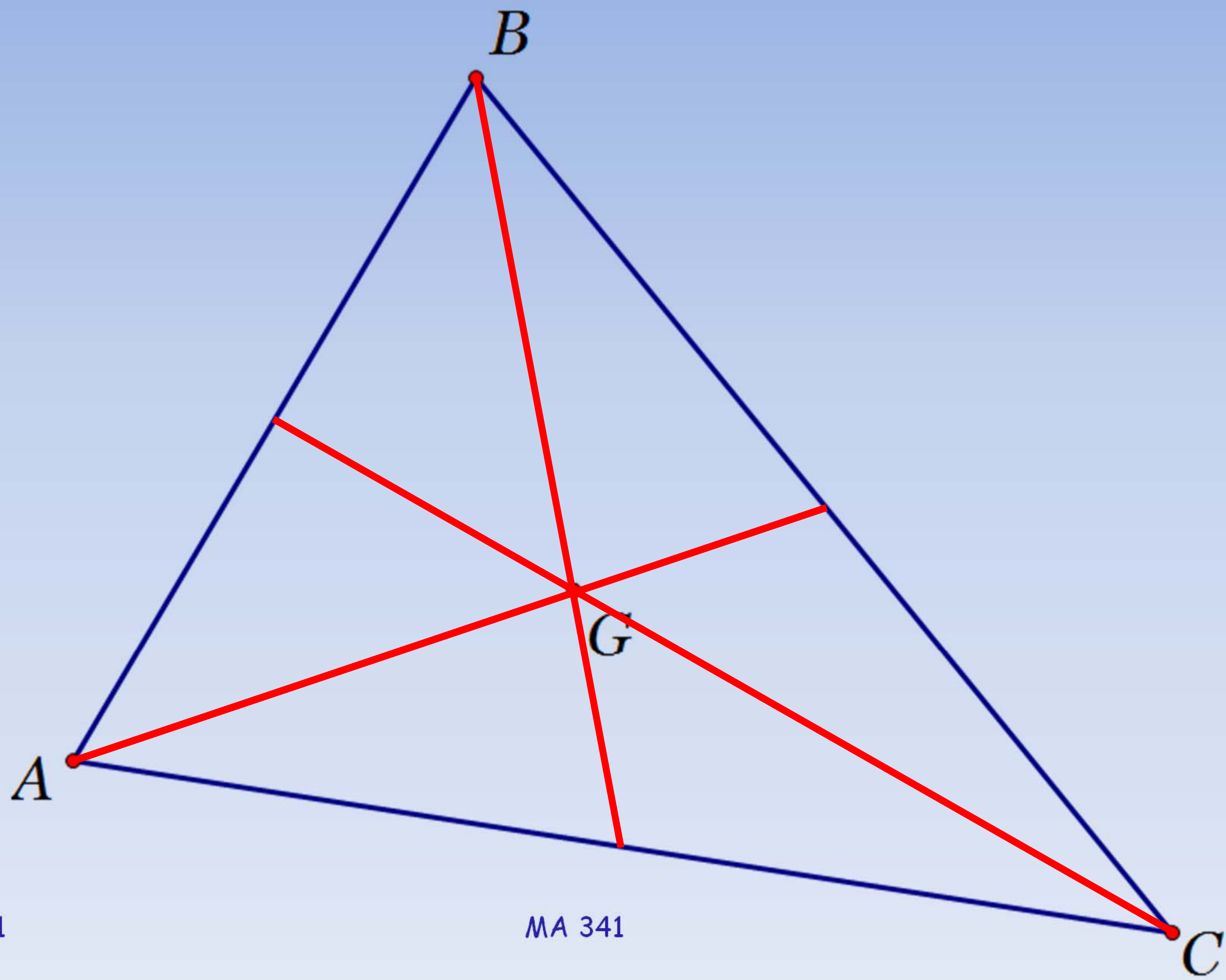
Nagel , Speiker, Napoleon, Torricelli

MA 341 – Topics in Geometry
Lecture 17

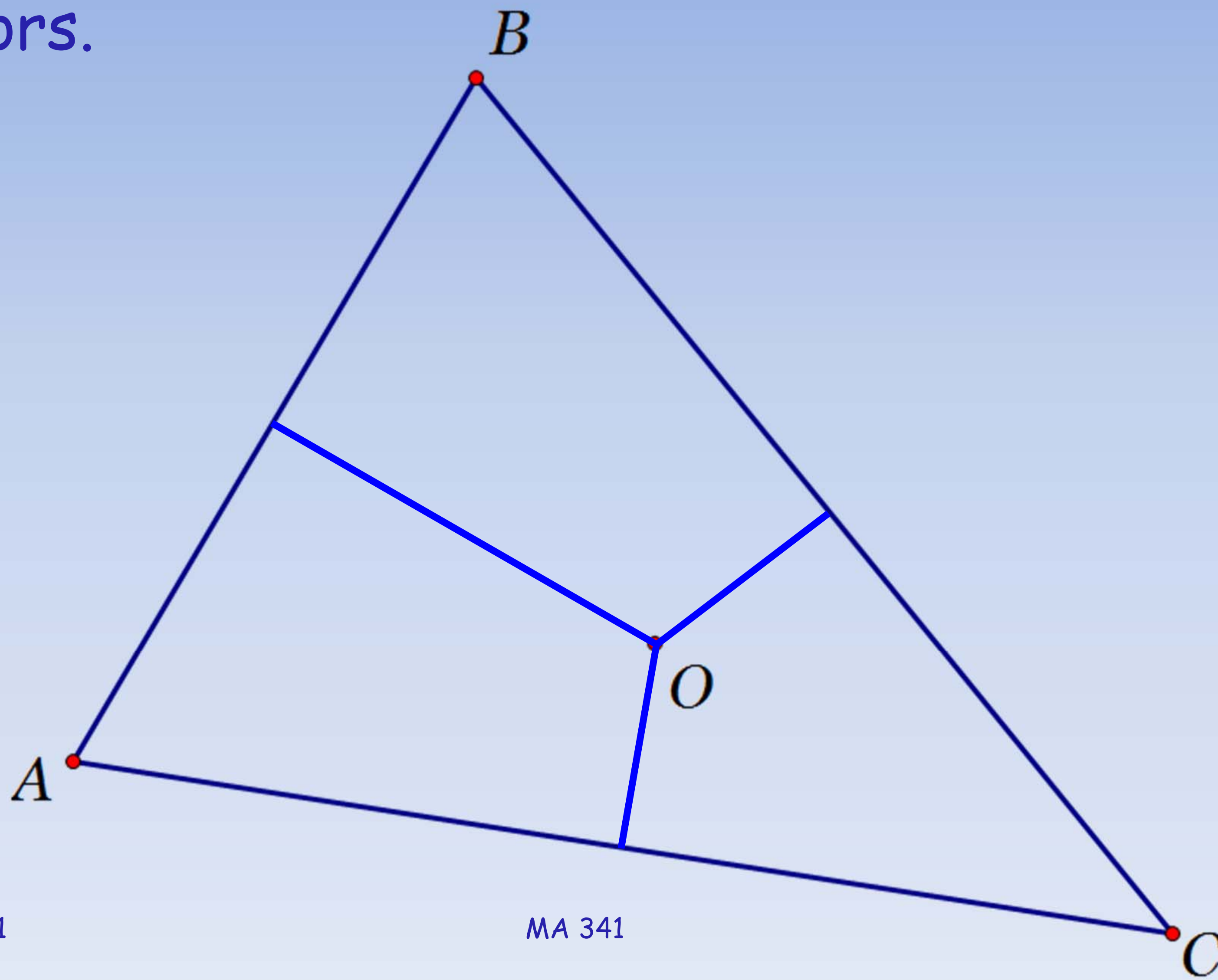
Centroid

The point of concurrency of the three medians.



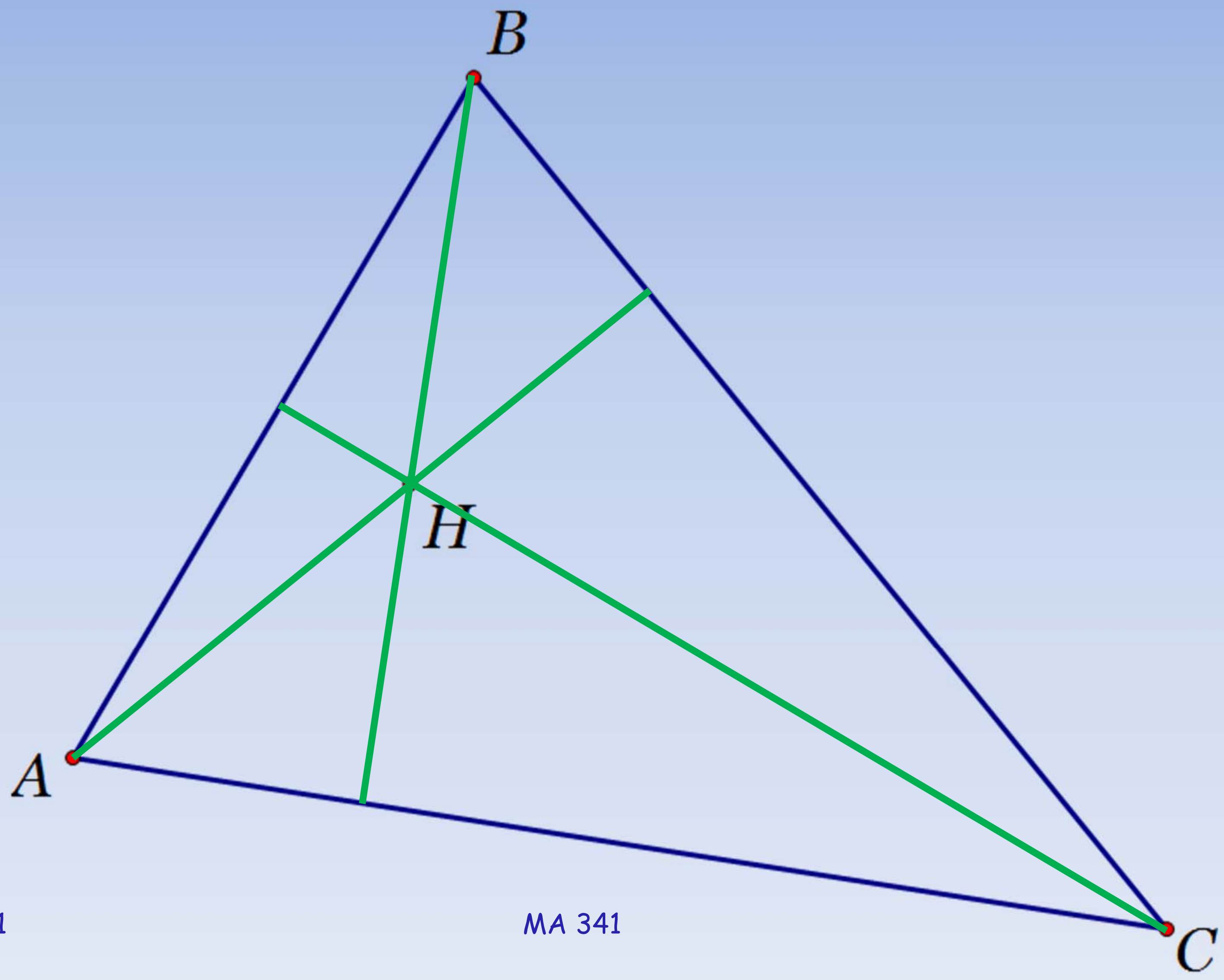
Circumcenter

Point of concurrency of the three perpendicular bisectors.



