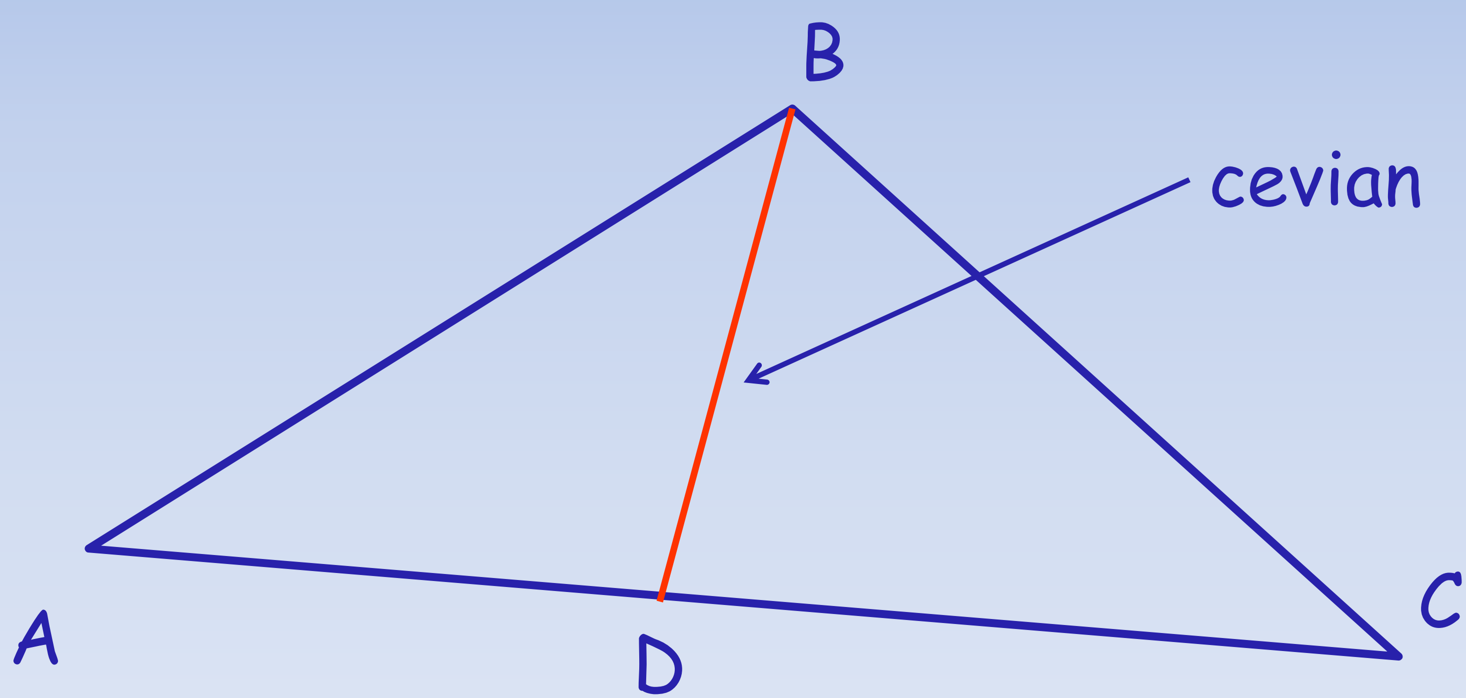
Cevians, Symmedians, and Excircles

MA 341 – Topics in Geometry
Lecture 16

UK

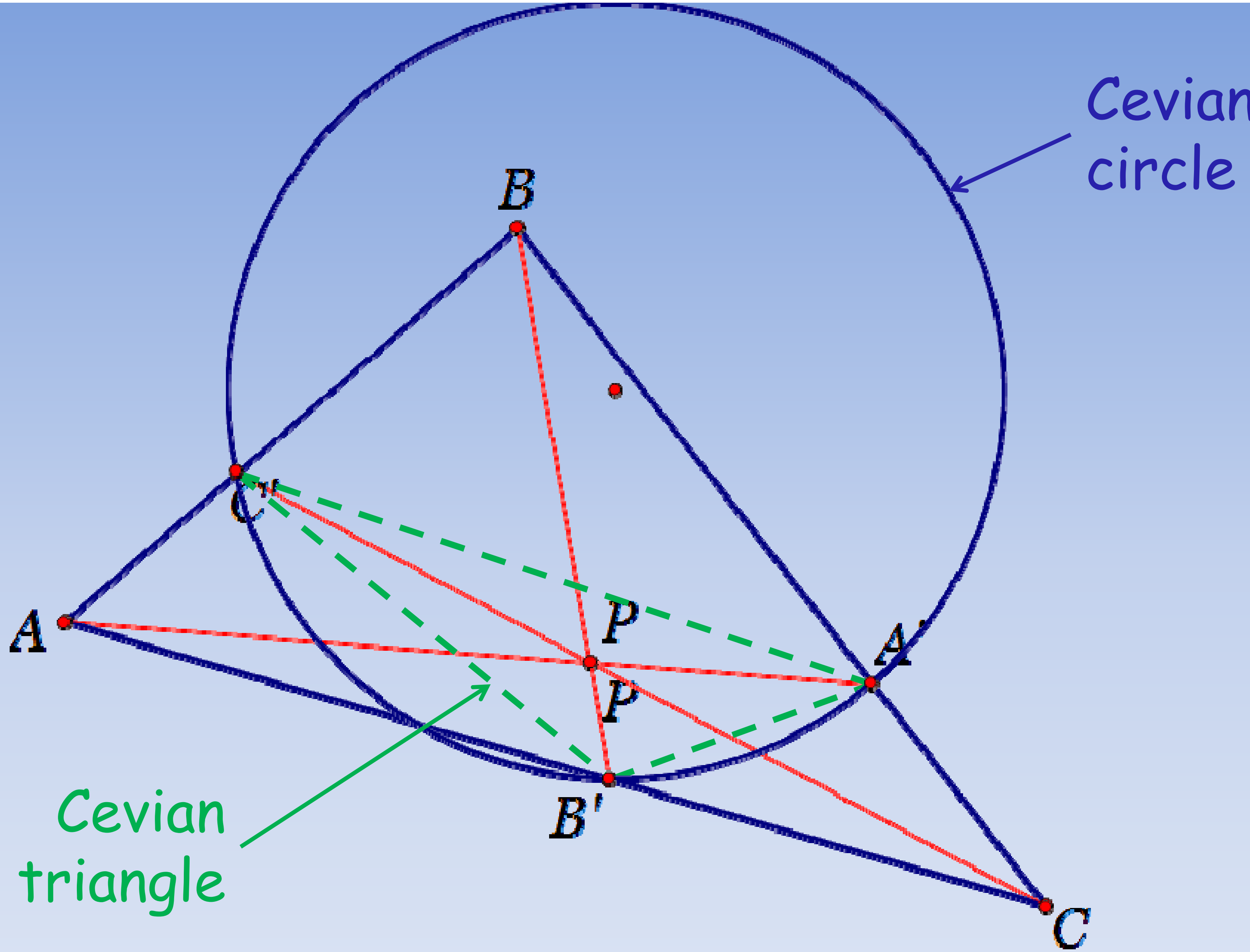
Cevian

A cevian is a line segment which joins a vertex of a triangle with a point on the opposite side (or its extension).



Cevian Triangle & Circle

- Pick P in the interior of ΔABC
- Draw cevians from each vertex through P to the opposite side
- Gives set of three intersecting cevians AA', BB', and CC' with respect to that point.
- The triangle $\Delta A'B'C'$ is known as the cevian triangle of ΔABC with respect to P
- Circumcircle of $\Delta A'B'C'$ is known as the evian circle with respect to P.


Cevian
circle
B
C'
A
P
A'
P
B'
C
Cevian
triangle

Cevians

In ΔABC examples of cevians are:

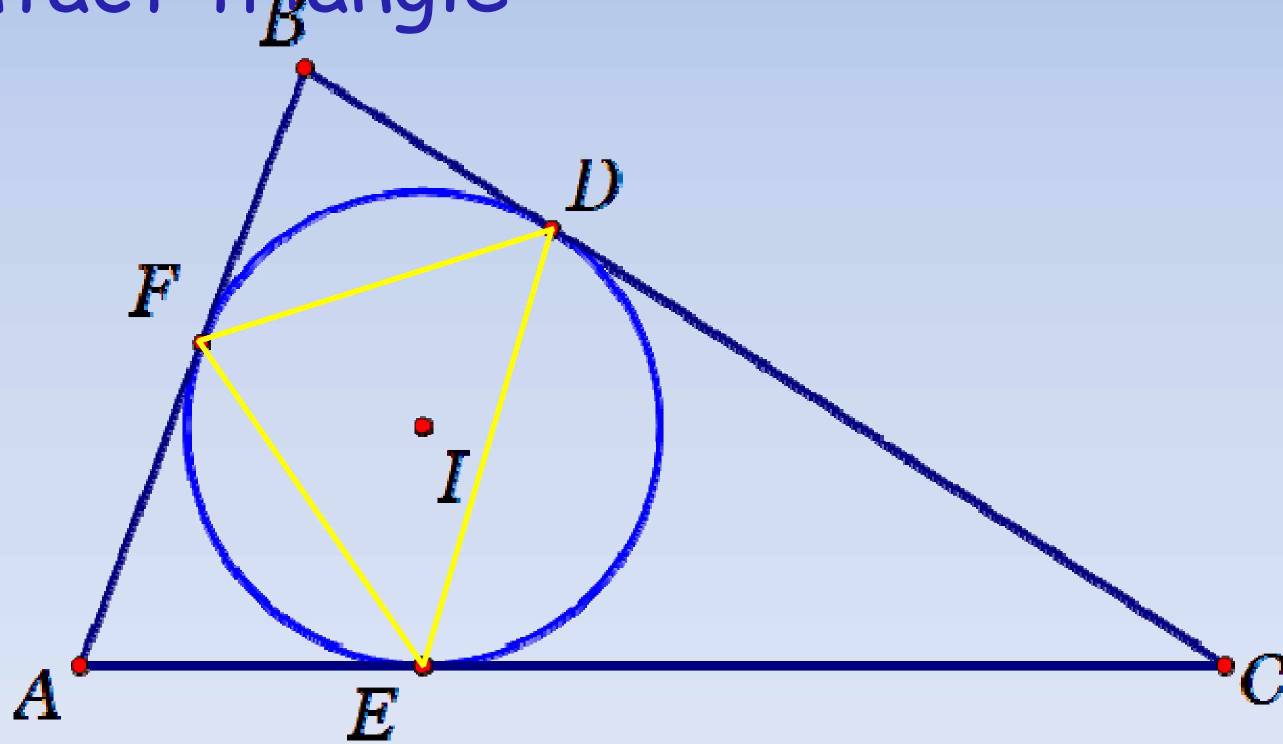
- medians – cevian point = G
- perpendicular bisectors – cevian point = O
- angle bisectors – cevian point = I (incenter)
- altitudes – cevian point = H

Ceva's Theorem deals with concurrence of any set of cevians.

Gergonne Point

In ΔABC find the incircle and points of tangency of incircle with sides of ΔABC.

Known as contact triangle

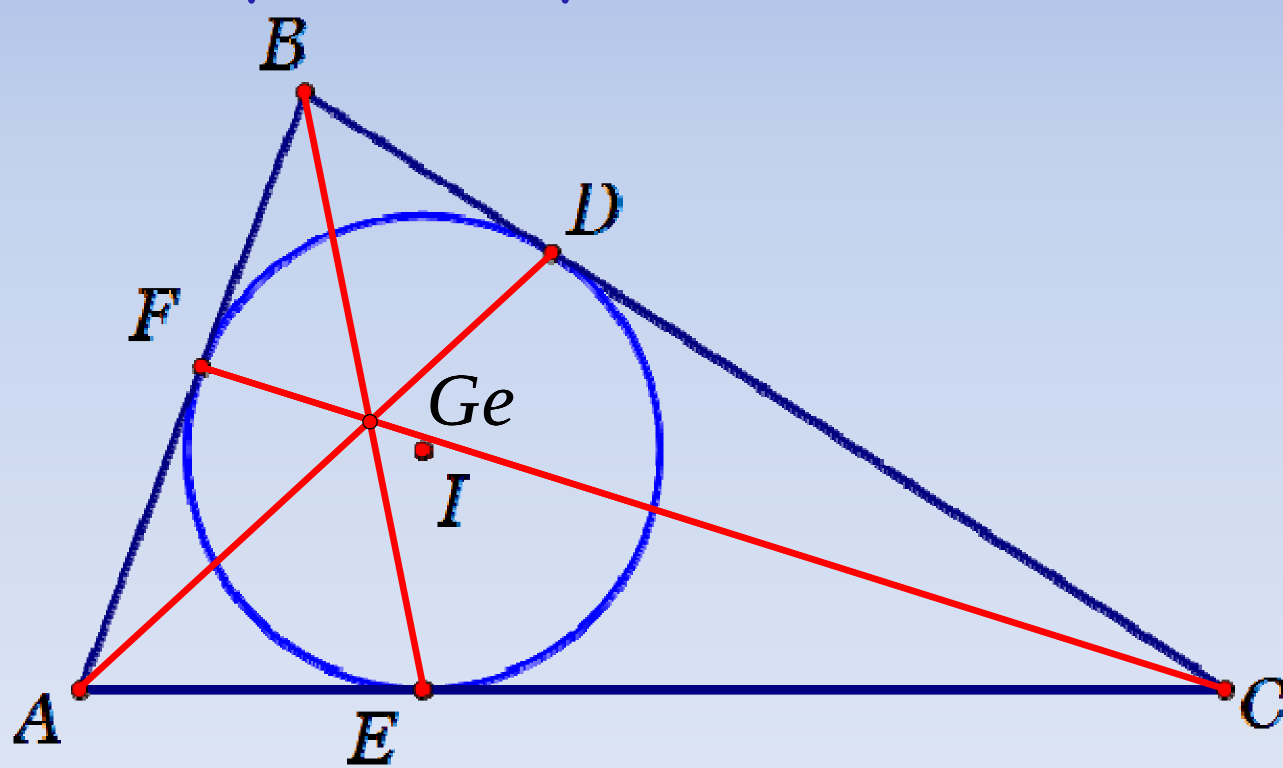


Gergonne Point

These cevians are concurrent!

