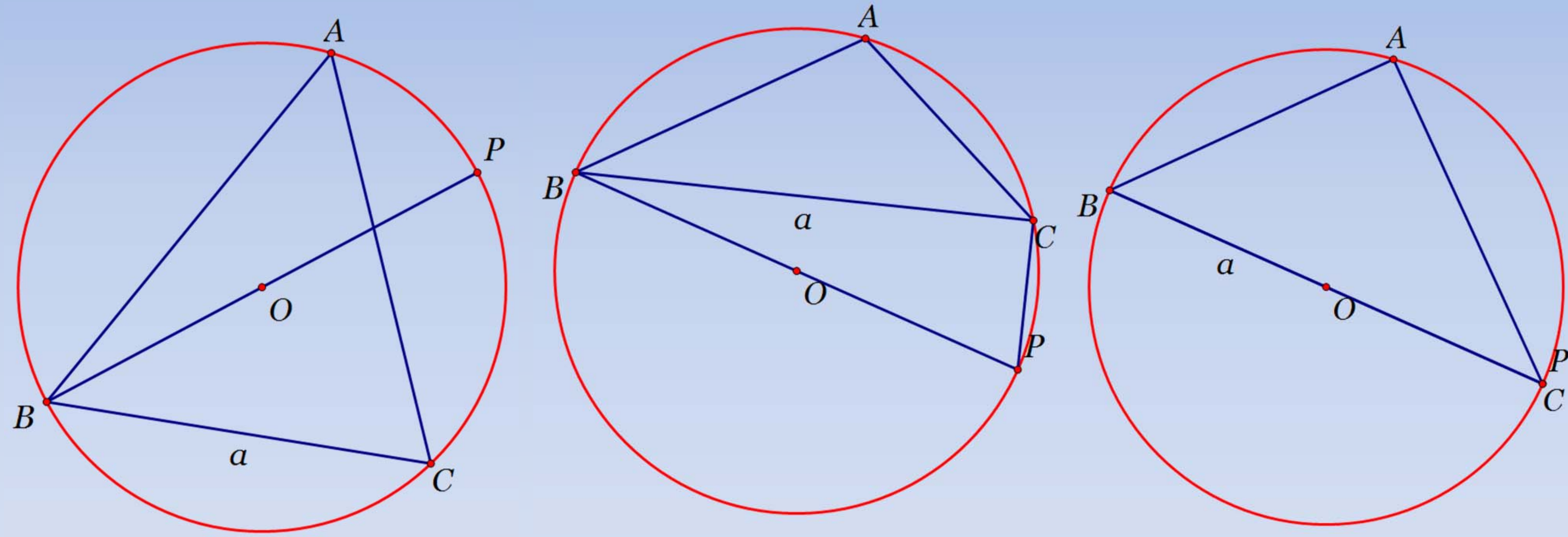
Extended Law of Sines

Theorem: Given ΔABC with circumradius R, let a, b, and c denote the lengths of the sides opposite angles $\angle A$, $\angle B$, and $\angle C$, respectively. Then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

Proof

Three cases:



Proof

Case I: $\angle A < 90°$

BP = diameter

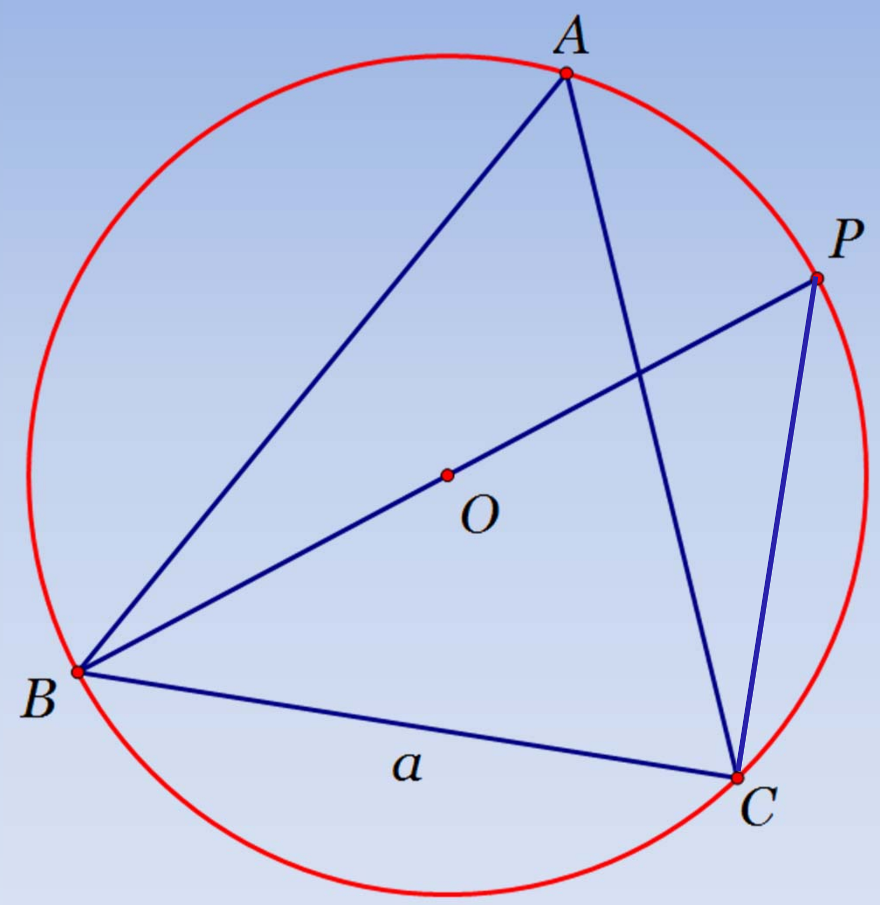
$\Rightarrow \Delta BCP$ right triangle

BP = 2R

$\Rightarrow \sin P = a/2R$

$\angle A = \angle P$

$\Rightarrow 2R = a/\sin A$



Proof

Case II: $\angle A > 90°$

BP = diameter

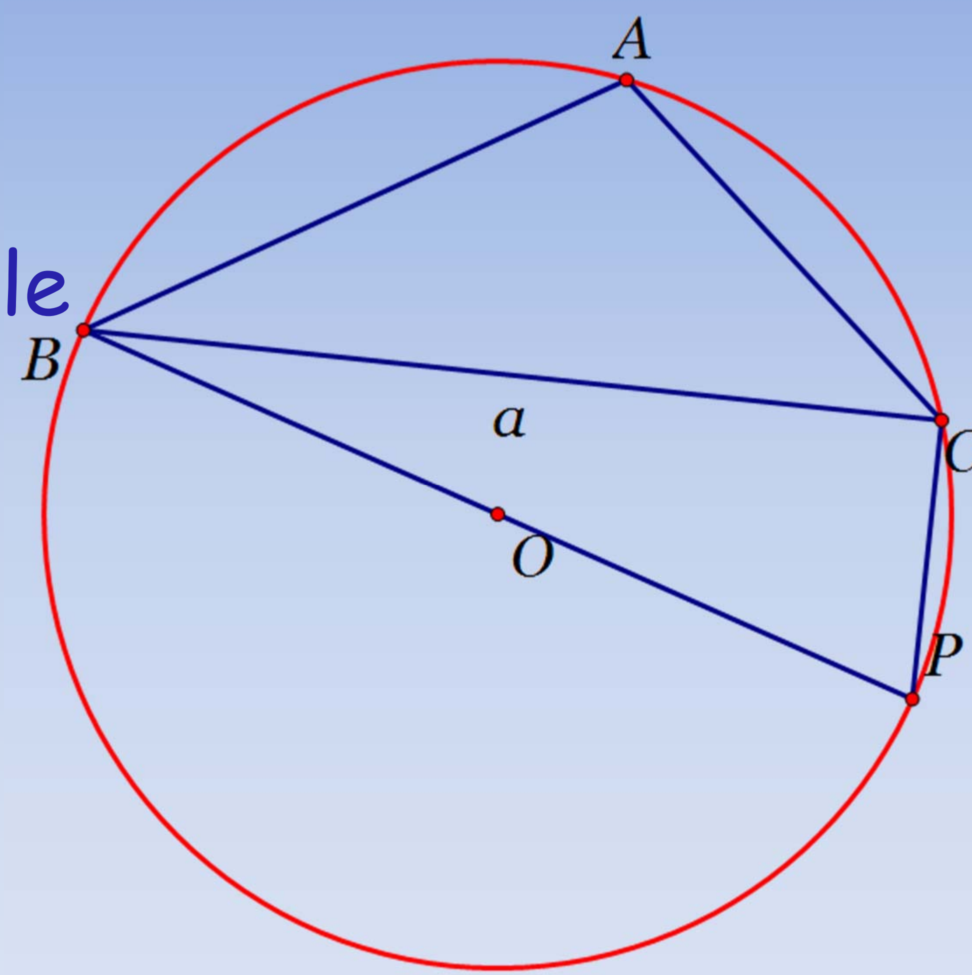
$\Rightarrow \Delta BCP$ right triangle

BP = 2R

$\Rightarrow \sin P = a/2R$

$\angle A = \angle P$

$\Rightarrow 2R = a/\sin A$