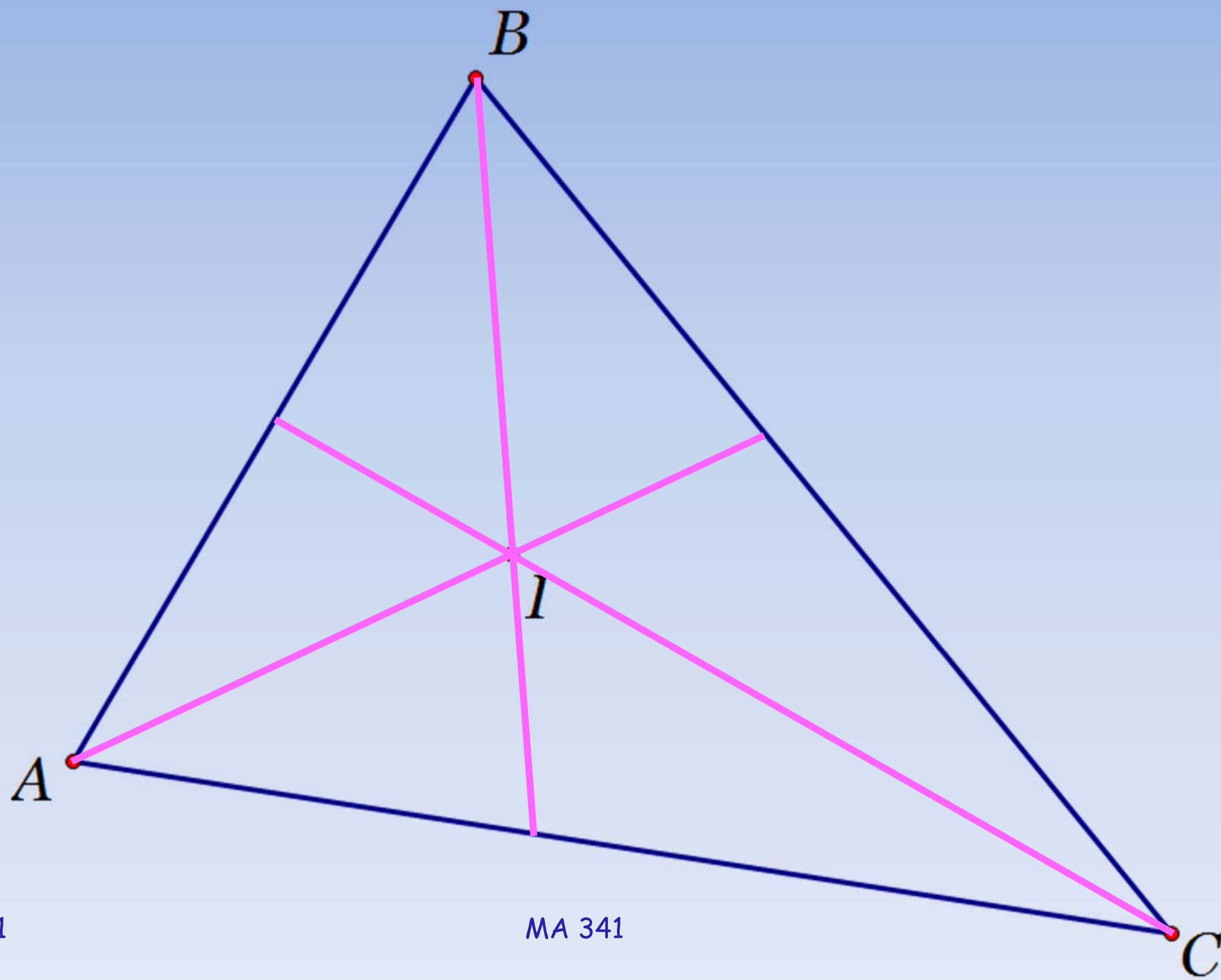
Orthocenter

Point of concurrency of the three altitudes.



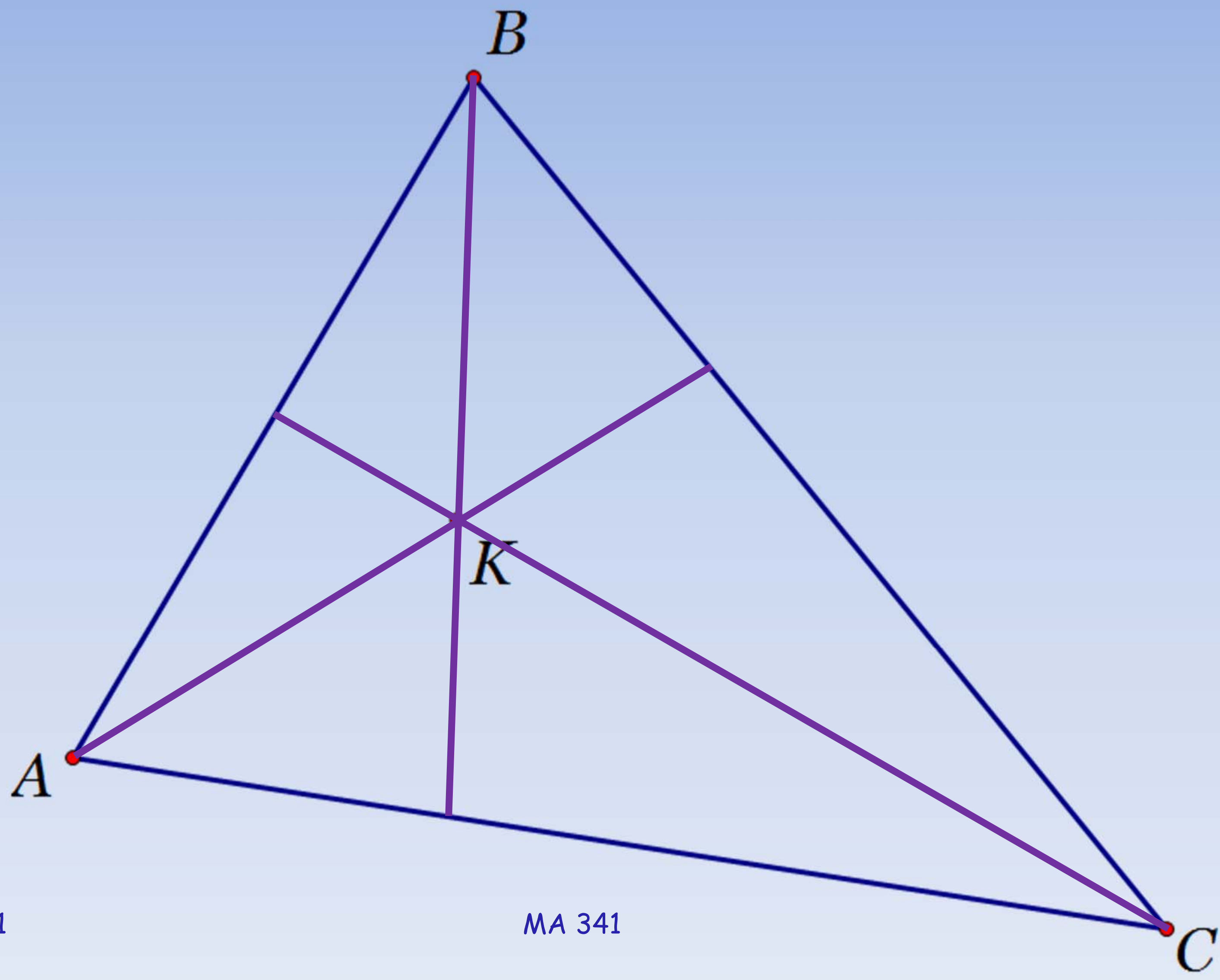
Incenter

Point of concurrency of the three angle bisectors.