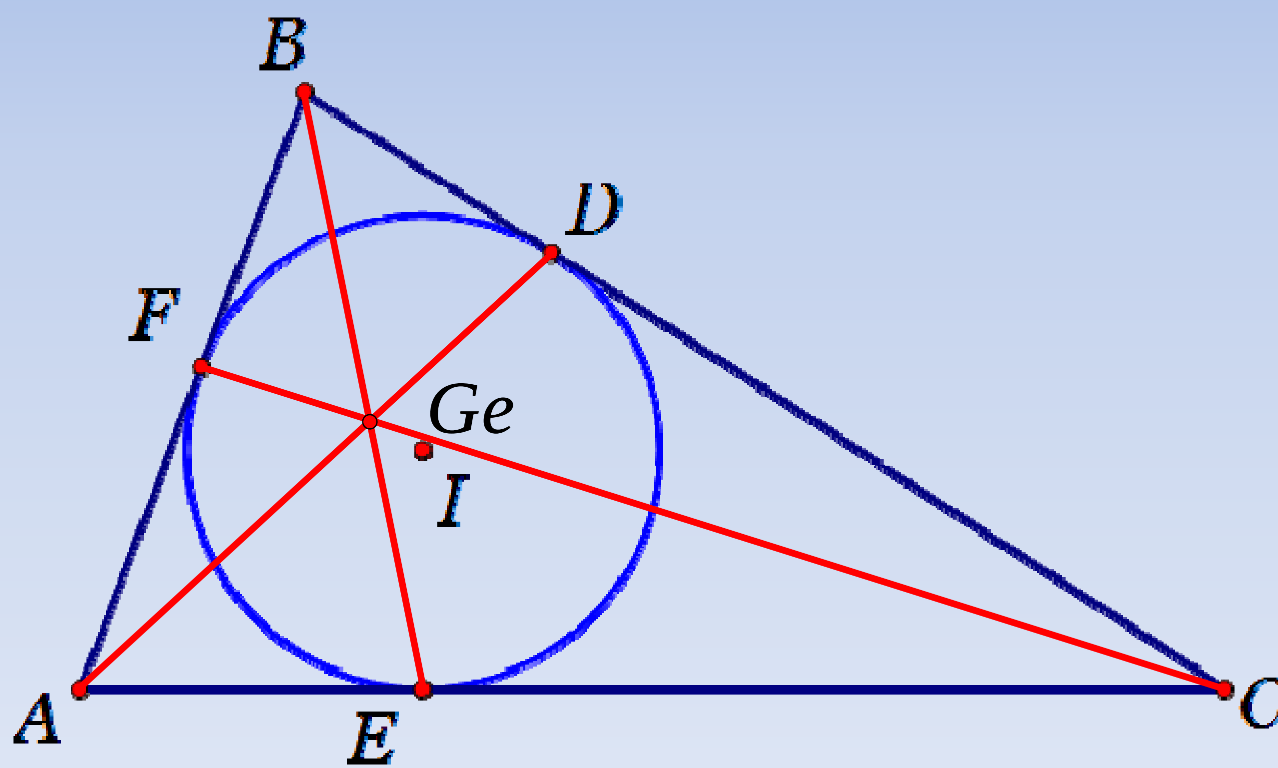
Why?

Recall that AE=AF, BD=BF, and CD=CE



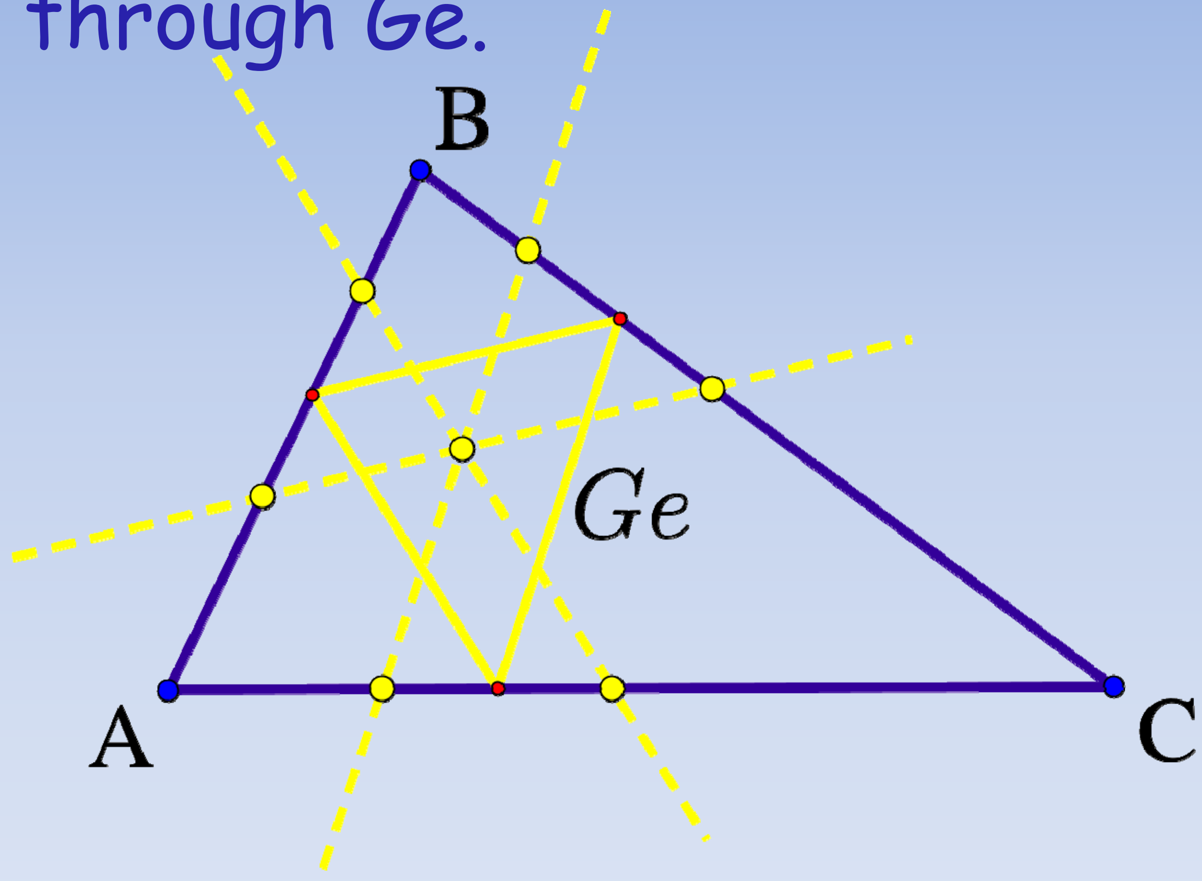
Gergonne Point

The point is called the Gergonne point, Ge.



Gergonne Point

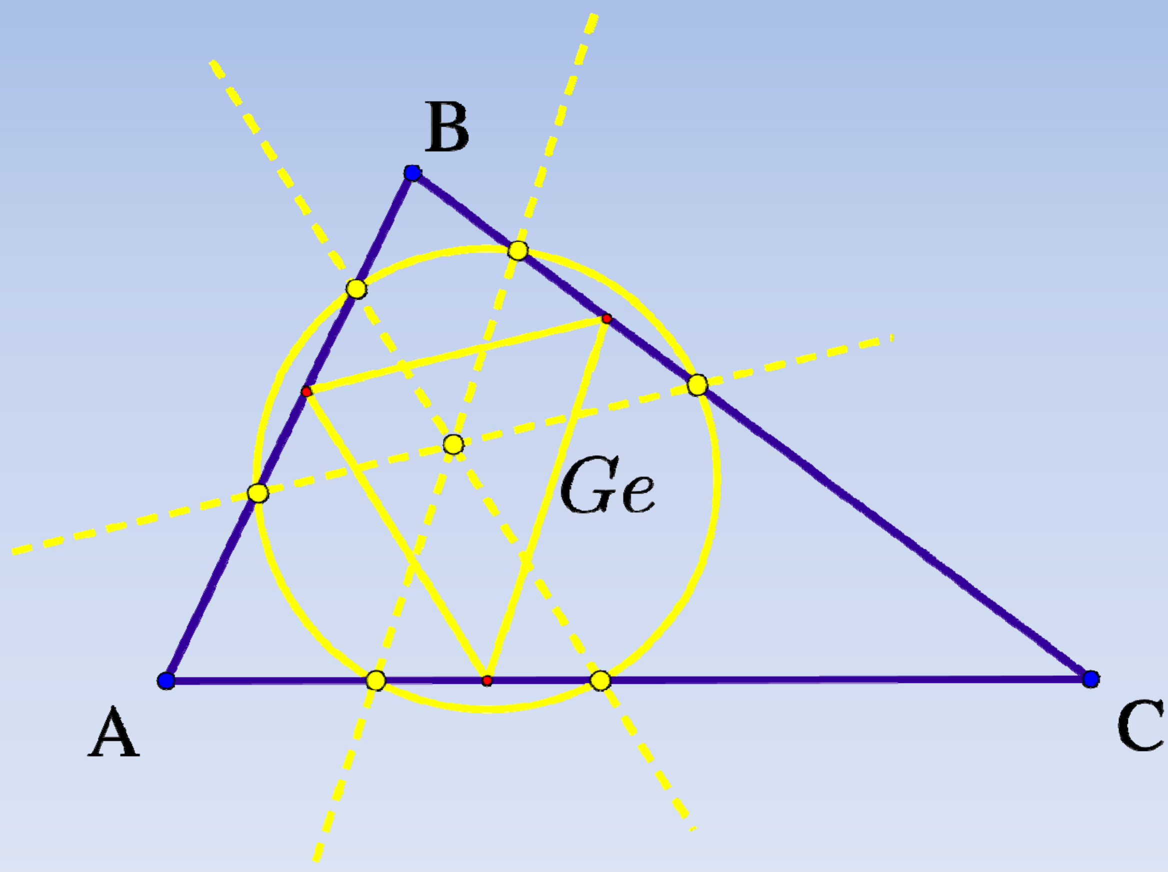
Draw lines parallel to sides of contact triangle through Ge.



Gergonne Point

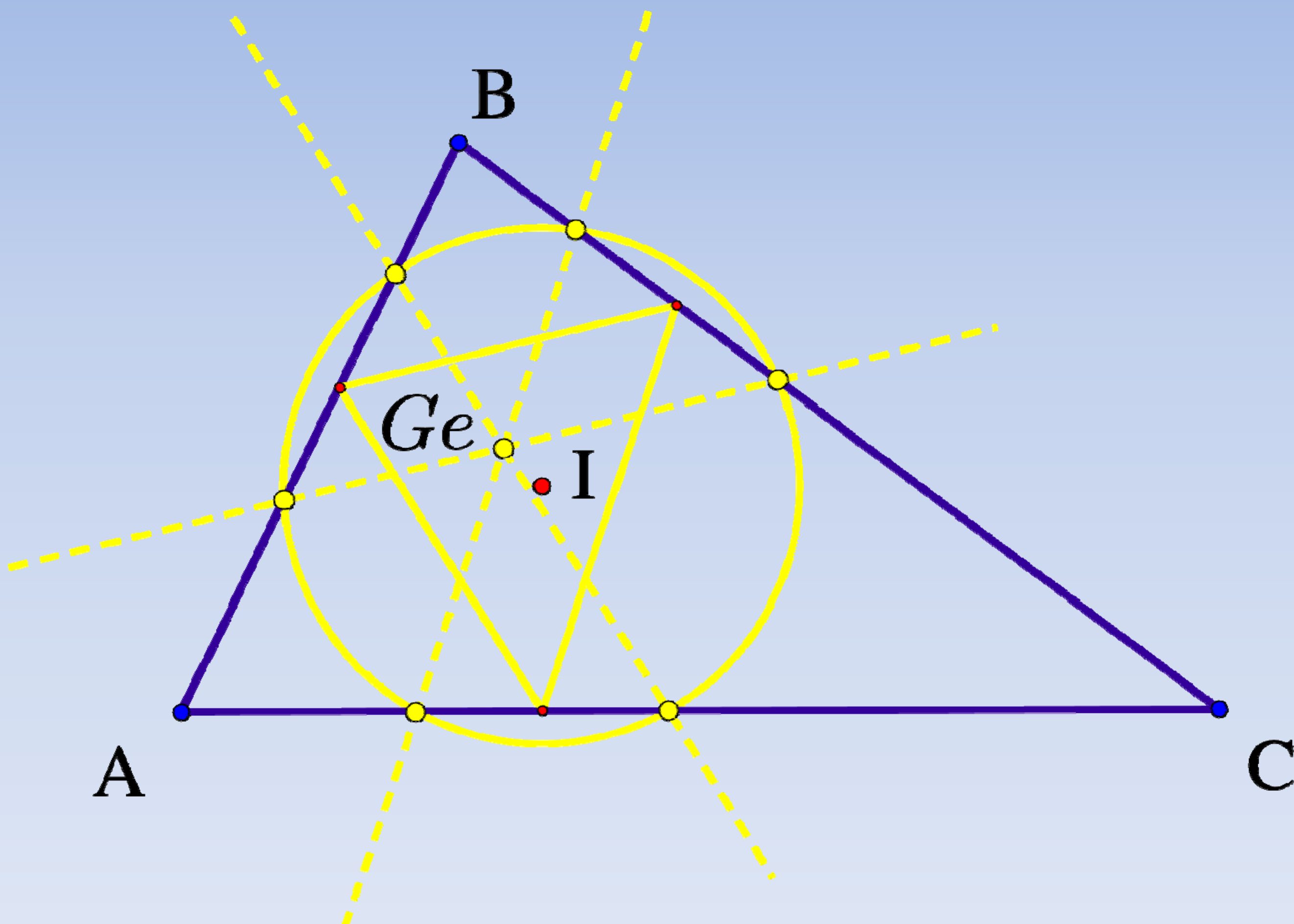
Six points are concyclic!!

Called the Adams Circle



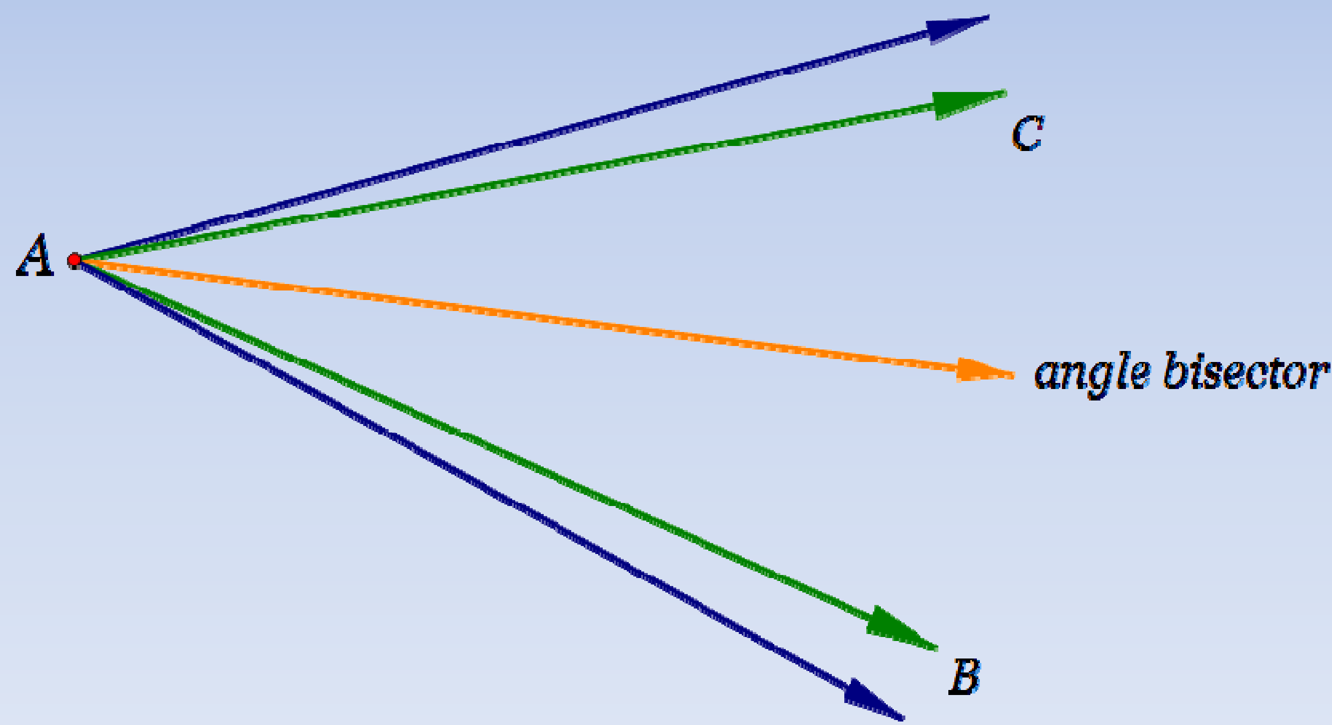
Gergonne Point

Center of Adams circle = incenter of ΔABC



Isogonal Conjugates

Two lines *AB* and *AC* through vertex *A* are said to be <u>isogonal</u> if one is the reflection of the other through the angle bisector.