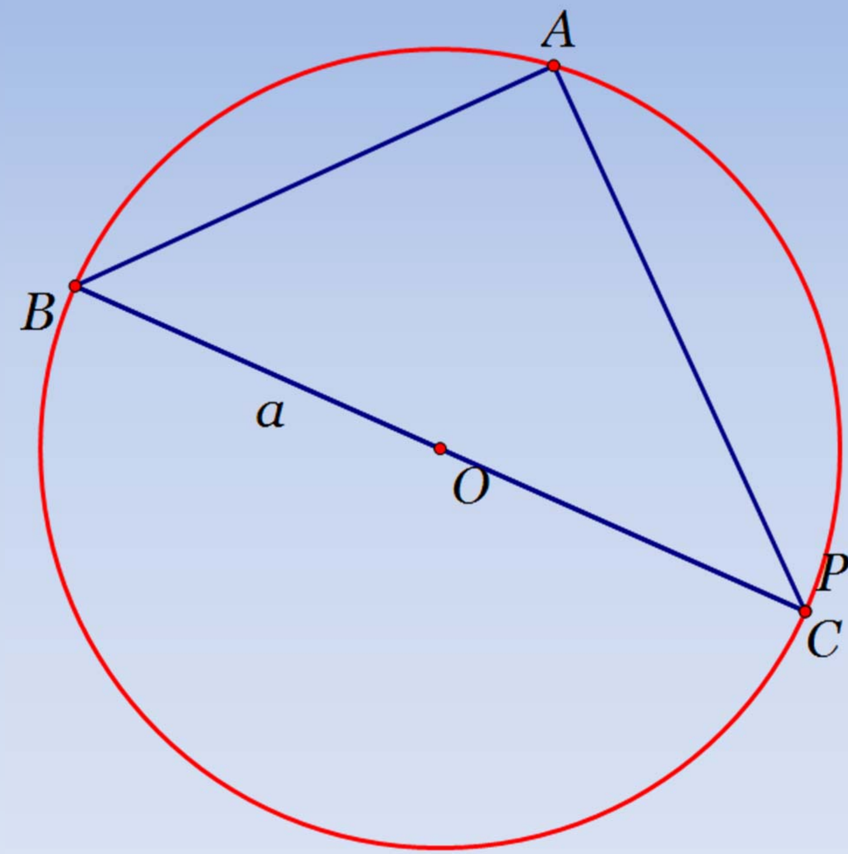
Proof

Case III: $\angle A = 90°$

BP = a = diameter

BP = 2R

2R = a = a/sin A



Circumradius and Area

Theorem: Let R be the circumradius and K be the area of ΔABC and let a, b, and c denote the lengths of the sides as usual. Then $4KR=abc$

$$K = \frac{abc}{4R}$$

Proof

$$K = \frac{1}{2} ab \sin C$$

$$2K = ab \sin C$$

$$c/\sin C = 2R$$

$$\sin C = c/2R$$

$$2K = abc/2R$$

$$4KR = abc$$