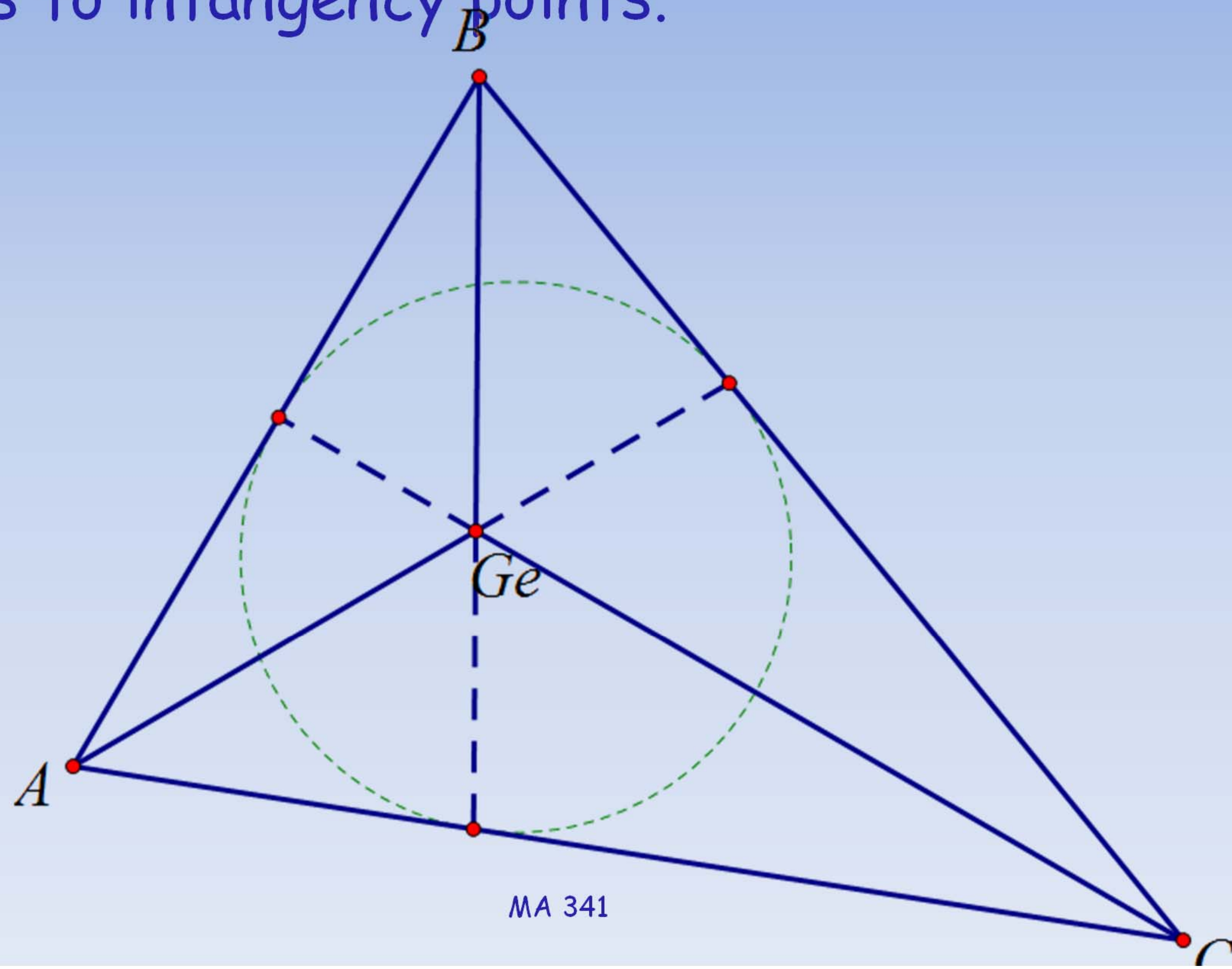
Symmedian Point

Point of concurrency of the three symmedians.



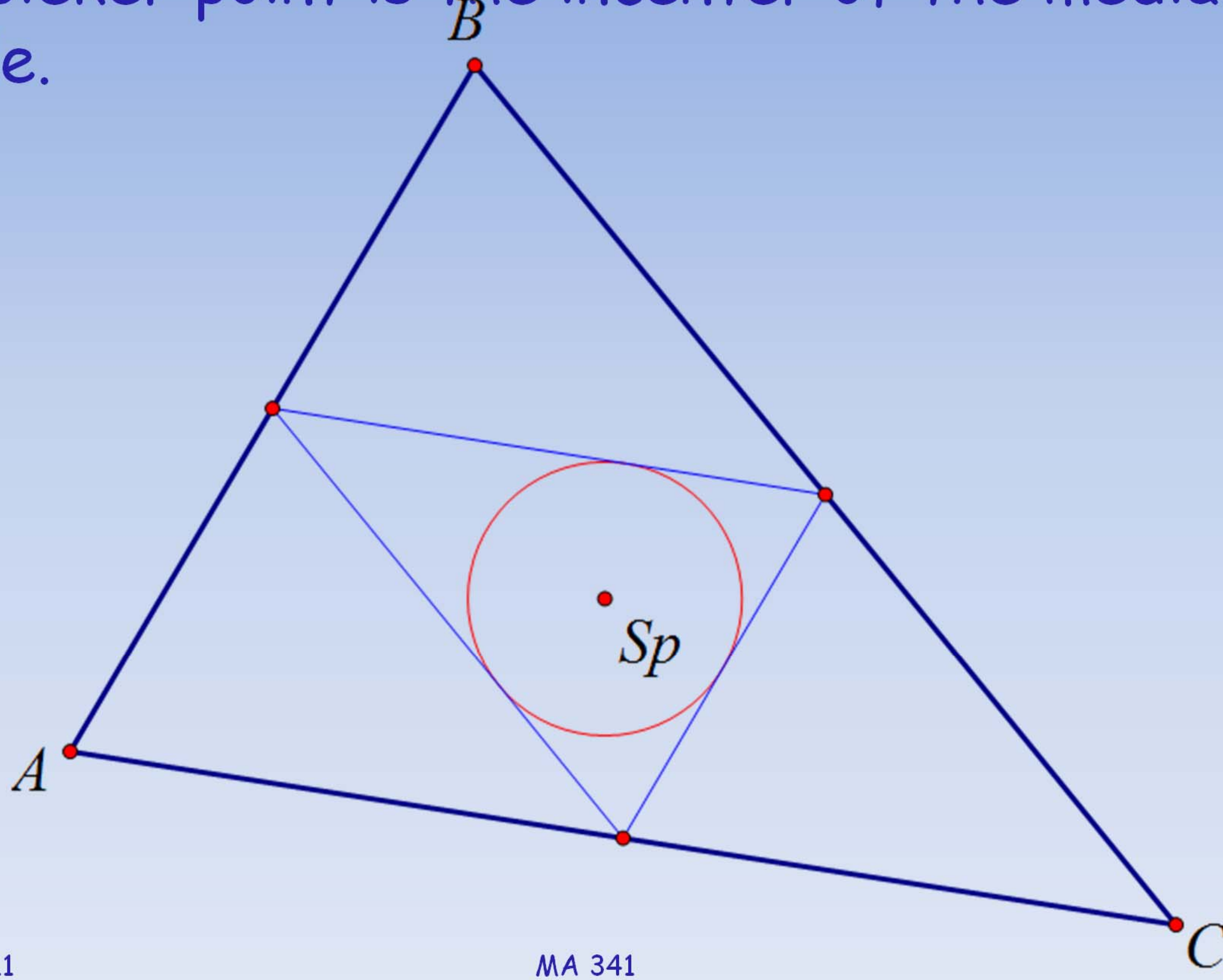
Gergonne Point

Point of concurrency of the three segments from vertices to intangency points.



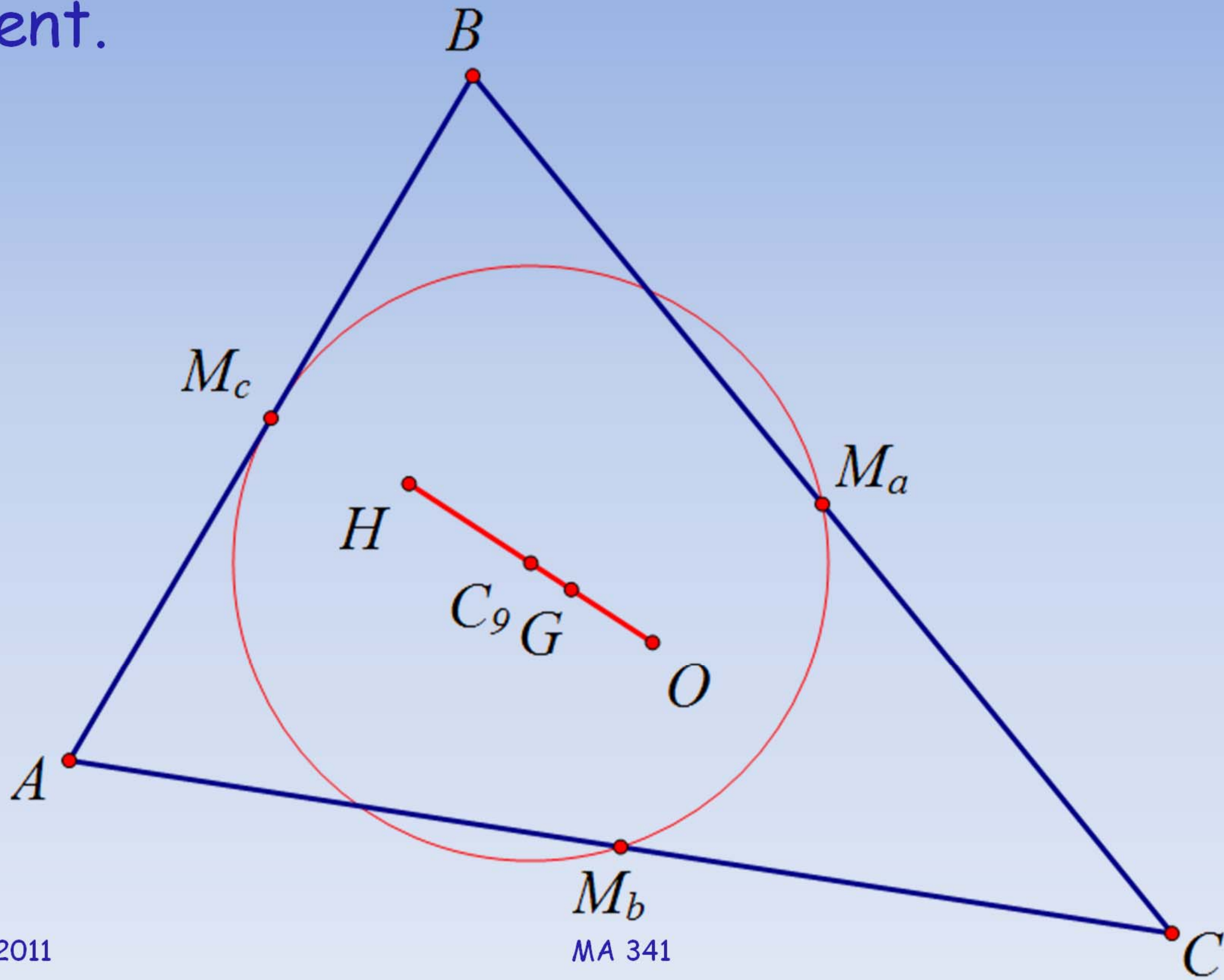
Spieker Point

The Spieker point is the incenter of the medial triangle.