Isogonal Conjugates

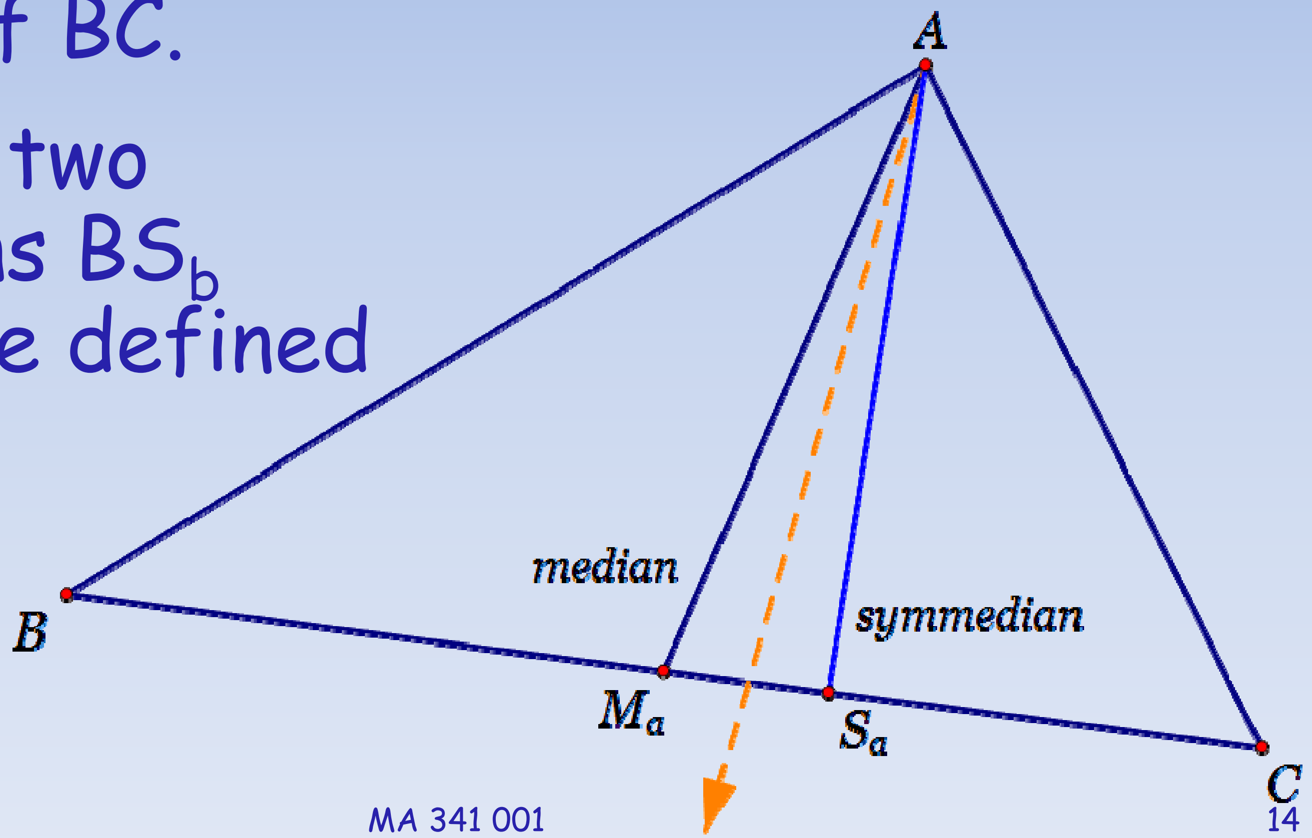
If lines through A, B, and C are concurrent at P, then the isogonal lines are concurrent at Q.

Points P and Q are isogonal conjugates.

Symmedians

In ΔABC, the symmedian AS_a is a cevian through vertex A ($S_a \in BC$) isogonally conjugate to the median AM_a, M_a being the midpoint of BC.

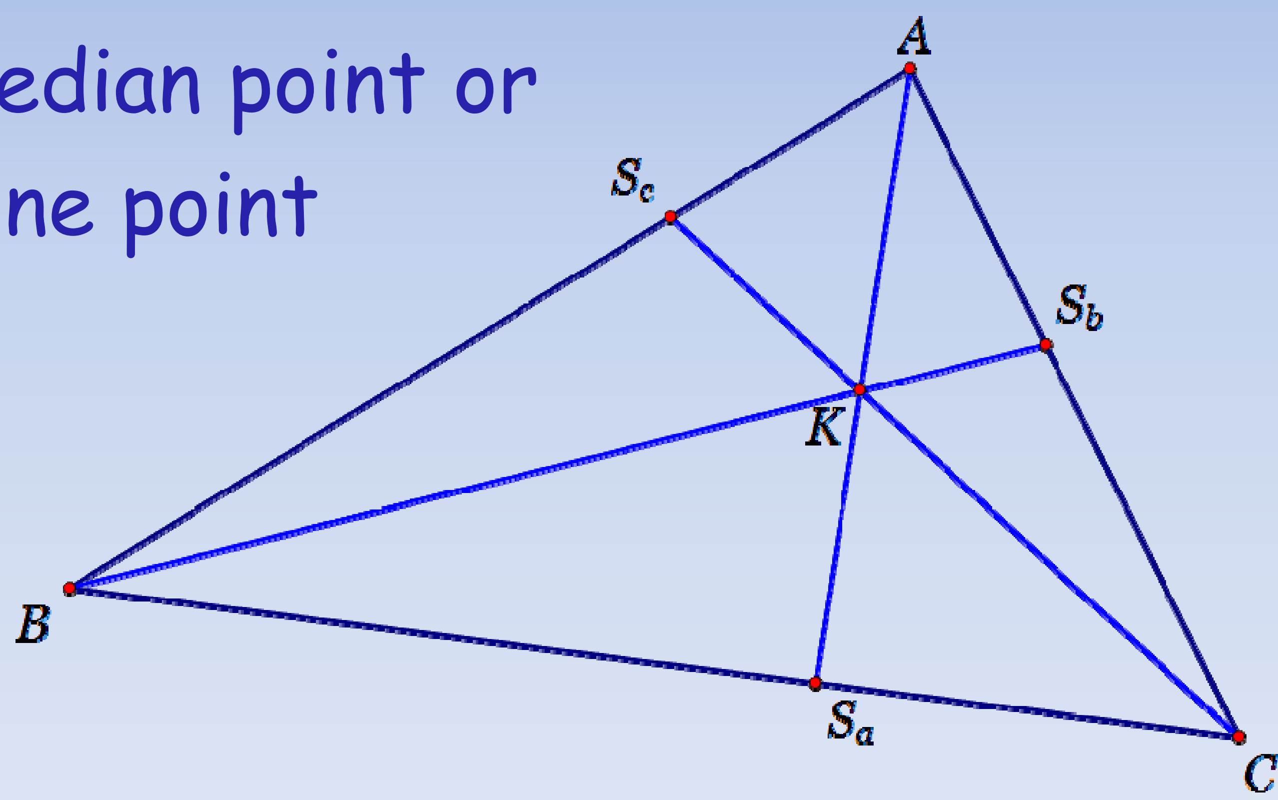
The other two symmedians BS_b and CS_c are defined similarly.



Symmedians

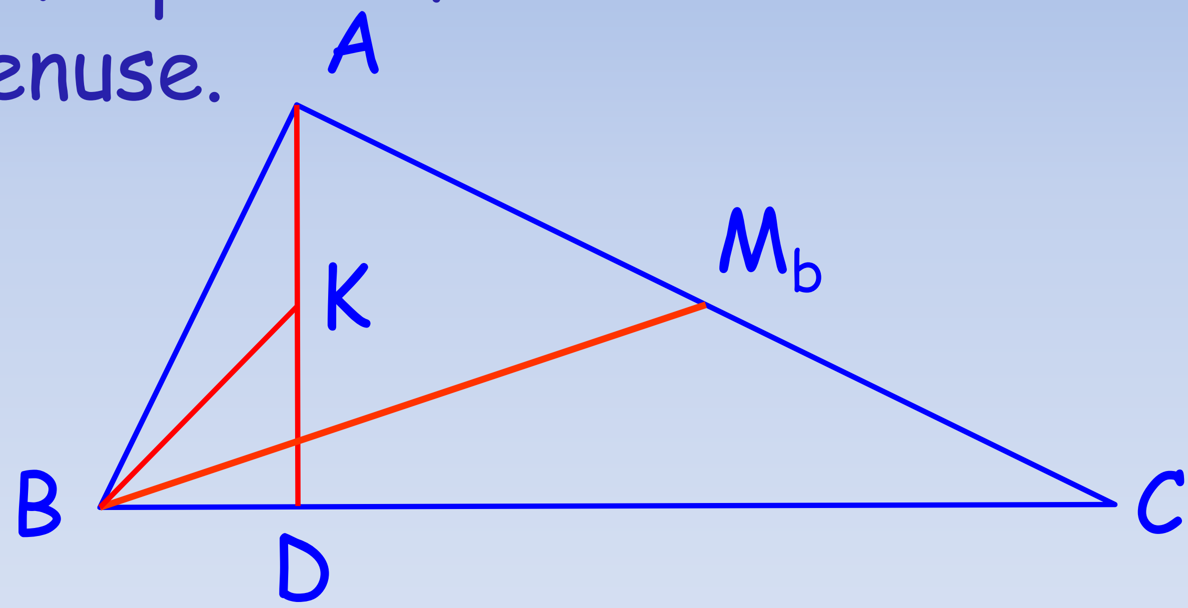
The three symmedians AS_a, BS_b and CS_c concur in a point commonly denoted K and variably known as either

- the symmedian point or
- the Lemoine point



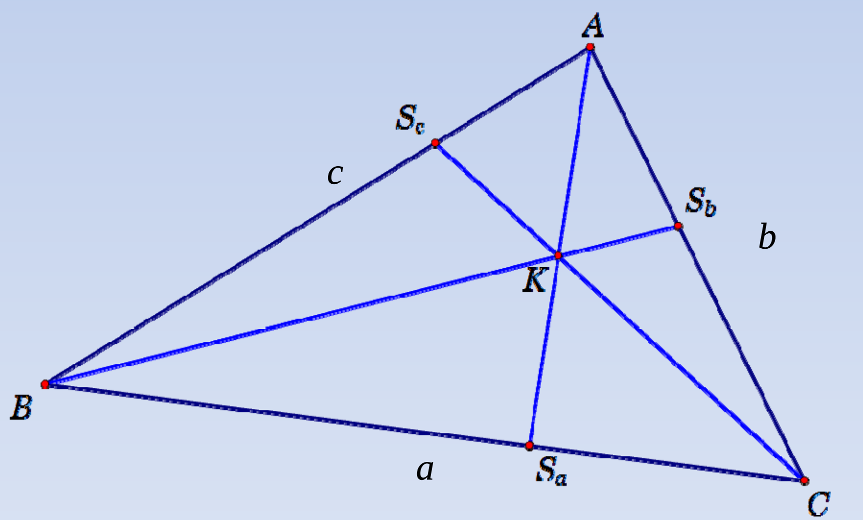
Symmedian of Right Triangle

The symmedian point K of a right triangle is the midpoint of the altitude to the hypotenuse.



Proportions of the Symmedian

Draw the cevian from vertex A, through the symmedian point, to the opposite side of the triangle, meeting BC at S_a. Then



$$\frac{BS_a}{CS_a} = \frac{c^2}{b^2}$$

Length of the Symmedian

Draw the cevian from vertex C, through the symmedian point, to the opposite side of the triangle. Then this segment has length

$$CS_c = \frac{ab\sqrt{2a^2 + 2b^2 - c^2}}{a^2 + b^2}$$

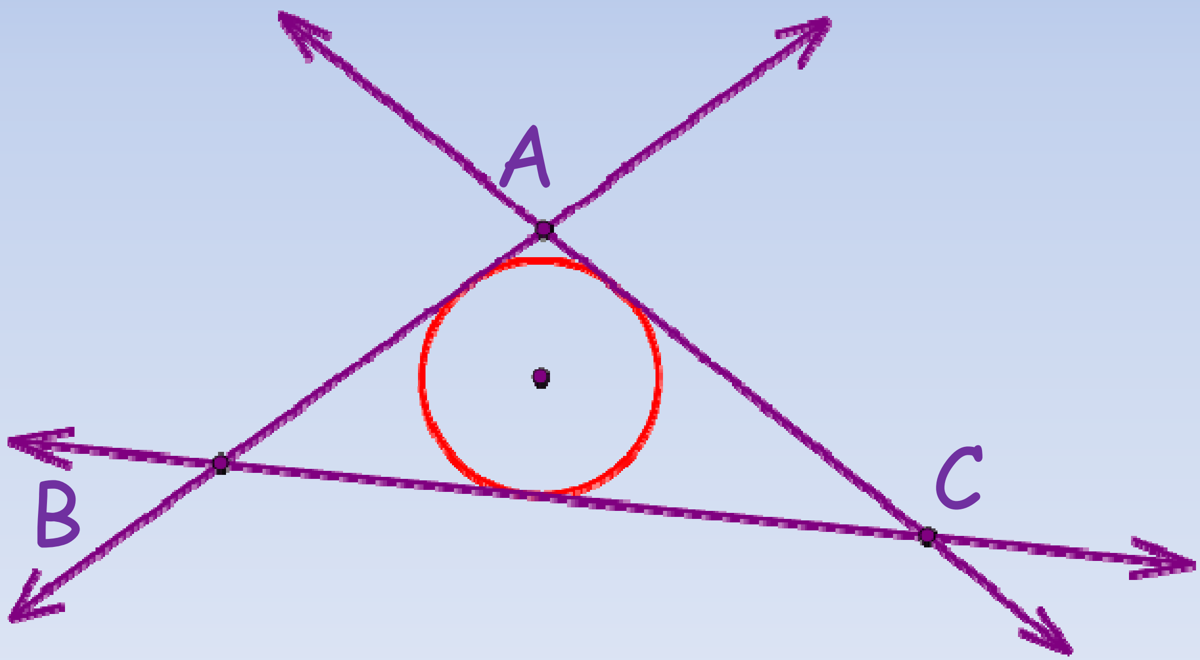
Likewise

$$AS_a = \frac{bc\sqrt{2b^2 + 2c^2 - a^2}}{b^2 + c^2}$$

$$BS_b = \frac{ac\sqrt{2a^2 + 2c^2 - b^2}}{a^2 + c^2}$$

Excircles

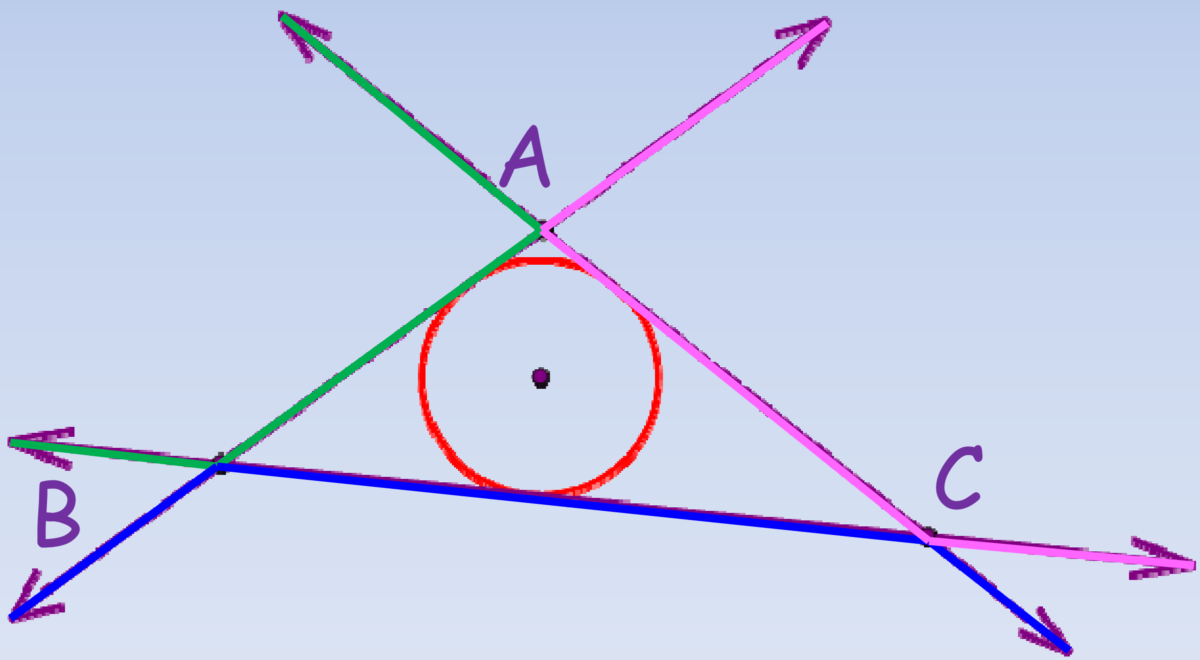
In several versions of geometry triangles are defined in terms of lines not segments.