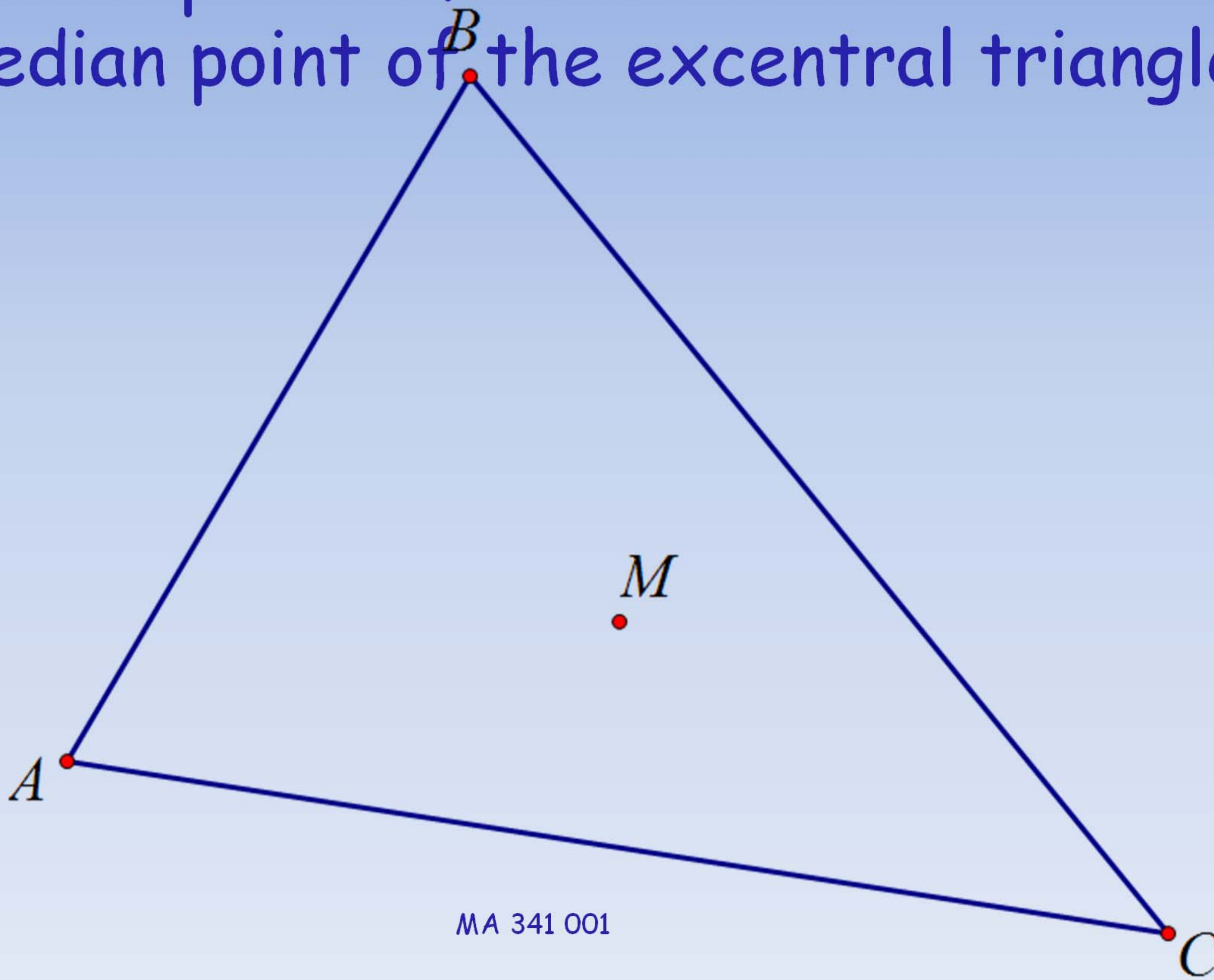
Nine Point Circle Center

The 9 point circle center is midpoint of the Euler segment.



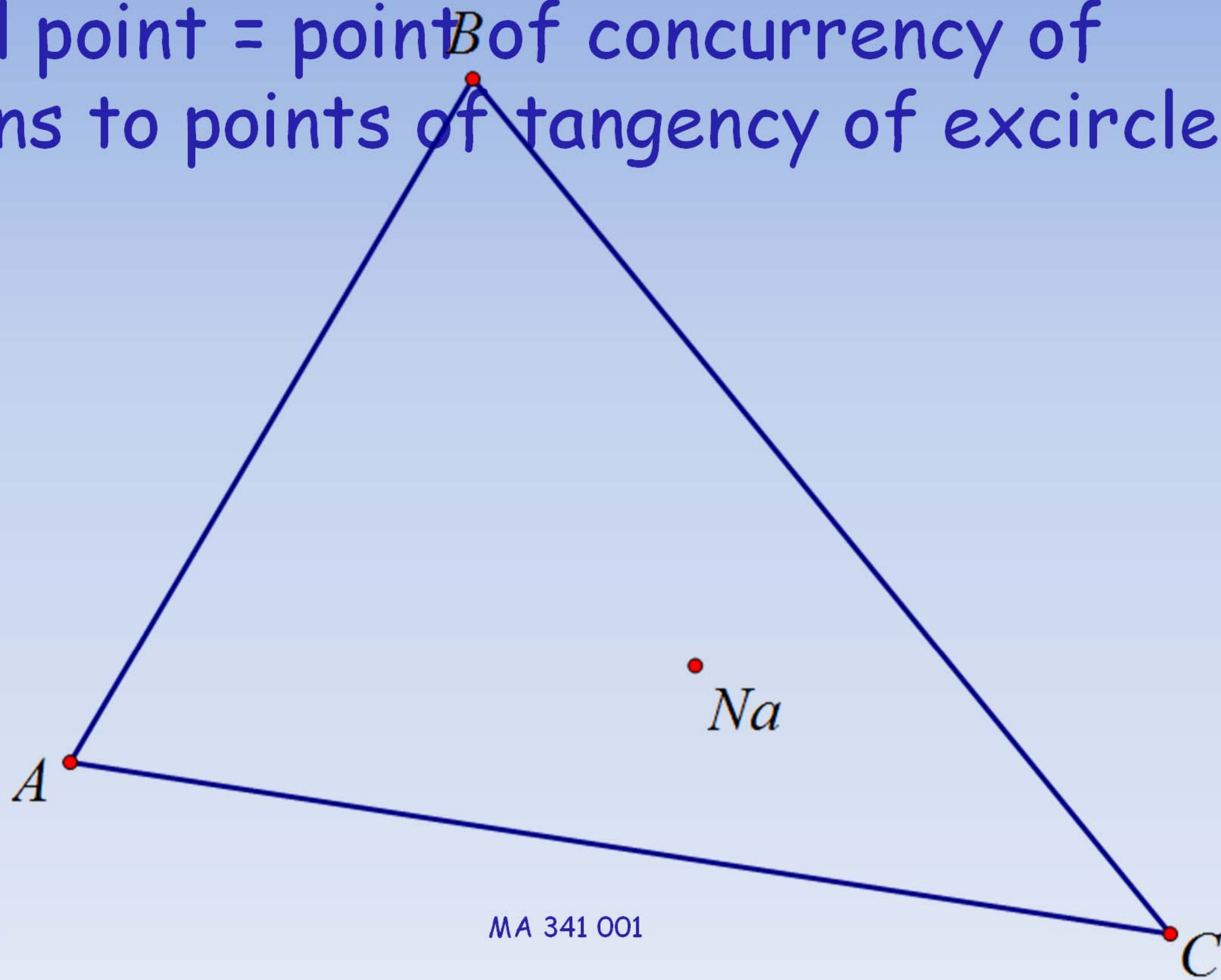
Mittenpunkt Point

The mittenpunkt of ΔABC is the symmedian point of the excentral triangle

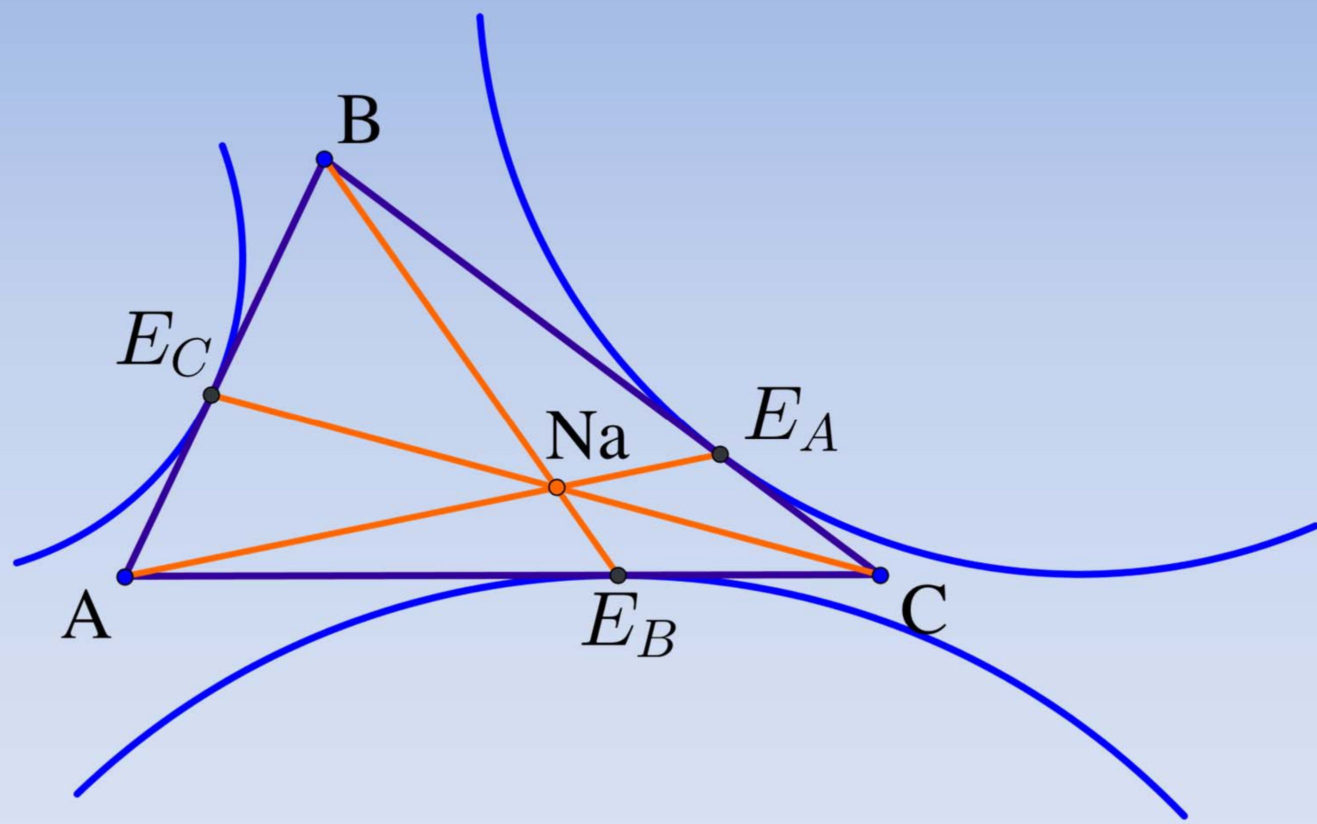


Nagel Point

Nagel point = point of concurrency of cevians to points of tangency of excircles

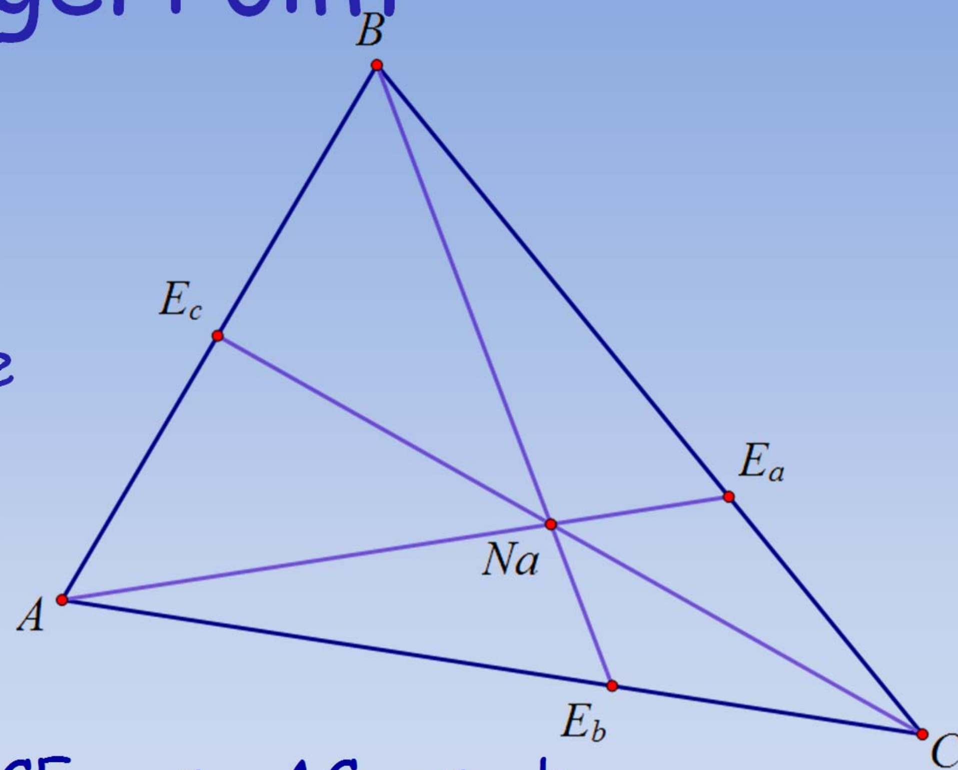


Nagel Point



The Nagel Point

E_a has the unique property of being the point on the perimeter that is exactly half way around the triangle from A.



$$AB+BE_a = AC+CE_a = s$$

Then

$$BE_a = s - AB = s - c \text{ and } CE_a = s - AC = s - b$$

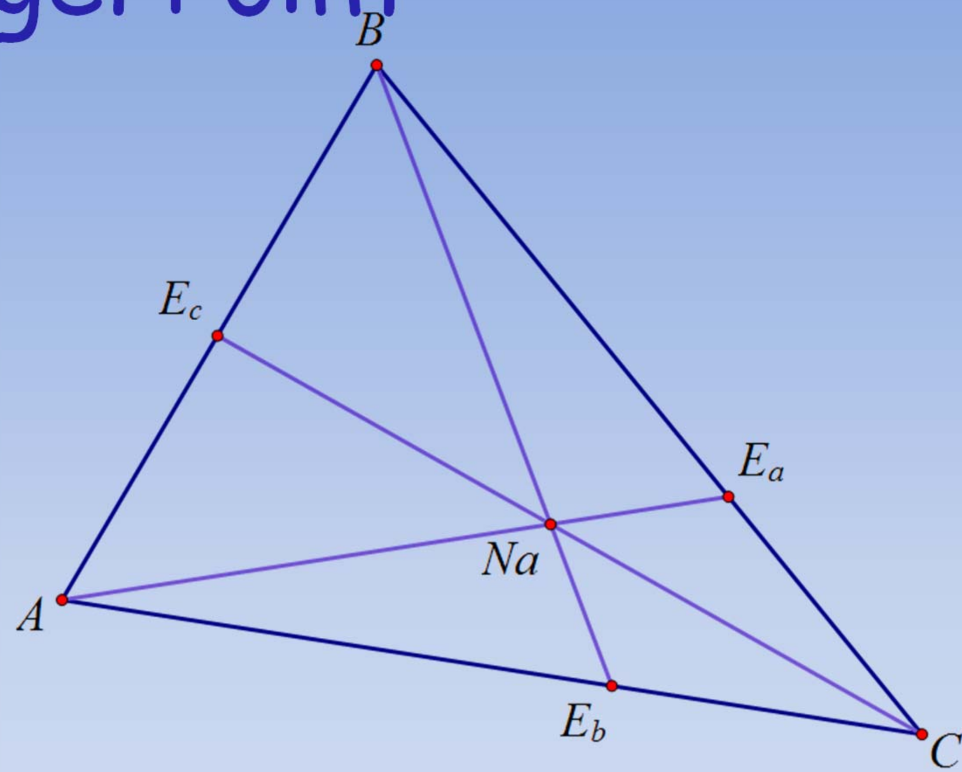
$$\frac{BE_a}{CE_a} = \frac{s-c}{s-b}$$

The Nagel Point

Likewise

$$\frac{CE_b}{AE_b} = \frac{s-a}{s-c}$$

$$\frac{AE_c}{BE_c} = \frac{s-b}{s-a}$$

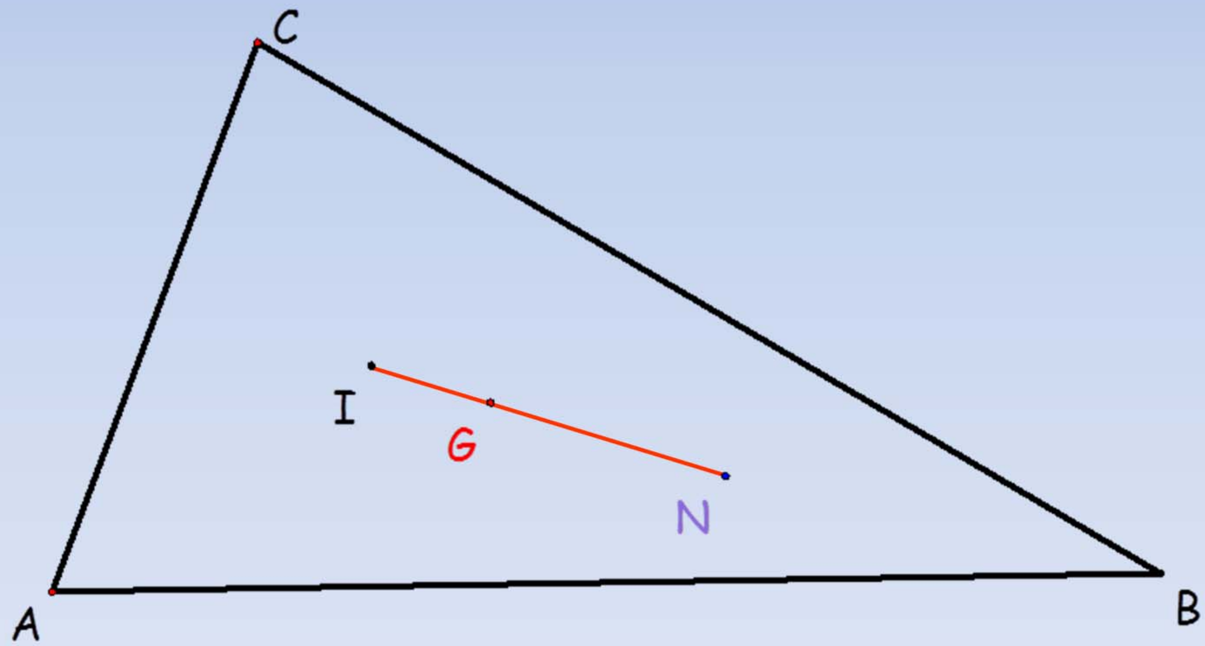


Apply Ceva's Theorem

$$\frac{AE_c}{BE_c}\frac{BE_a}{CE_a}\frac{CE_b}{AE_b} = \frac{s-b}{s-a} \times \frac{s-c}{s-b} \times \frac{s-a}{s-c} = 1$$