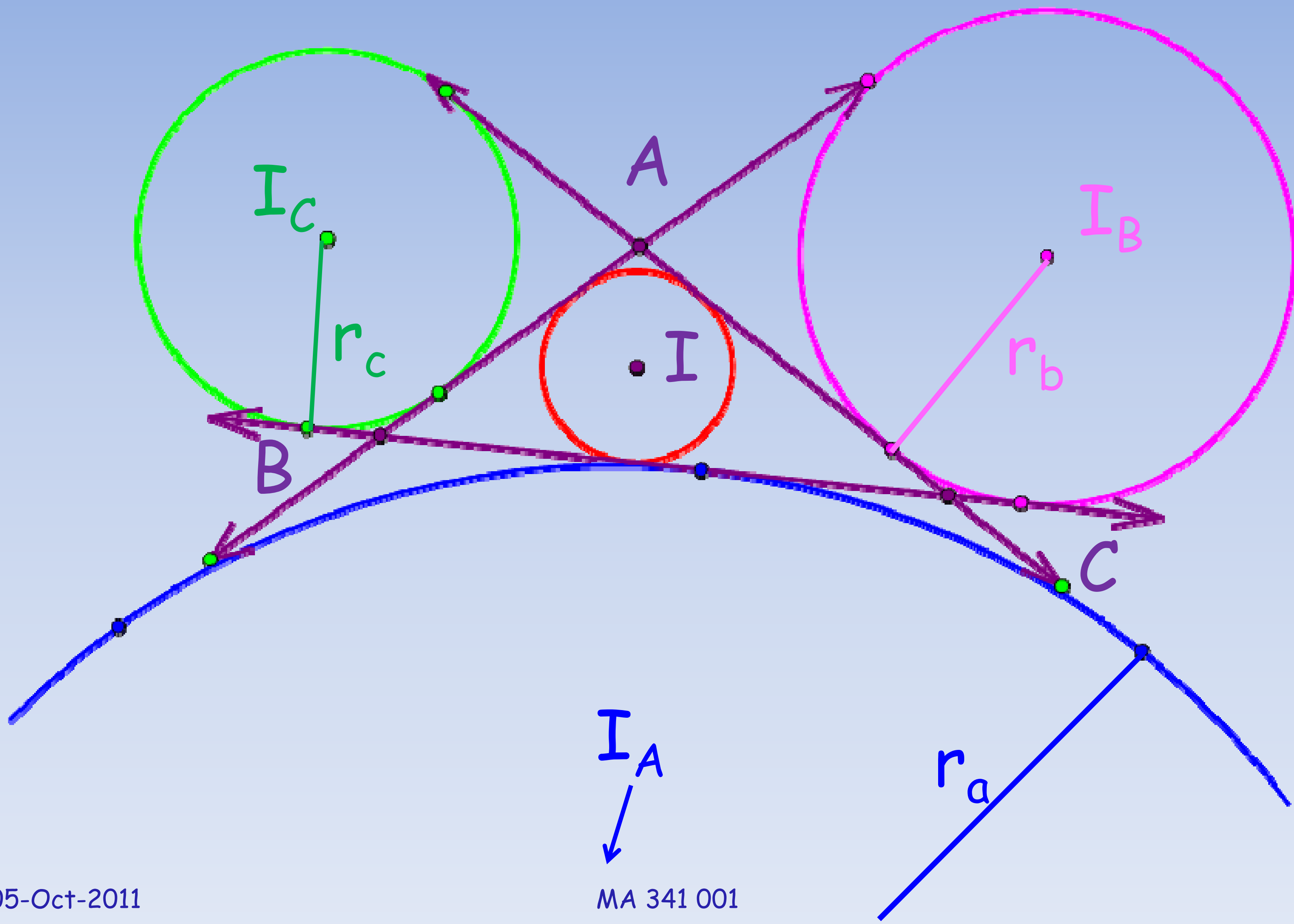
Excircles

Do these sets of three lines define circles?

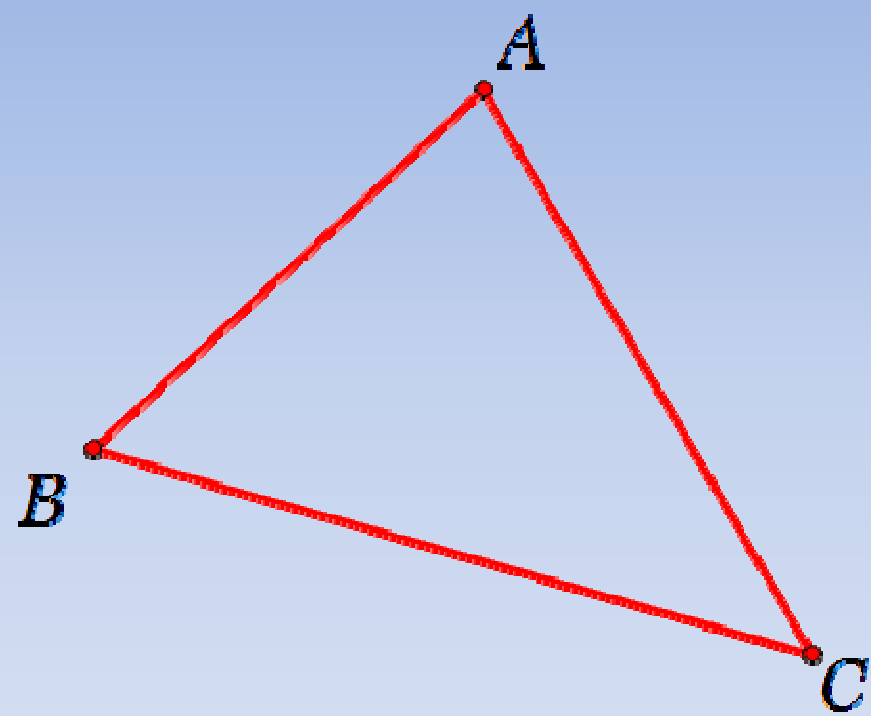
Known as tritangent circles



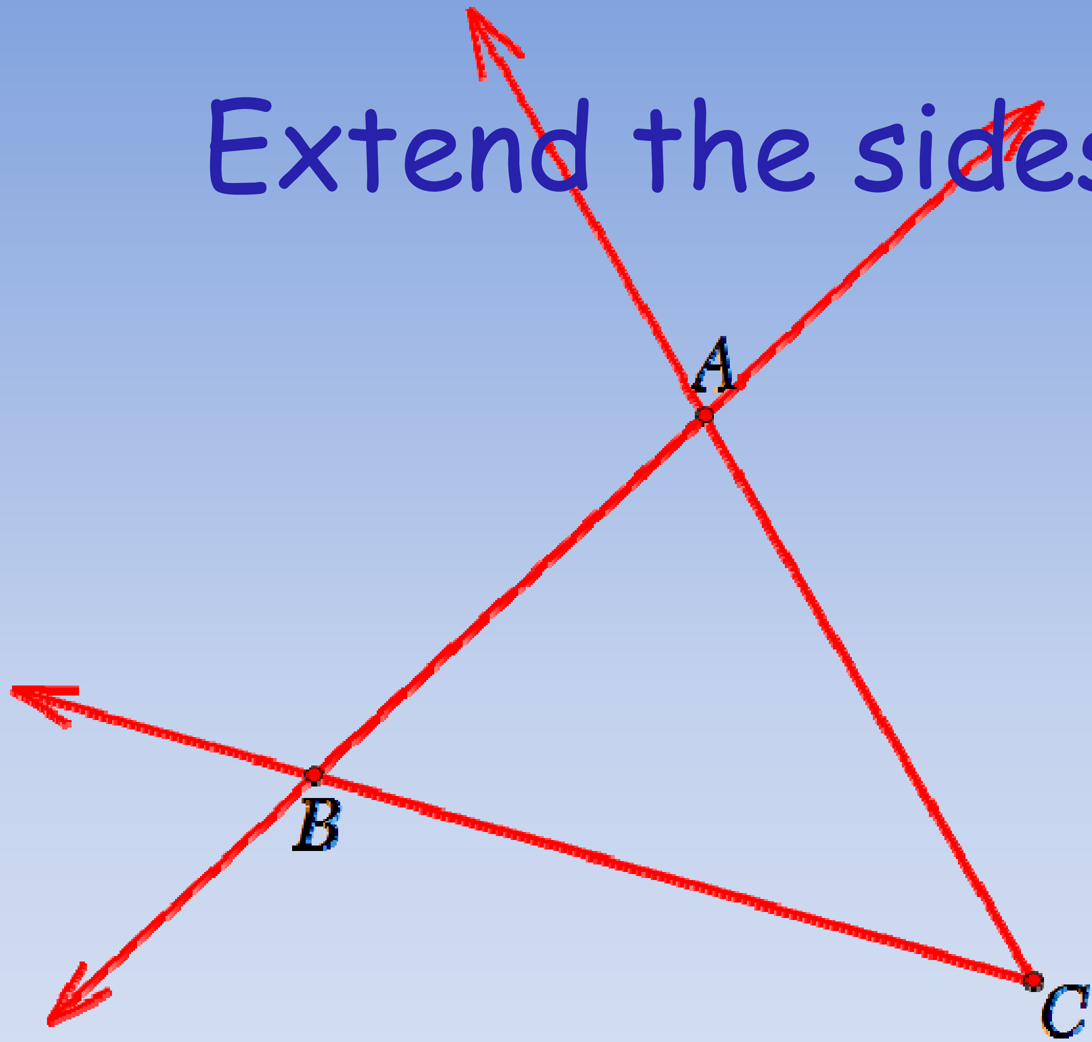
Excircles



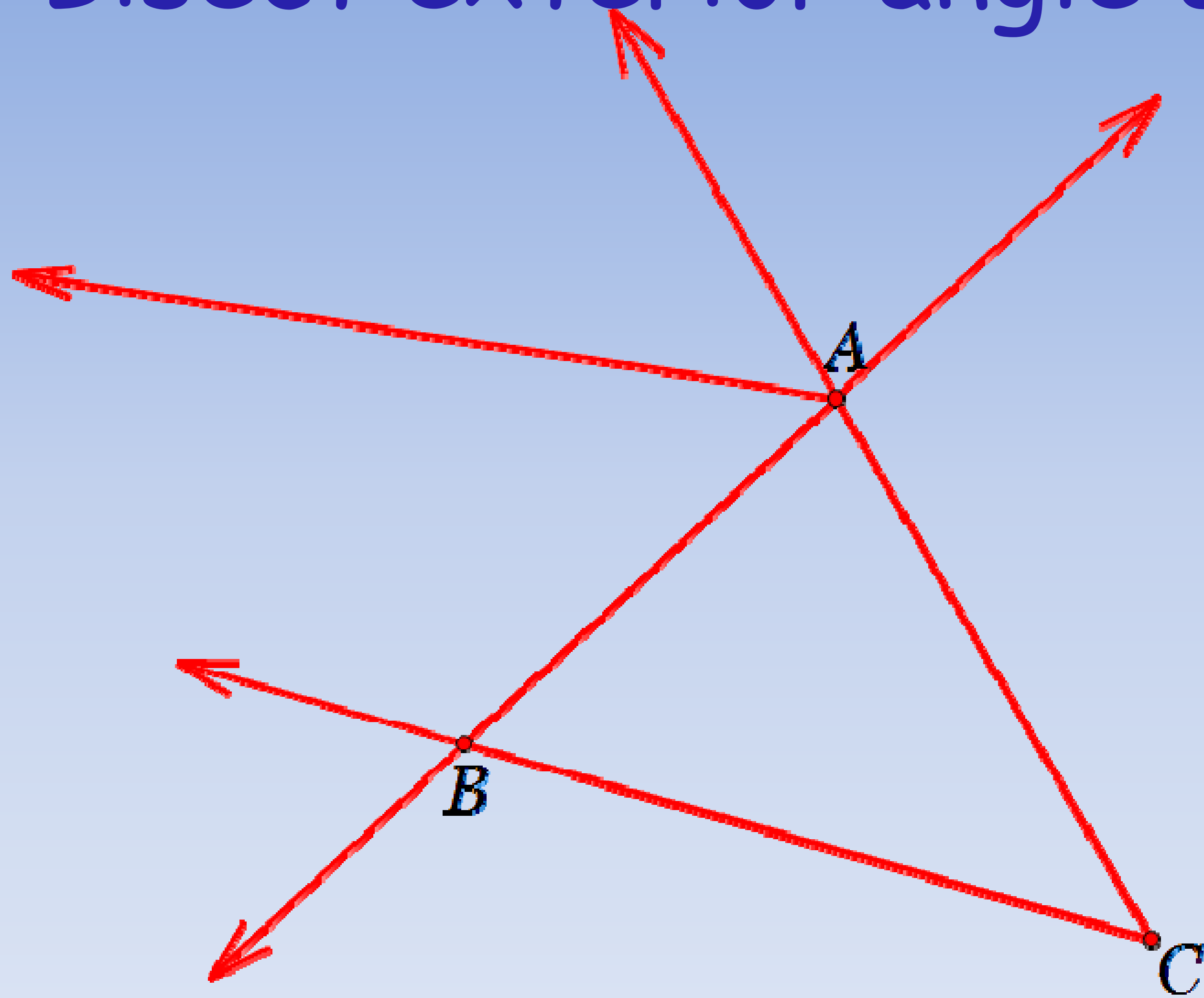
Construction of Excircles



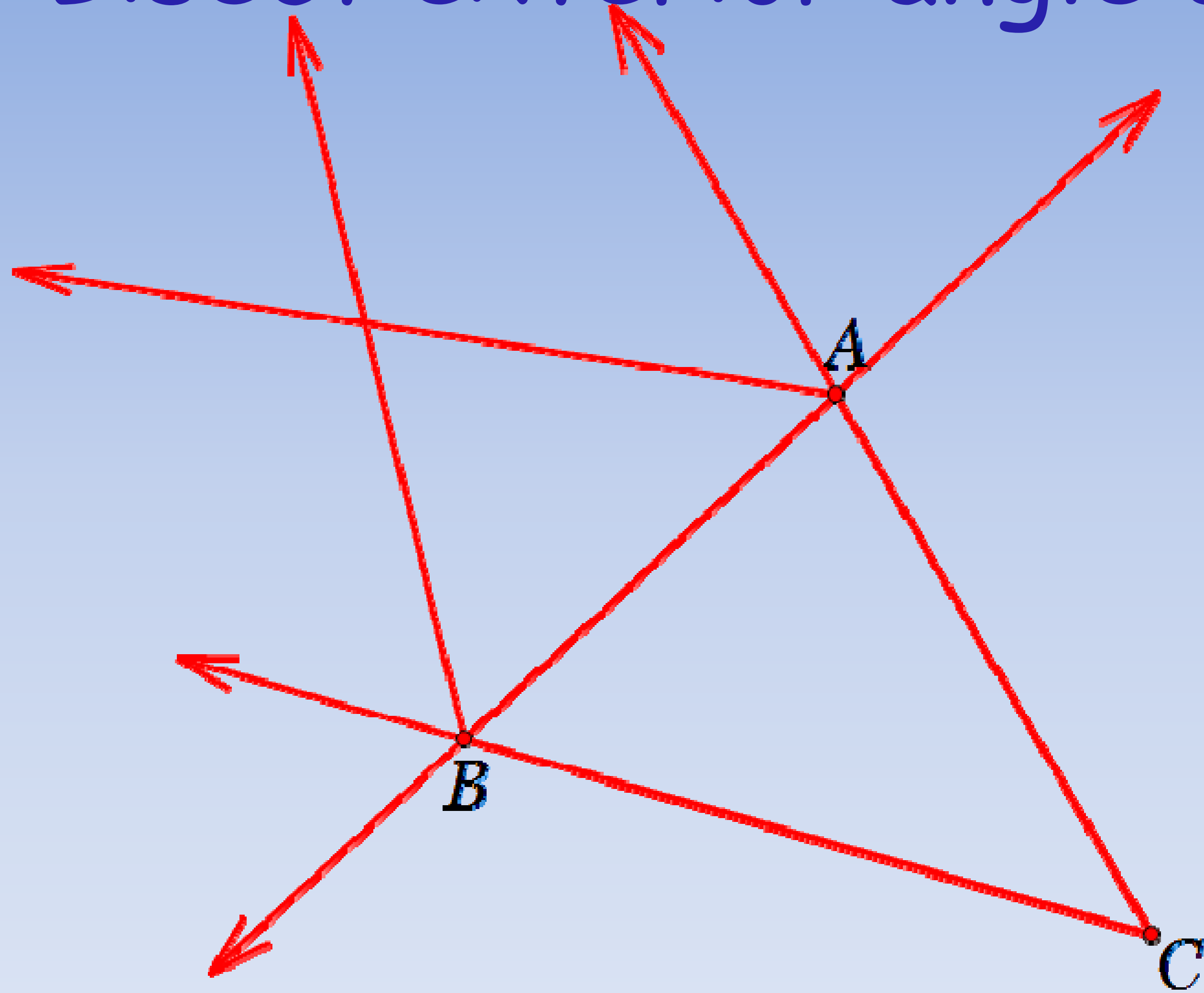
Extend the sides



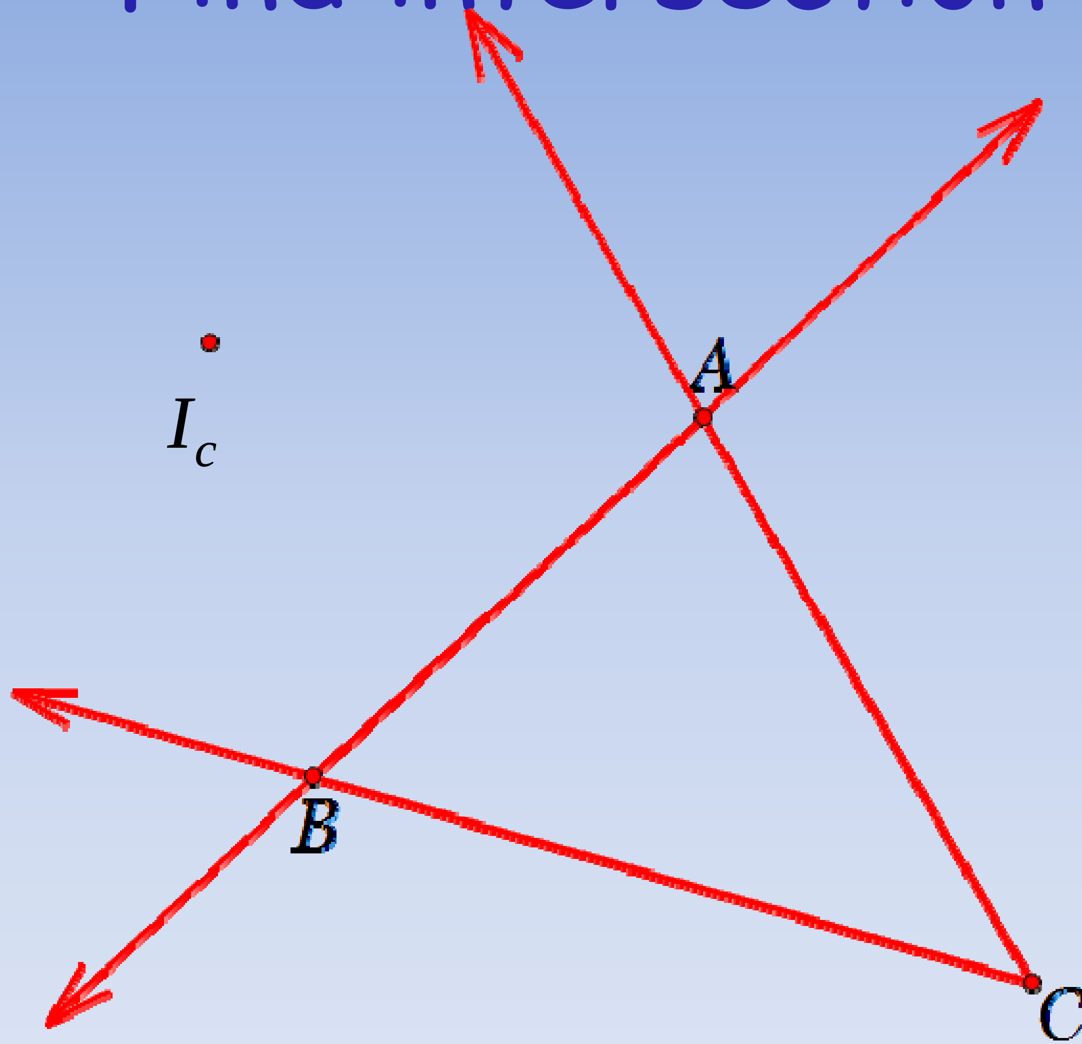
Bisect exterior angle at A



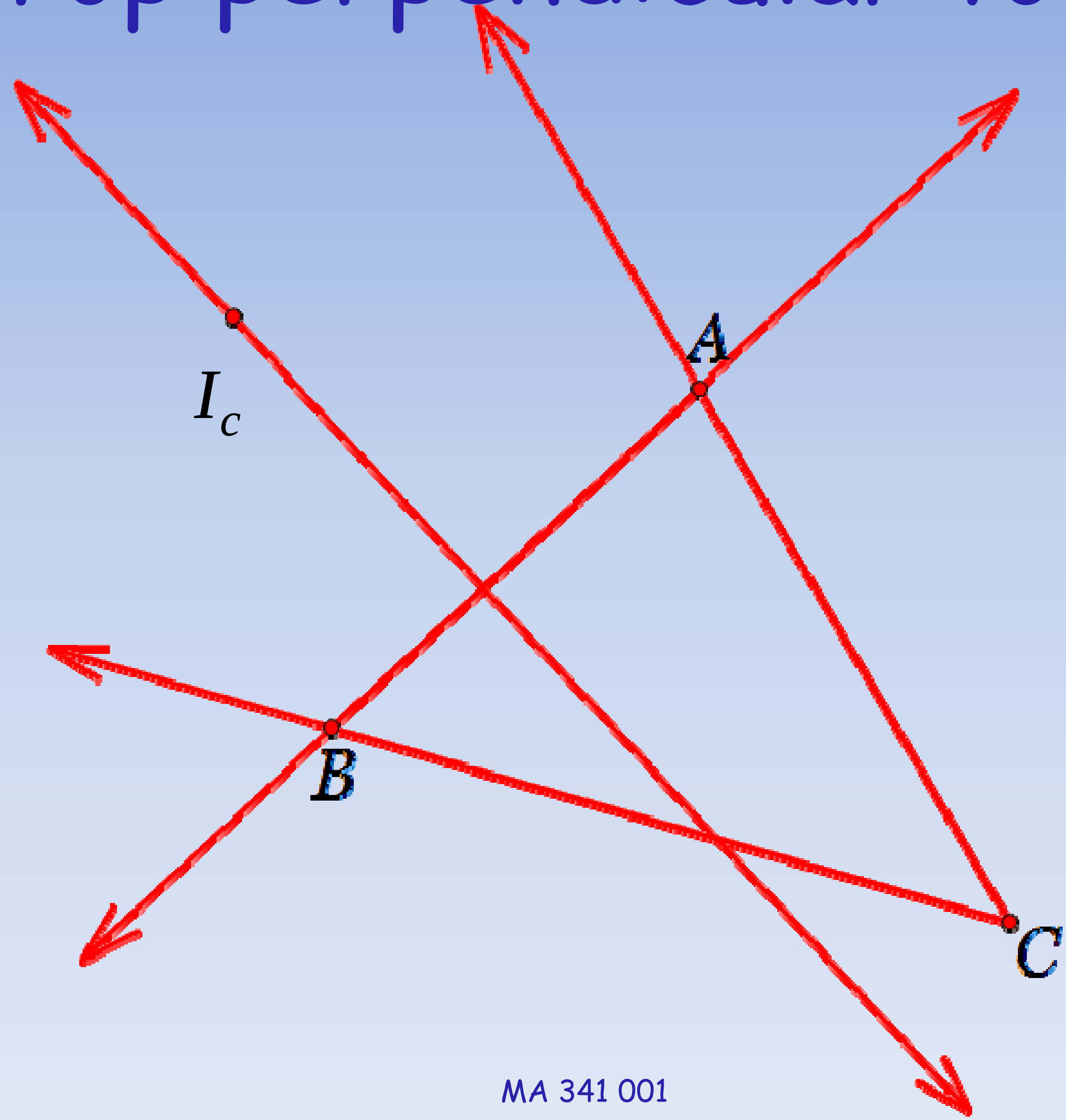
Bisect exterior angle at B



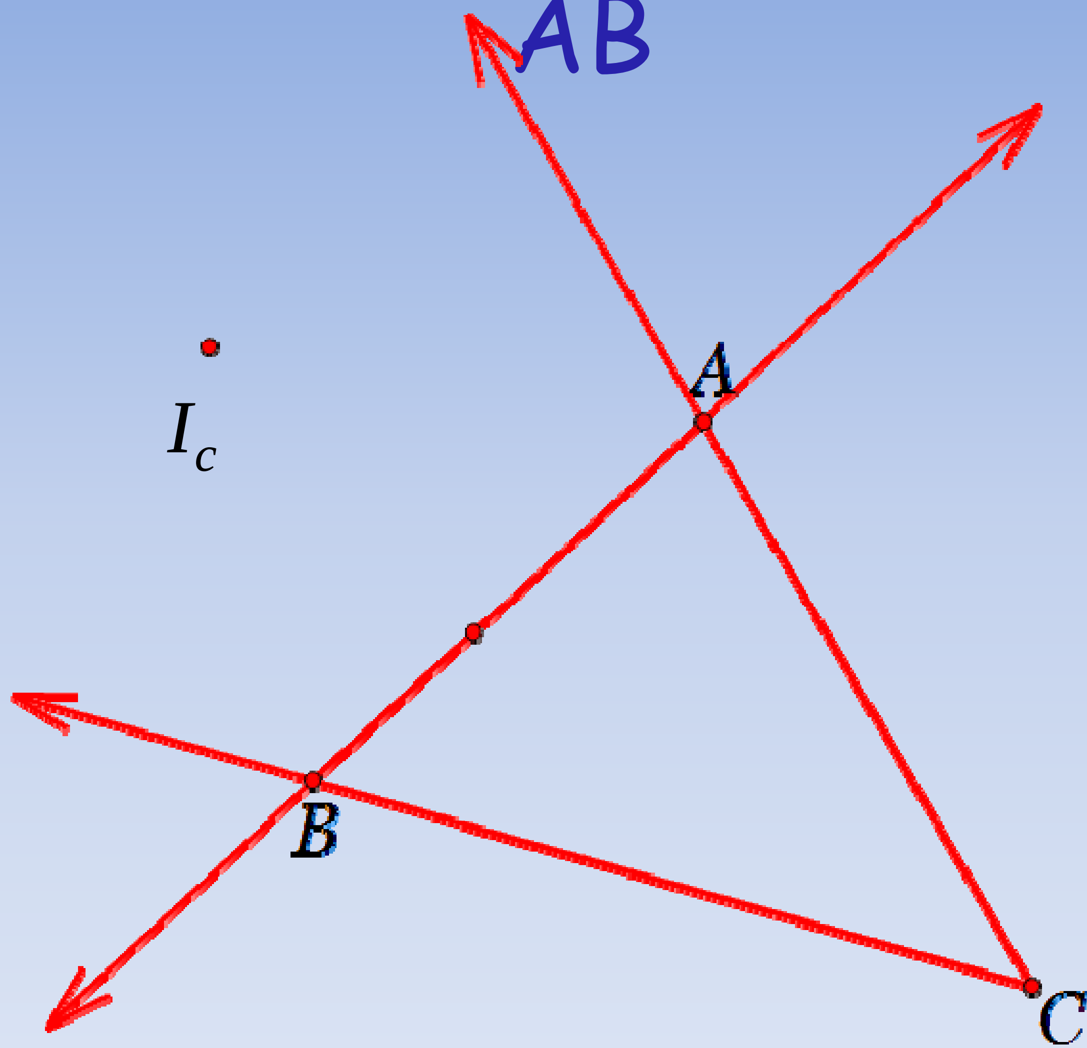
Find intersection



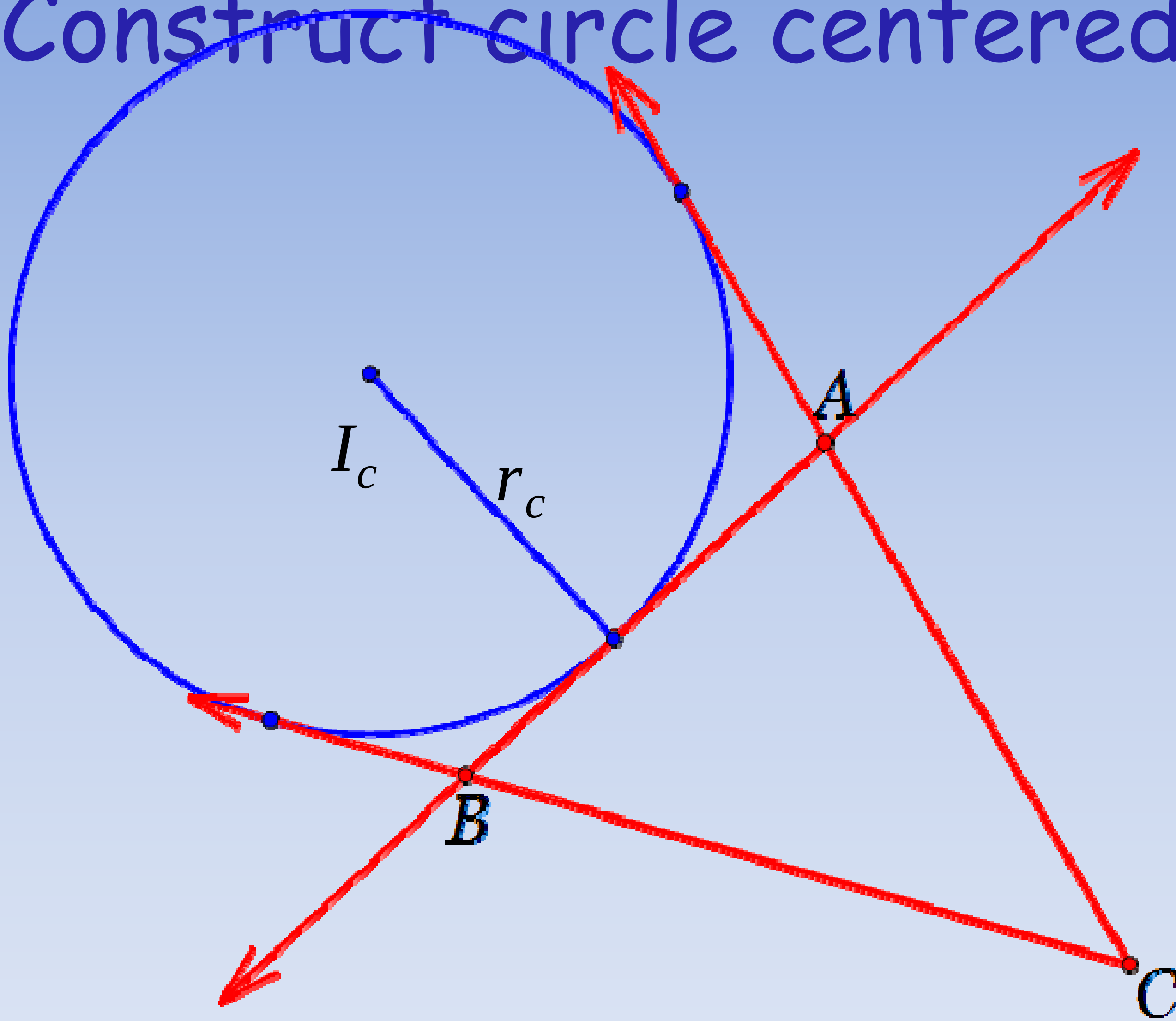
Drop perpendicular to AB

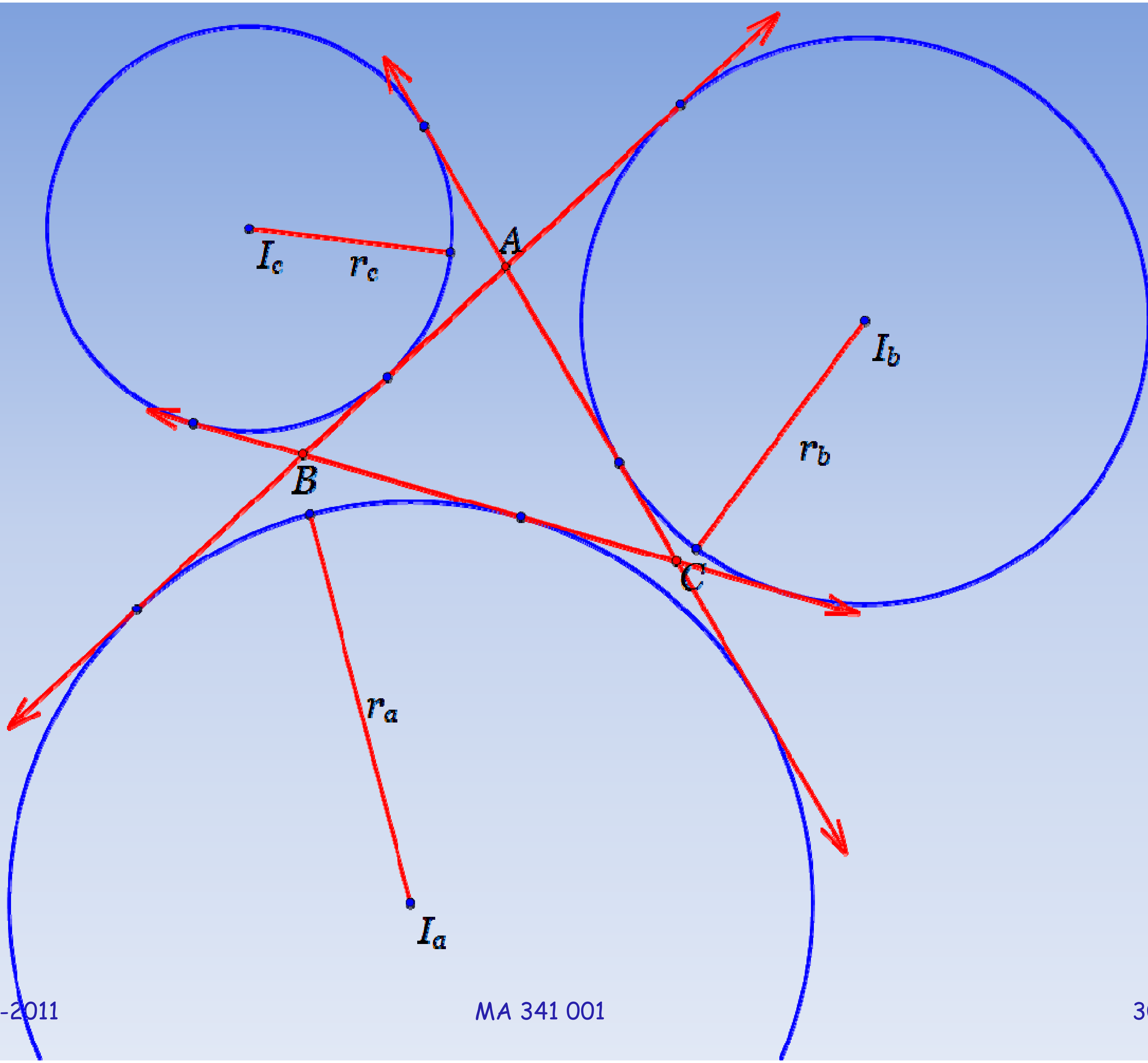


Find point of intersection with AB



Construct circle centered at I_c




A
B
C
I_c
r_c
I_b
r_b
r_a
I_a

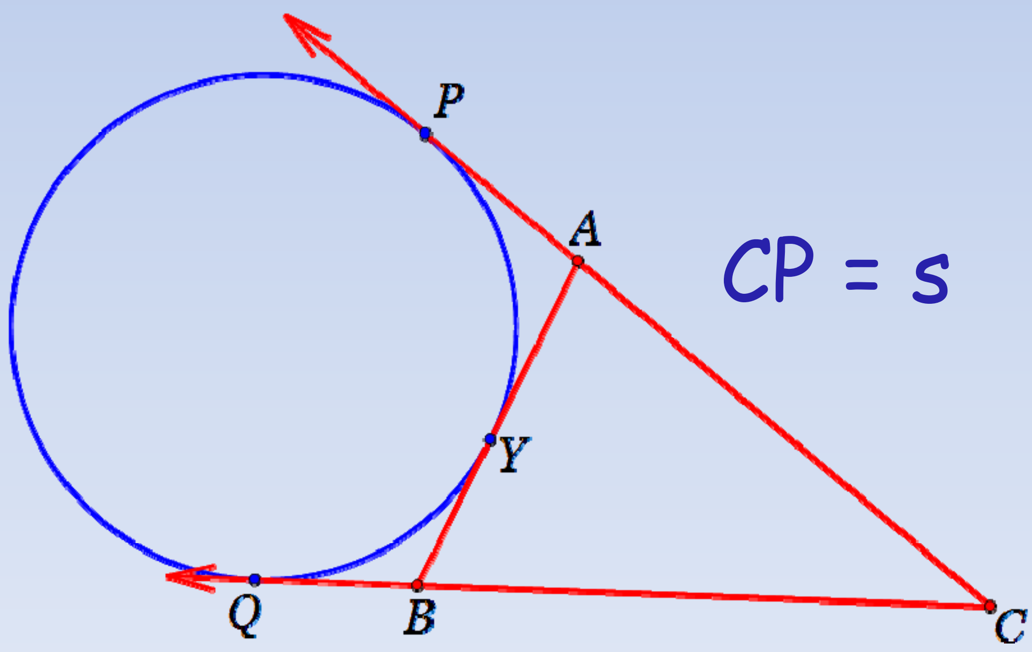
Excircles

The I_a, I_b, and I_c are called excenters.

r_a, r_b, r_c are called exradii

Excircles

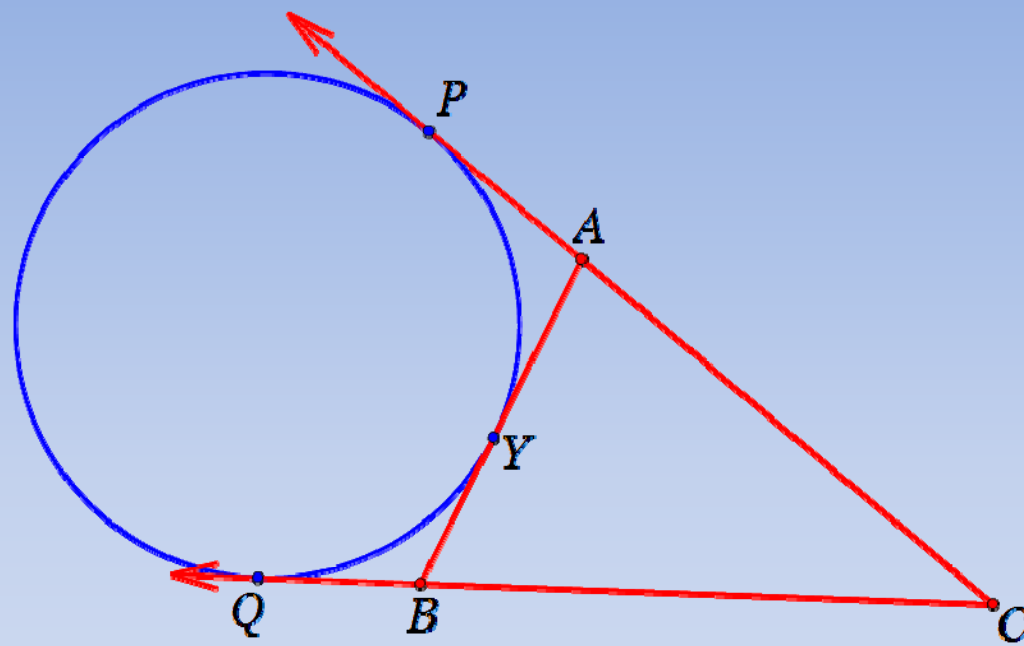
Theorem: The length of the tangent from a vertex to the opposite exscribed circle equals the semiperimeter, s.