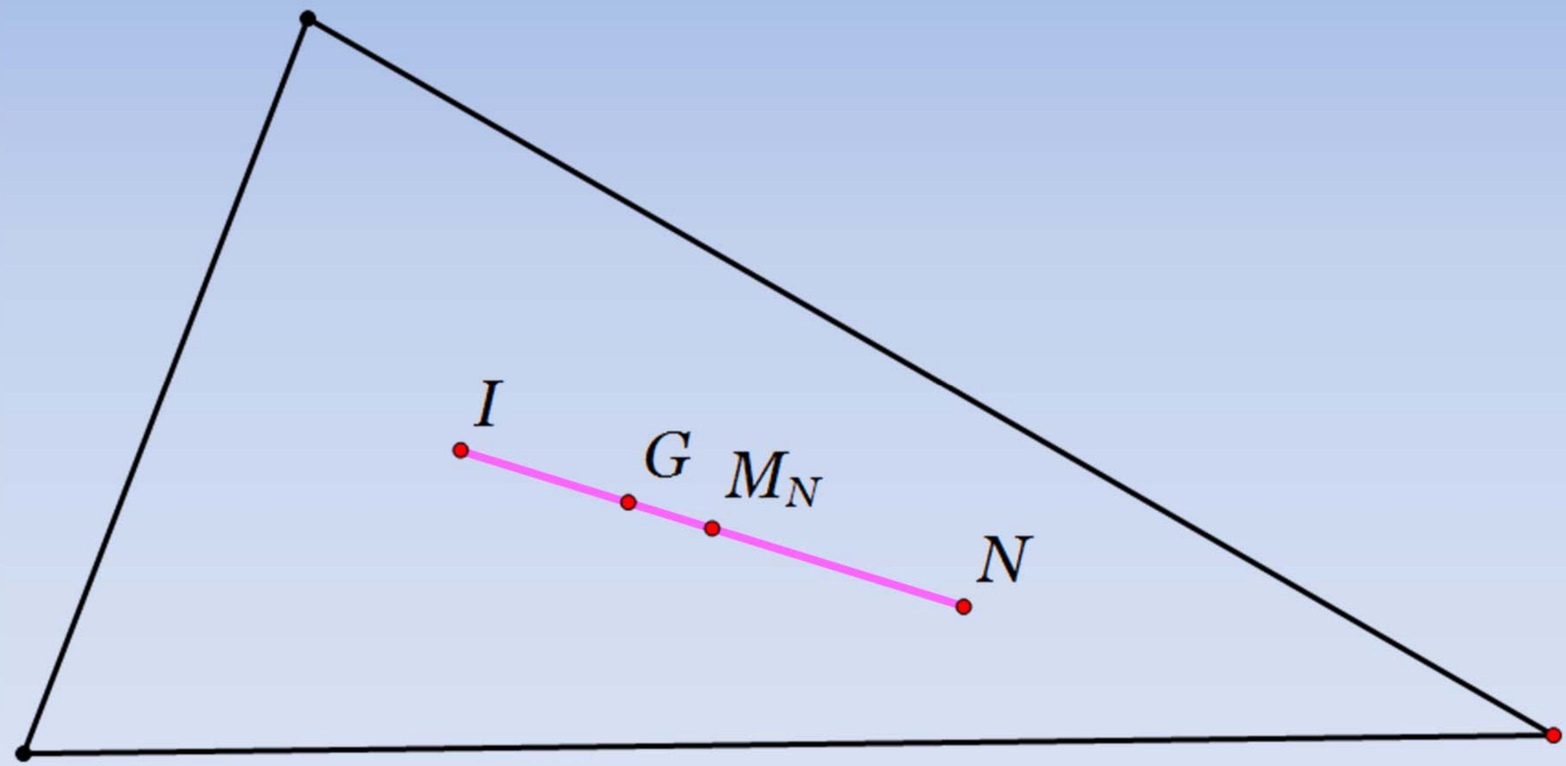
The Nagel Segment

1. In triangle ΔABC, let G, I, N be the centroid, incenter, and Nagel point, respectively. Then I, G, N lie on a line in that order.
2. The centroid is one-third of the way from the incenter to the Nagel point, NG = 2 IG.



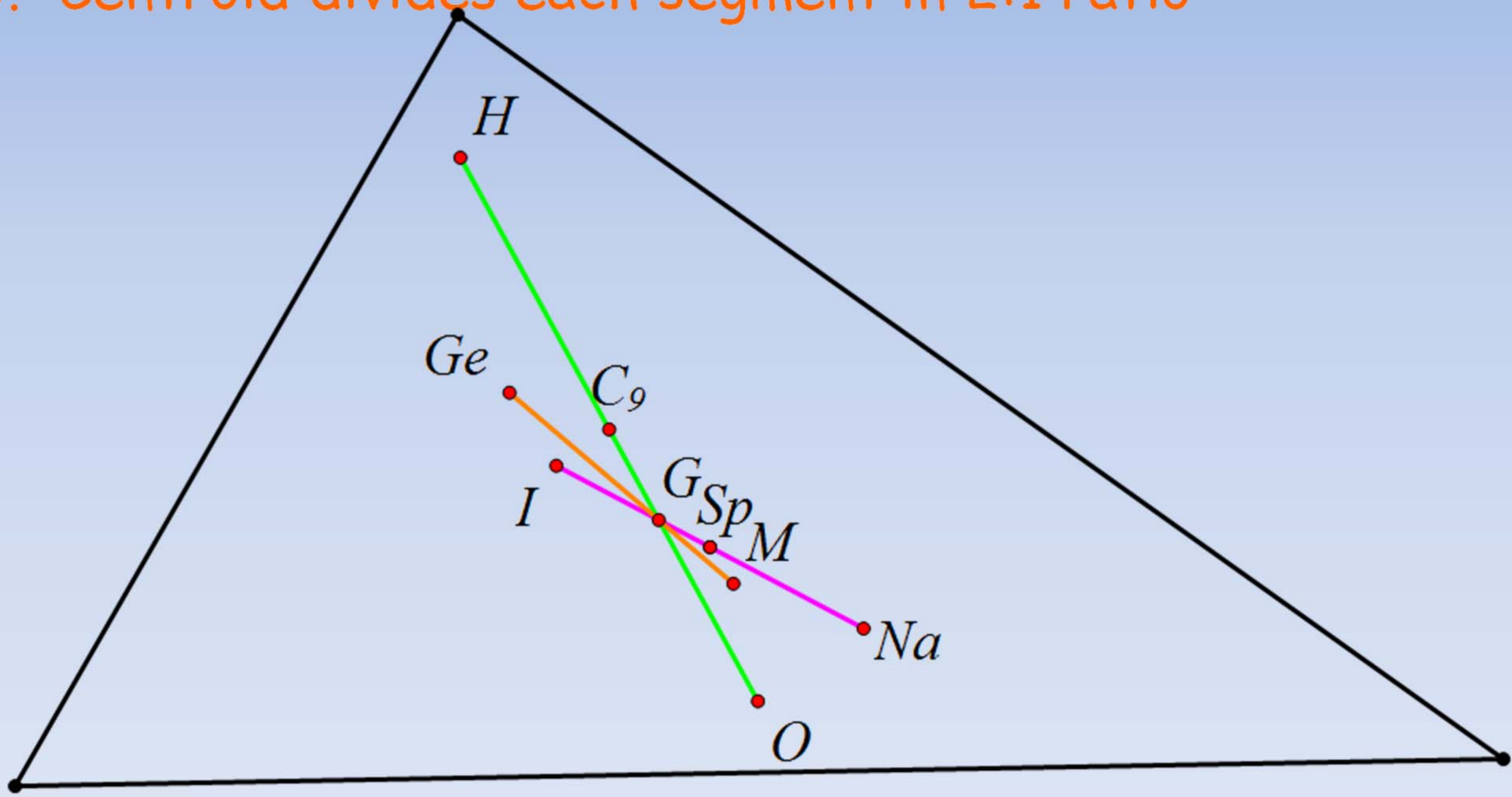
The Nagel Segment

1. Continuing the analogy if P, Q, R are the midpoints of sides BC,CA,AB, respectively, then the incenter of ΔPQR is the midpoint of IN.



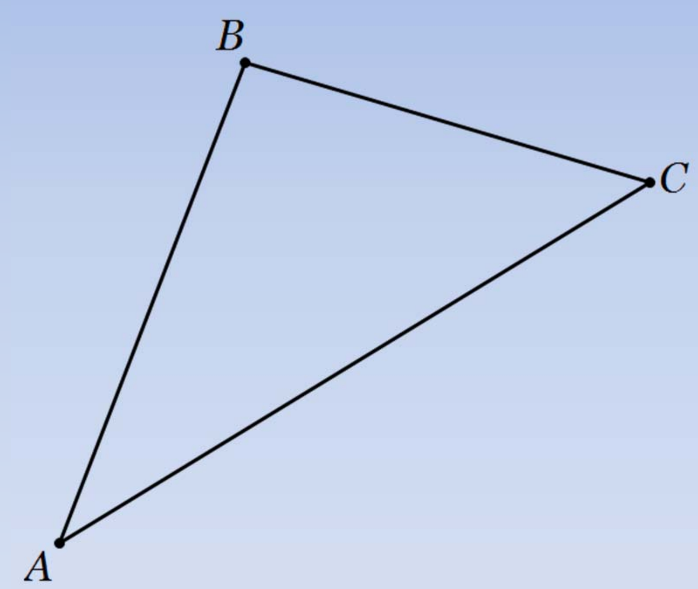
The Segments

1. The Euler Segment (midpoint = 9 pt circle center)
2. The Nagel Segment (midpoint = Spieker pt)
3. *Centroid divides each segment in 2:1 ratio*