CP = s

Excircles

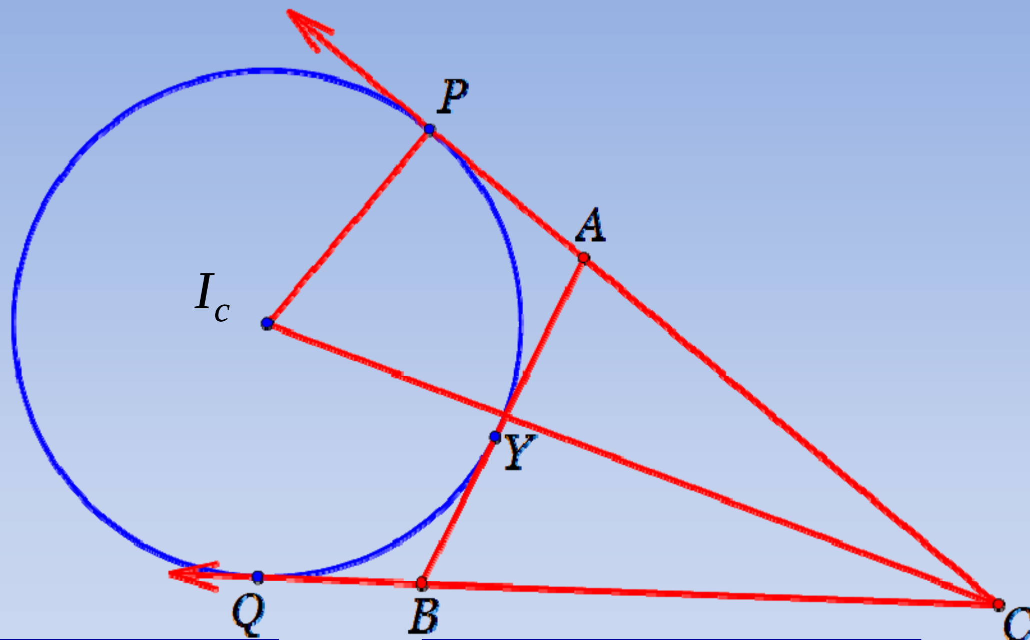
1. $CQ = CP$
2. $AP = AY$
3. $CP = CA+AP$
   $= CA+AY$
4. $CQ= BC+BY$



5. $CP + CQ = AC + AY + BY + BC$
6. $2CP = AB + BC + AC = 2s$
7. $CP = s$

Exradii

1. $CP \perp I_C P$
2. $\tan(C/2) = r_C/s$
3. Use Law of Tangents



$$r_c = s\tan\left(\frac{C}{2}\right) = s\sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \sqrt{\frac{s(s-a)(s-b)}{s-c}}$$

Exradii

Likewise

$$r_a = \sqrt{\frac{s(s-b)(s-c)}{s-a}}$$

$$r_b = \sqrt{\frac{s(s-a)(s-c)}{s-b}}$$

$$r_c = \sqrt{\frac{s(s-a)(s-b)}{s-c}}$$

Excircles

Theorem: For any triangle ΔABC

$$\frac{1}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}$$

Excircles

$$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\sqrt{\frac{s-a}{s(s-b)(s-c)}}+\sqrt{\frac{s-b}{s(s-a)(s-c)}}+\sqrt{\frac{s-c}{s(s-a)(s-b)}}$$

$$=\frac{s-a}{\sqrt{s(s-a)(s-b)(s-c)}}+\frac{s-b}{\sqrt{s(s-a)(s-b)(s-c)}}+\frac{s-c}{\sqrt{s(s-a)(s-b)(s-c)}}$$

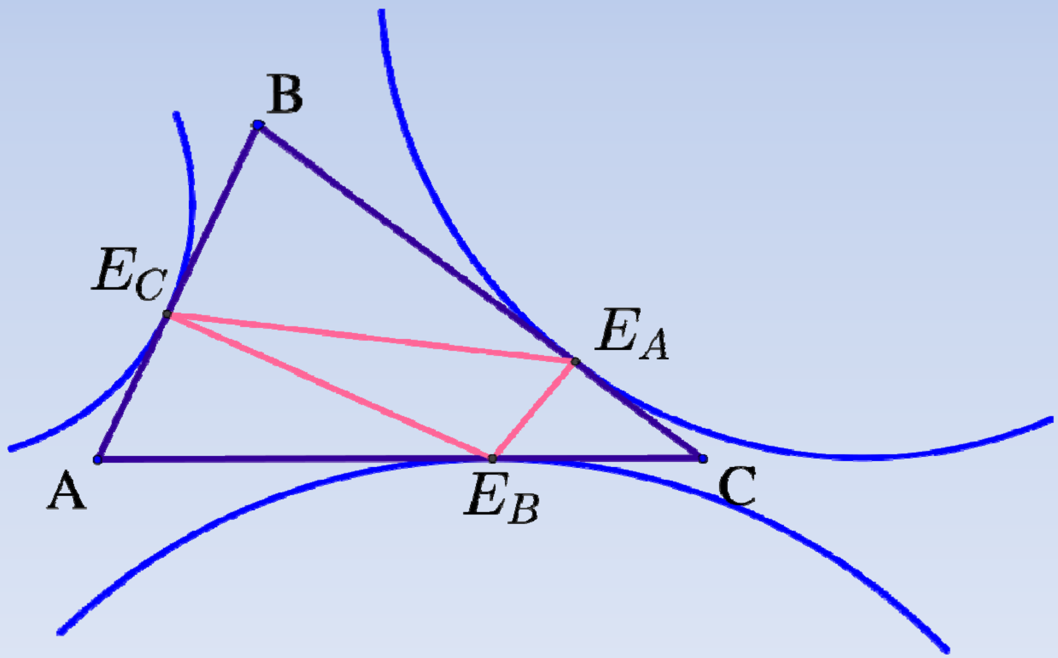
$$=\frac{3s-(a+b+c)}{\sqrt{s(s-a)(s-b)(s-c)}}$$

$$=\frac{s}{\sqrt{s(s-a)(s-b)(s-c)}}=\frac{s}{K}$$

$$=\frac{1}{r}$$

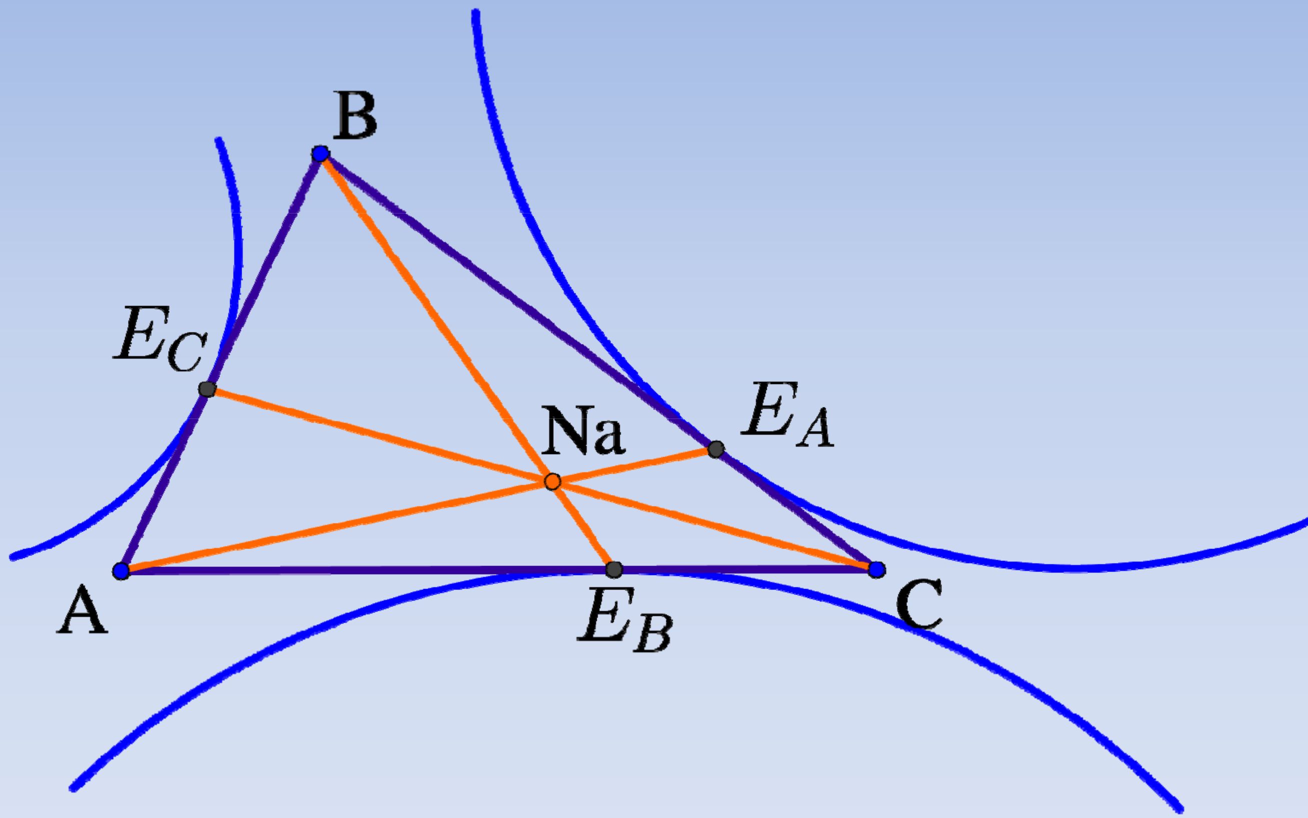
Nagel Point

In ΔABC find the excircles and points of tangency of the excircles with sides of ΔABC.



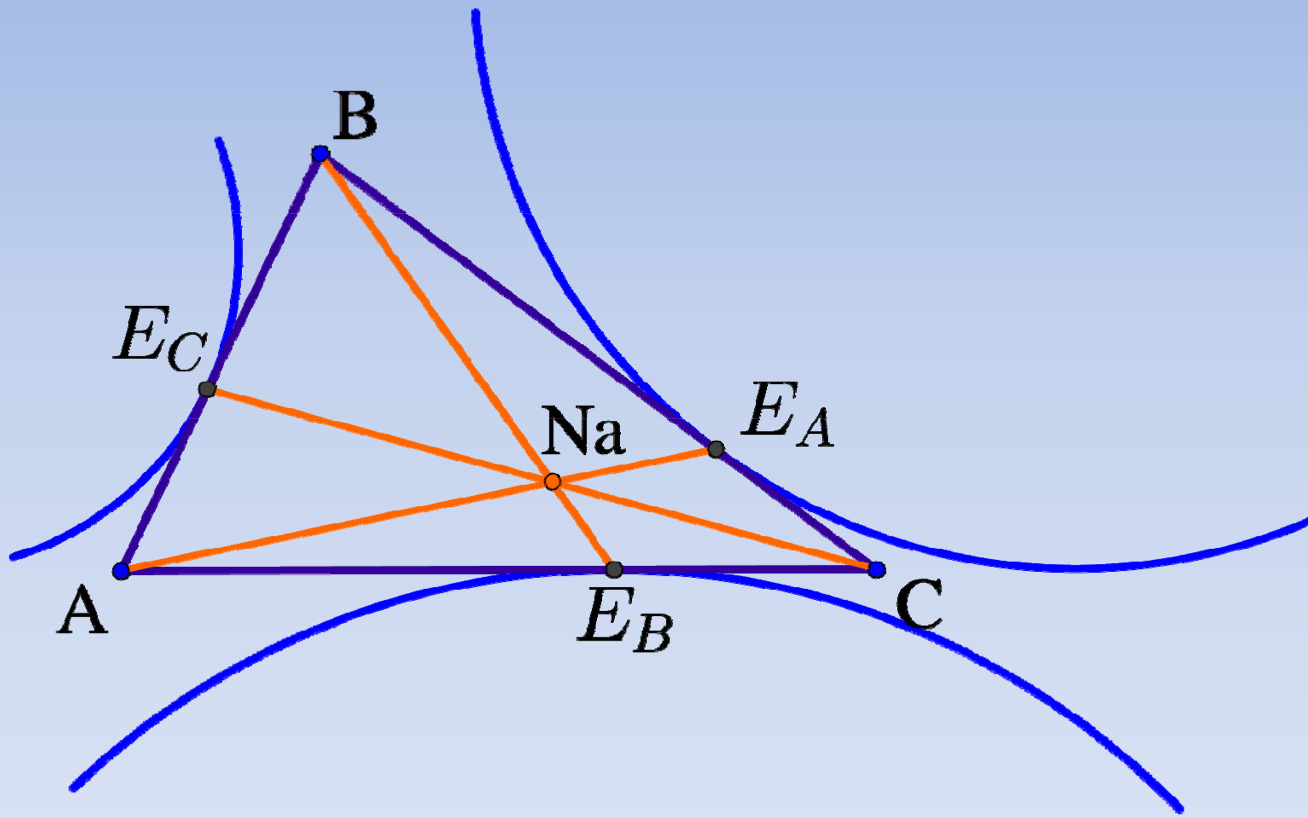
Nagel Point

These cevians are concurrent!



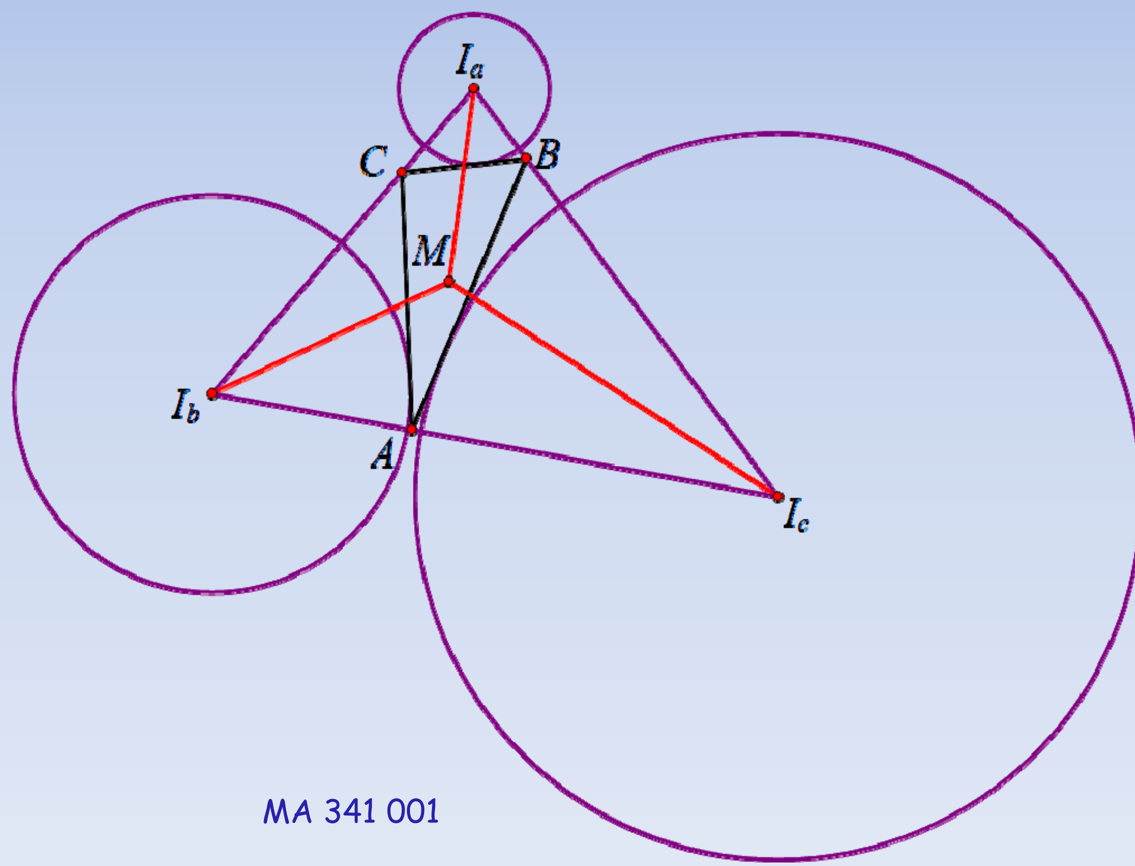
Nagel Point

Point is known as the Nagel point



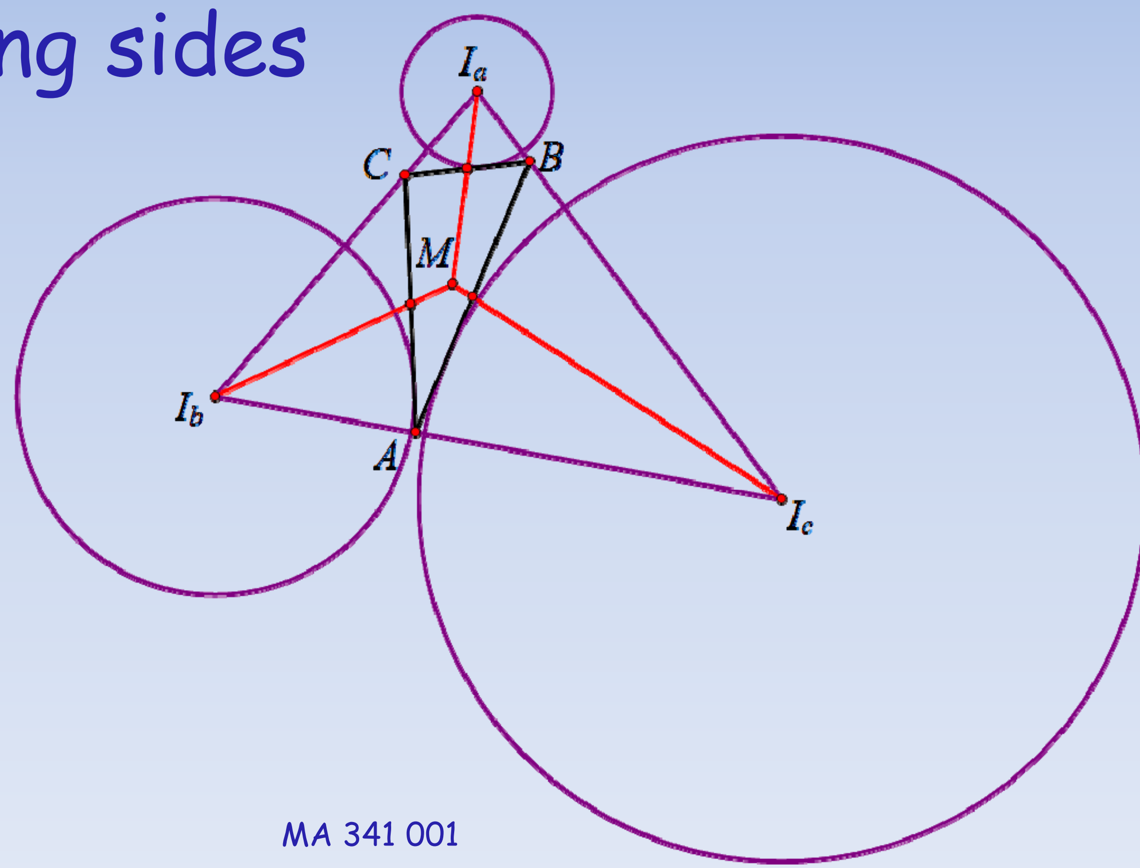
Mittenpunkt Point

The mittenpunkt of ΔABC is the symmedian point of the excentral triangle ($\Delta I_a I_b I_c$ formed from centers of excircles)



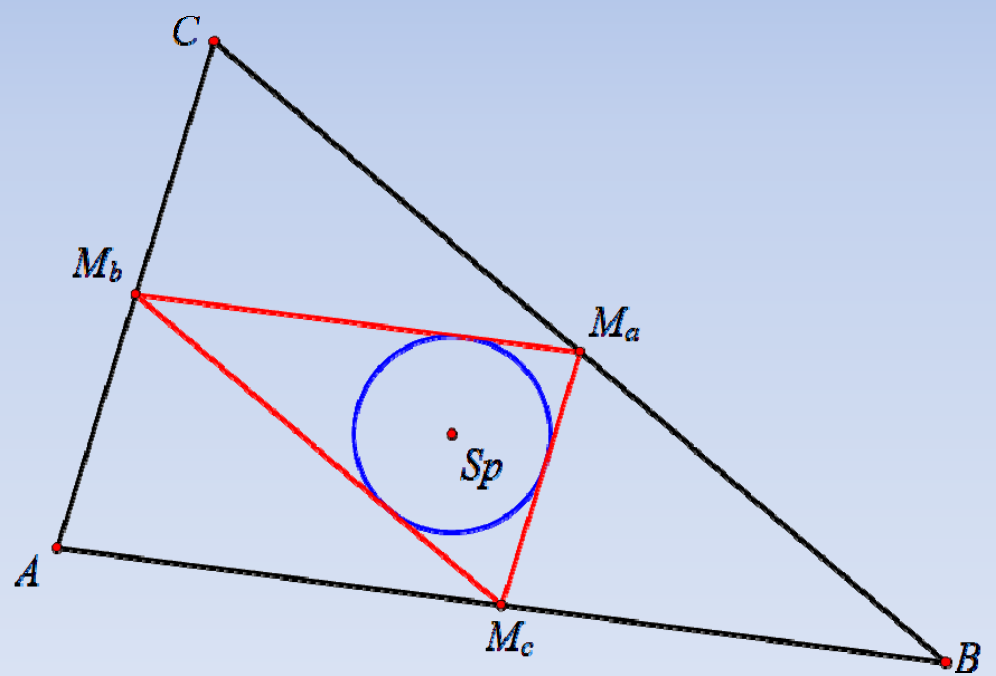
Mittenpunkt Point

The mittenpunkt of ΔABC is the point of intersection of the lines from the excenters through midpoints of corresponding sides