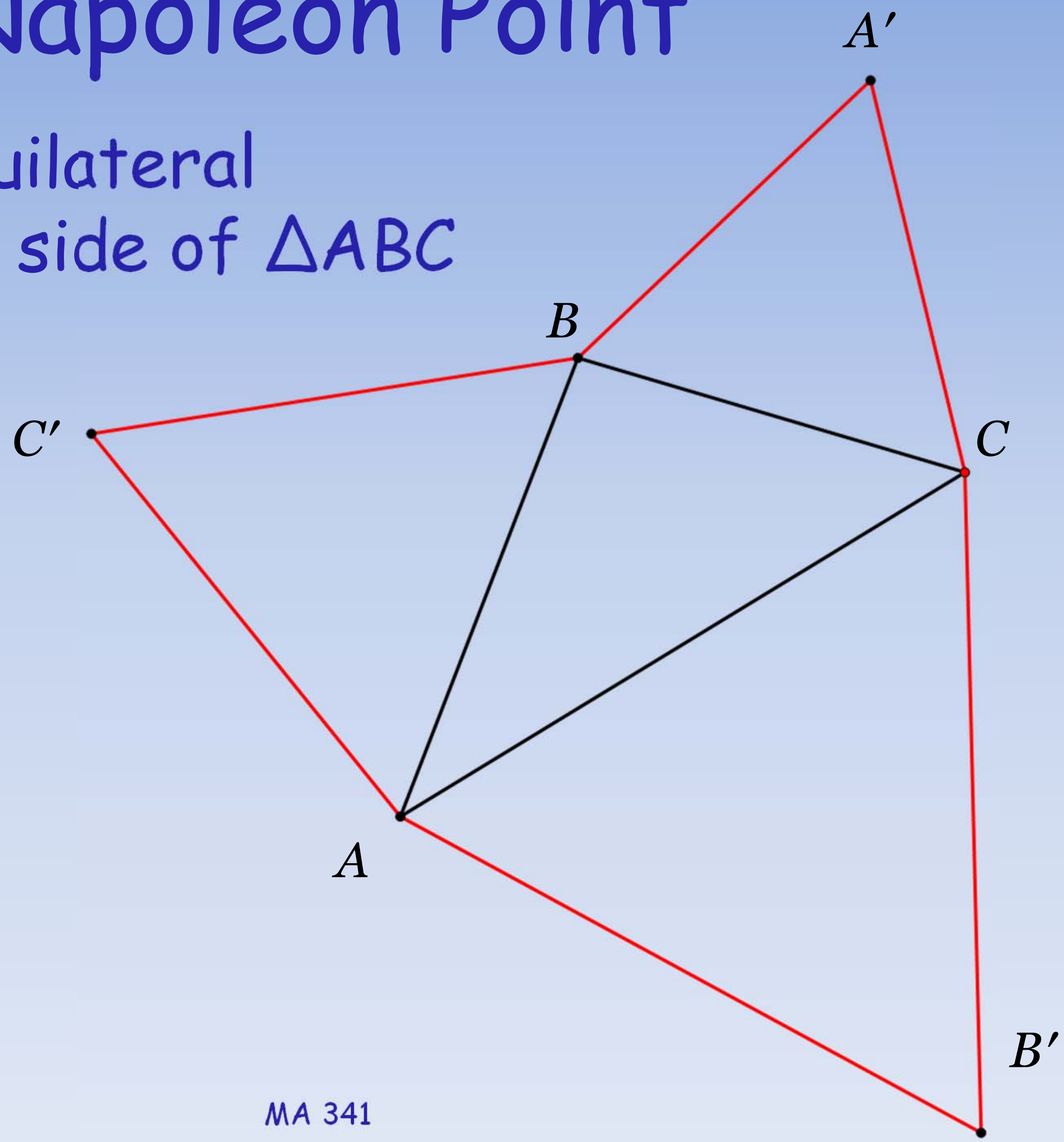
Napoleon Point

Construct a triangle ΔABC.



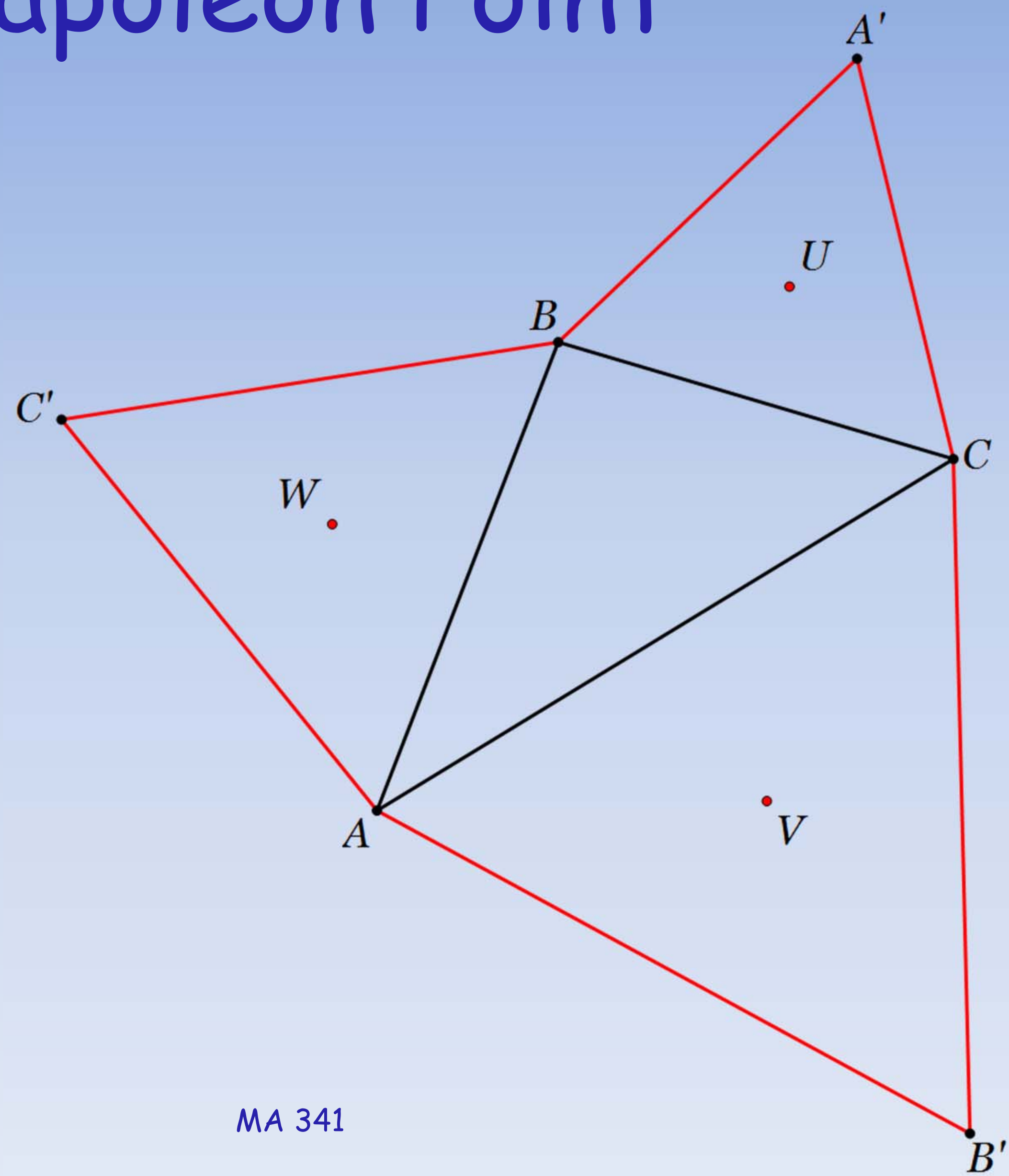
Napoleon Point

Construct an equilateral triangle on each side of ΔABC outside ΔABC .



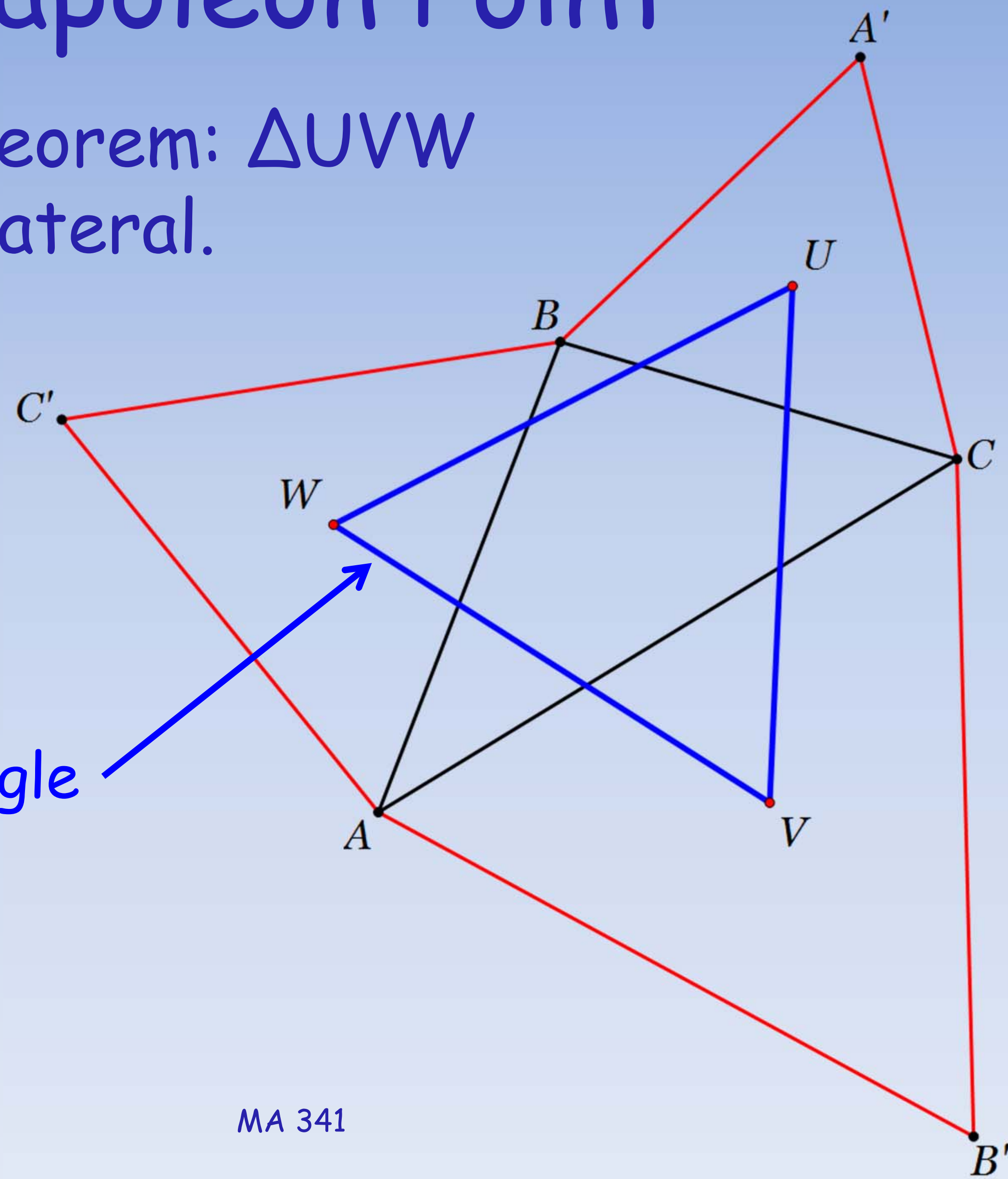
Napoleon Point

Construct the three centroids of triangles $\Delta A'BC$, $\Delta AB'C$, and $\Delta ABC'$ and label them U, V, and W, respectively.