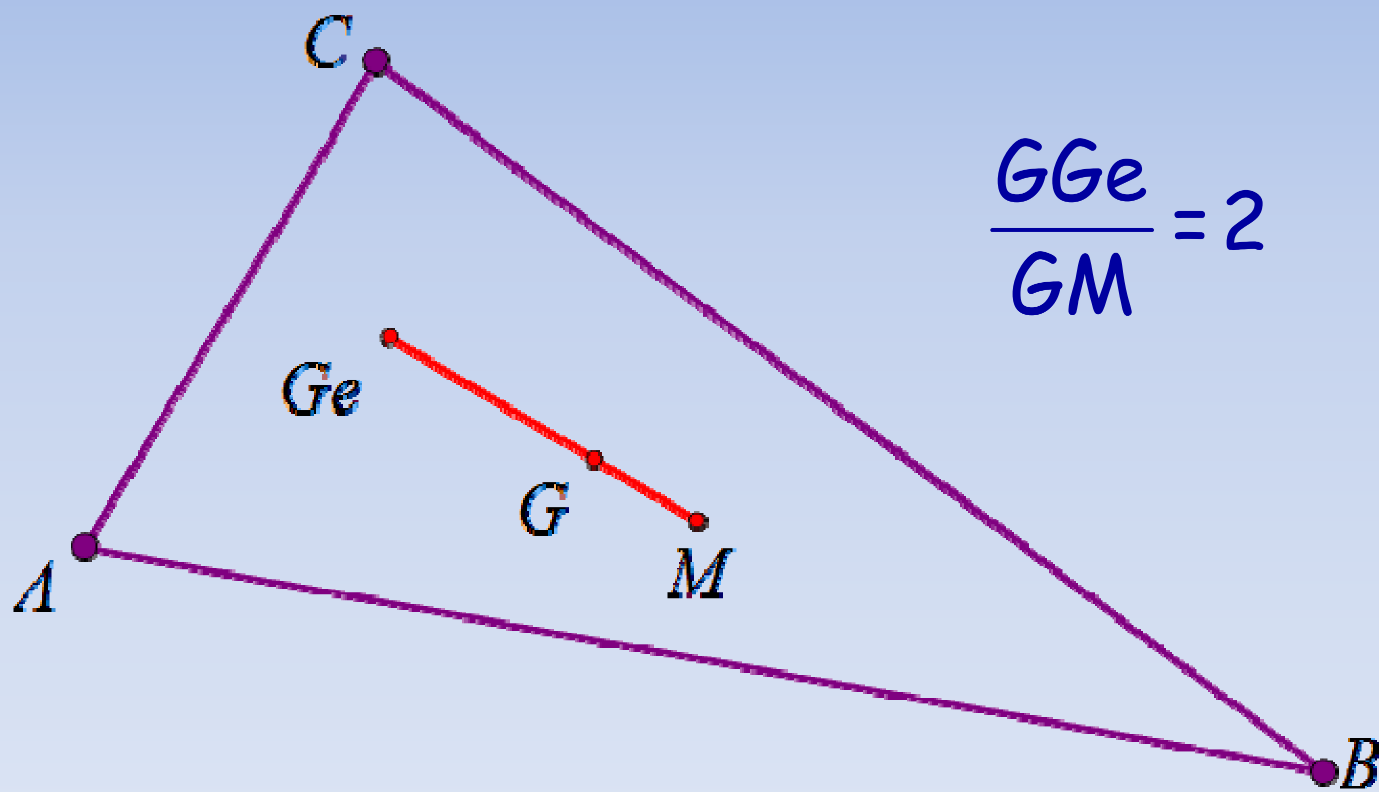
Spieker Point

The Spieker center is center of Spieker circle, i.e., the incenter of the medial triangle of the original triangle .



Special Segments

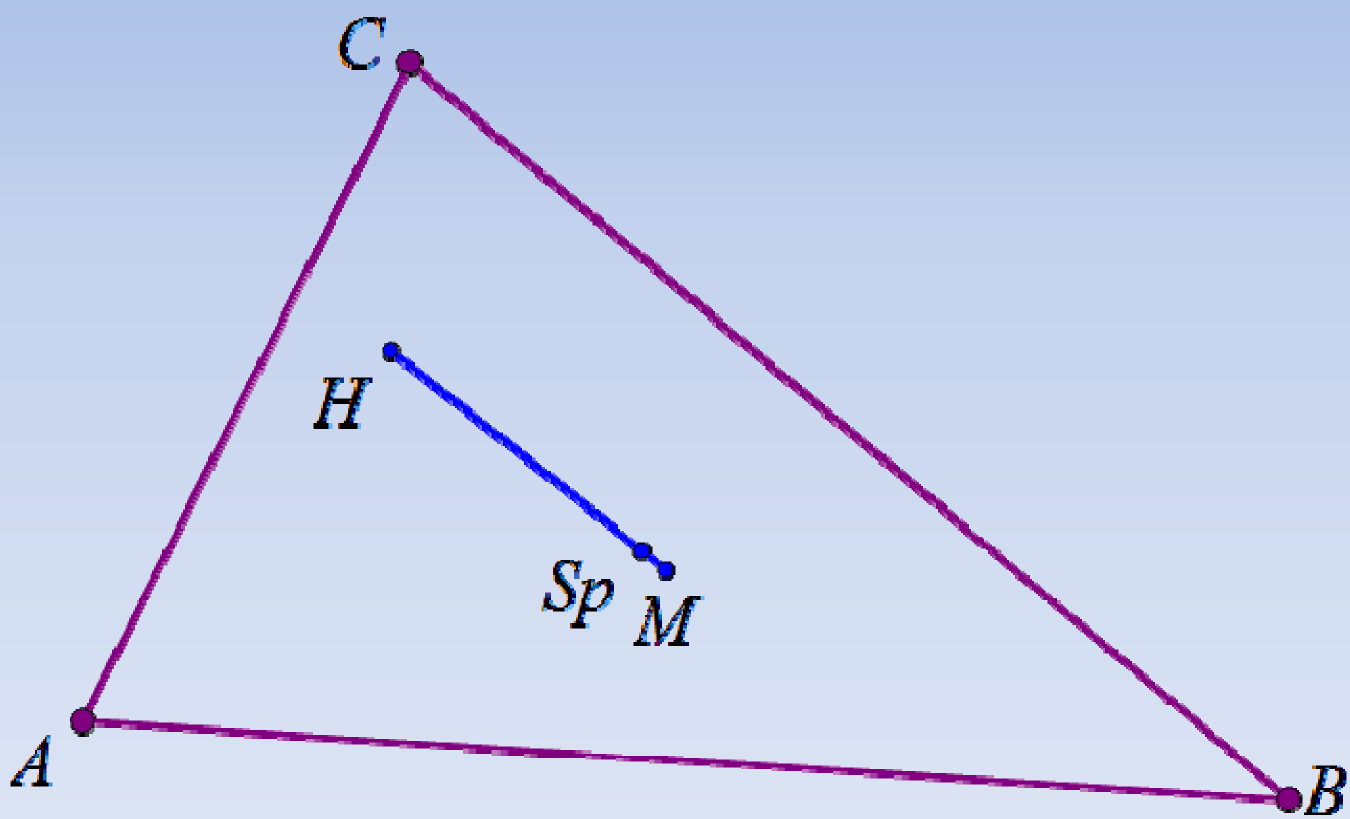
Gergonne point, centroid and mittenpunkt are collinear



$$\frac{GGe}{GM} = 2$$

Special Segments

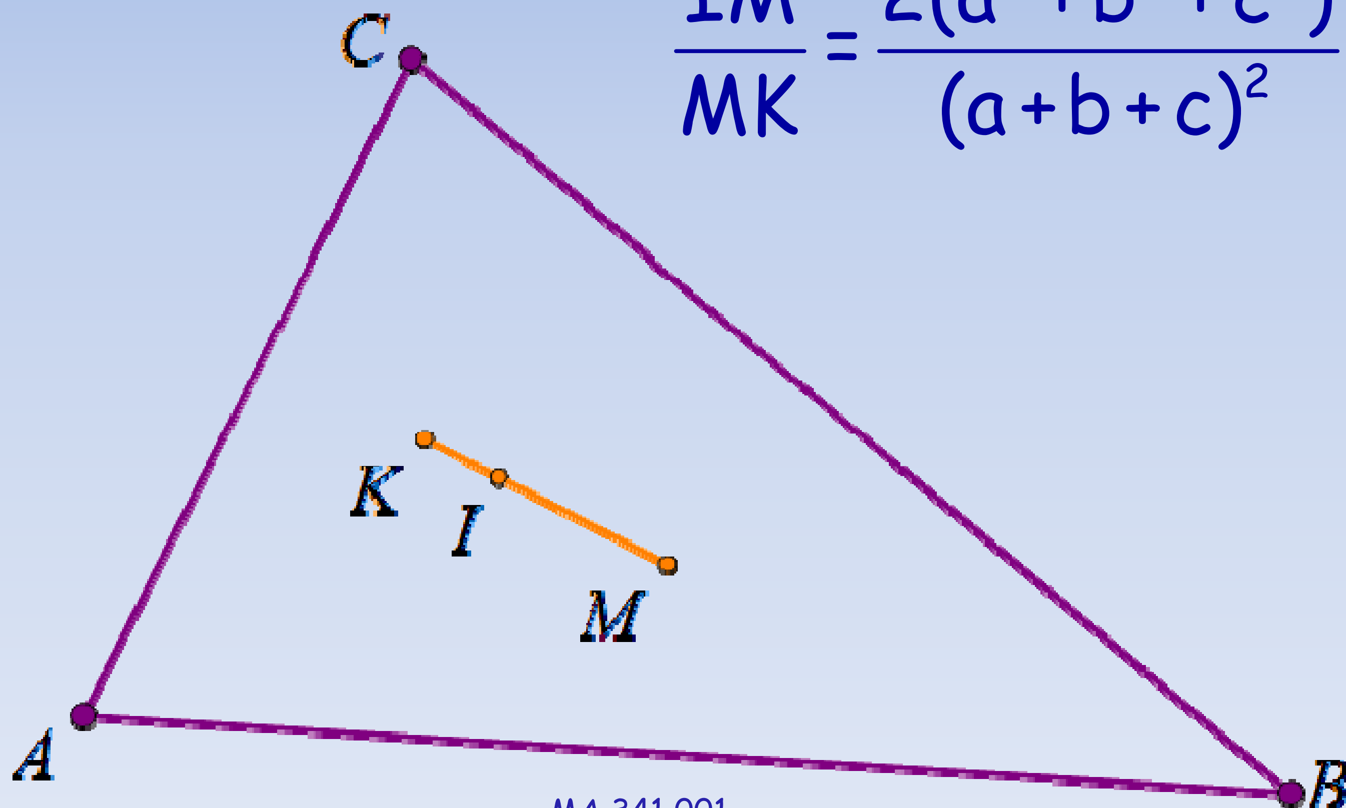
Mittenpunkt, Spieker center and orthocenter are collinear



Special Segments

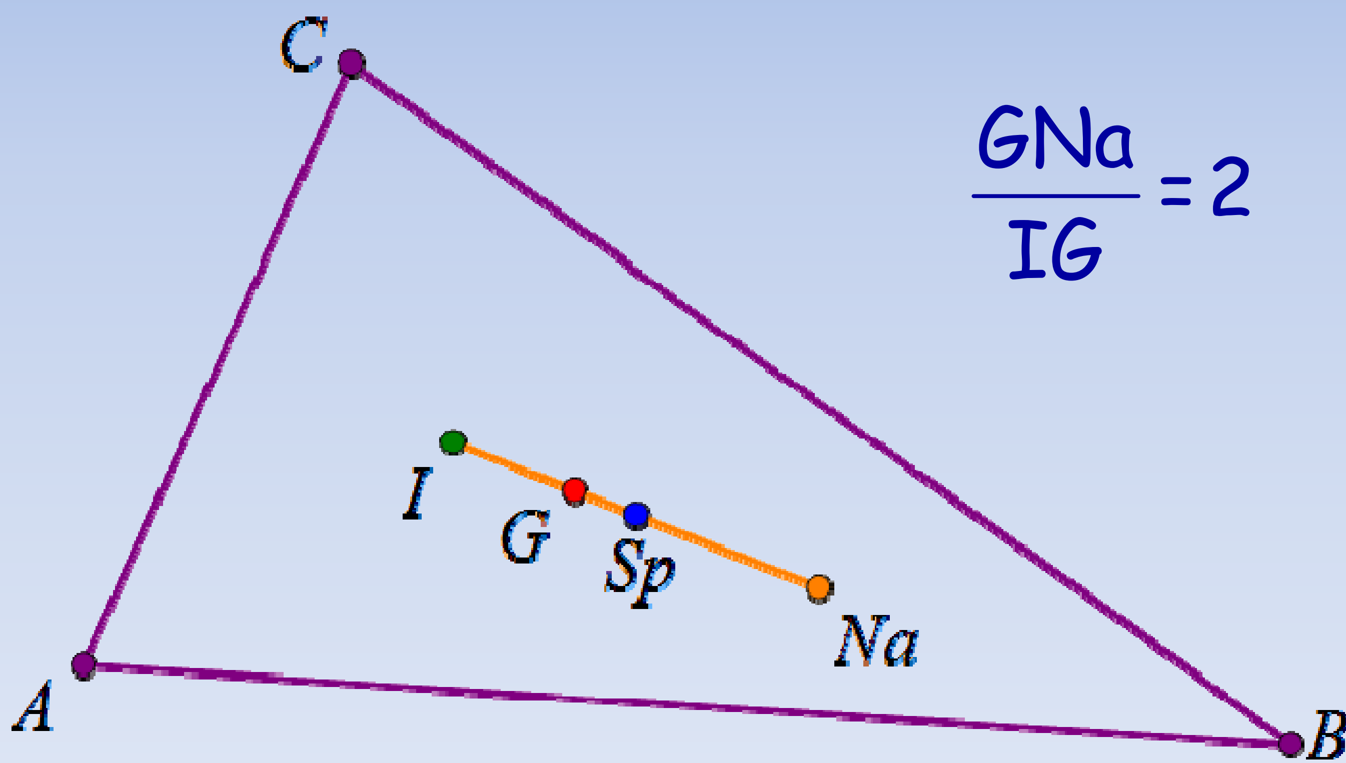
Mittenpunkt, incenter and symmedian point K are collinear with distance ratio

$$\frac{IM}{MK} = \frac{2(a^2+b^2+c^2)}{(a+b+c)^2}$$



Nagel Line

The Nagel line is the line on which the incenter , triangle centroid , Spieker center Sp, and Nagel point Na lie.



$$\frac{GNa}{IG} = 2$$

Various Centers