Napoleon Point

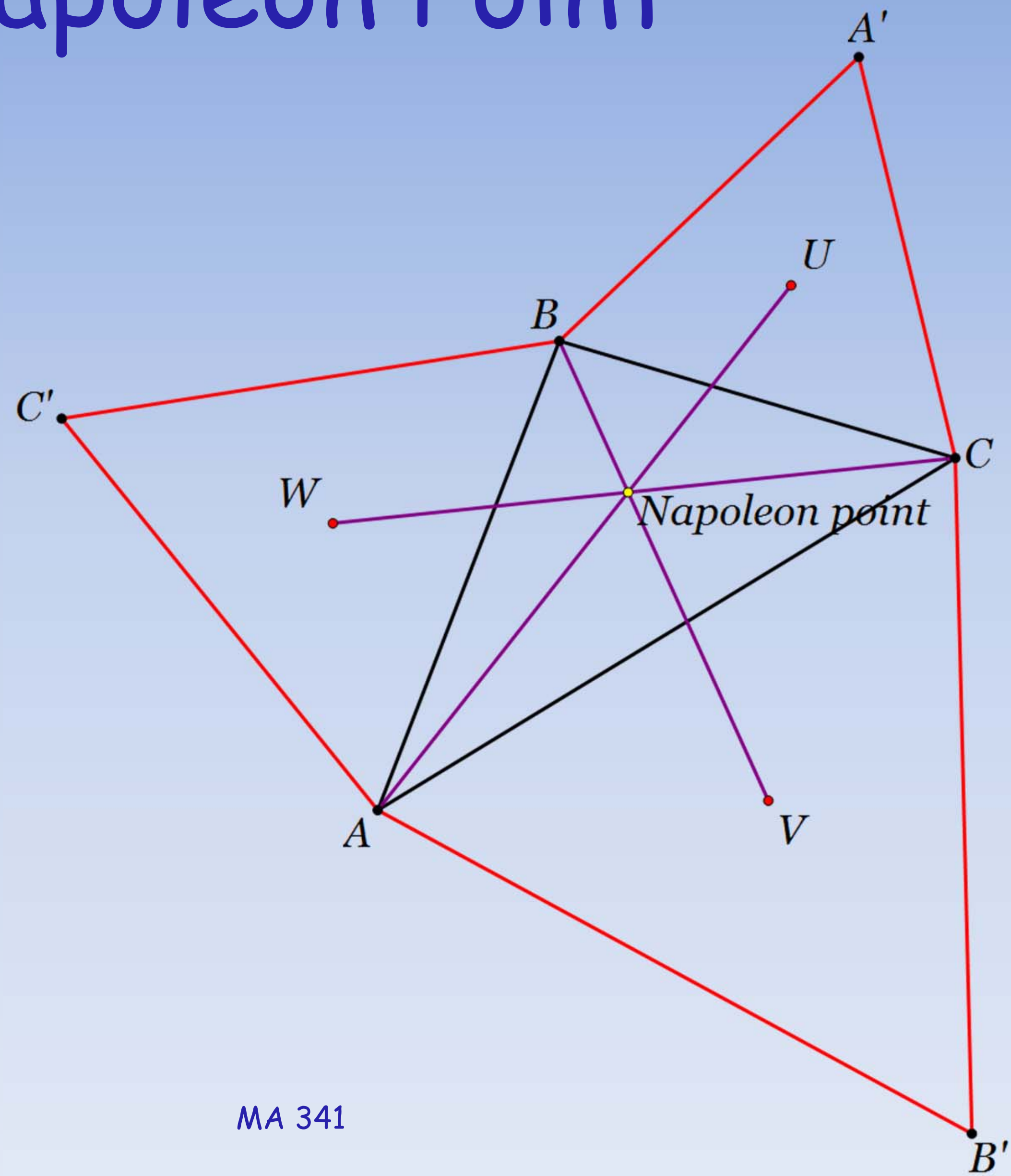
1. Napoleon's Theorem: ΔUVW is always equilateral.



Napoleon triangle

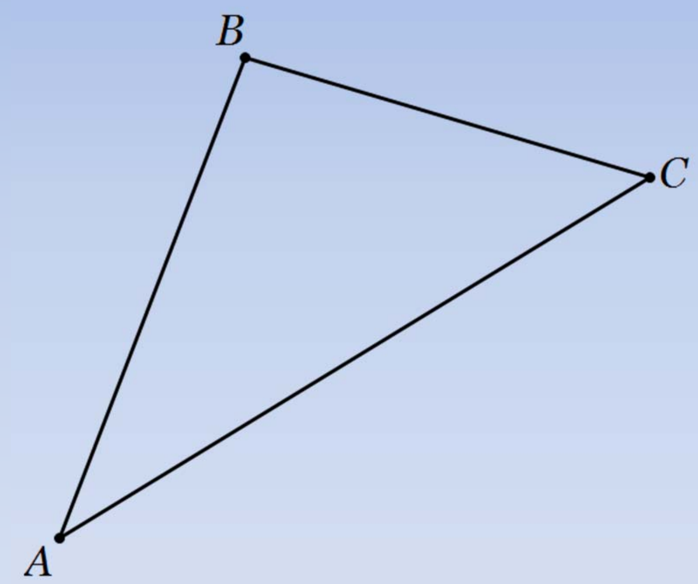
Napoleon Point

2. AU, BV, and CW are concurrent.



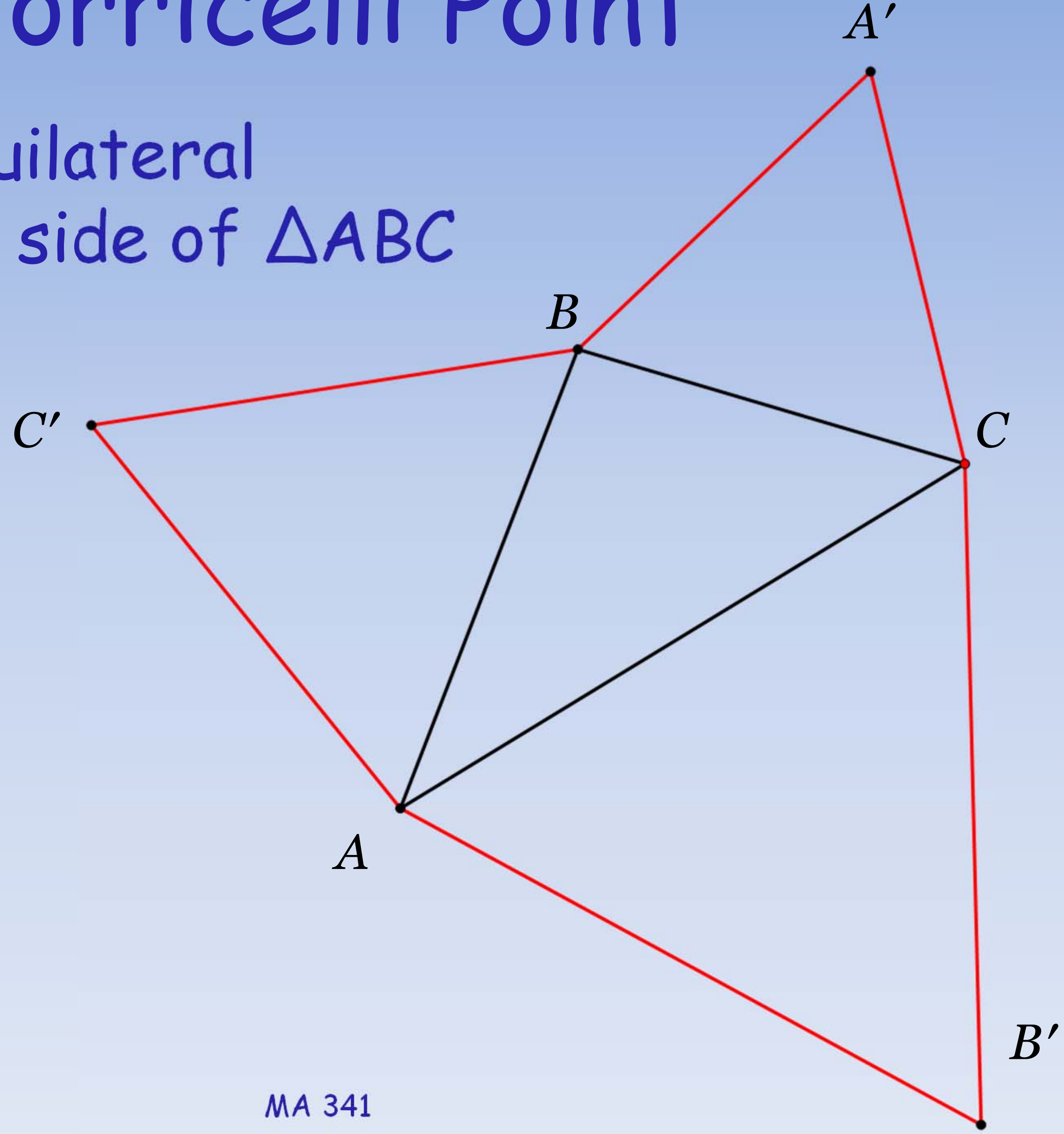
Torricelli Point aka Fermat Point

Construct a triangle ΔABC.



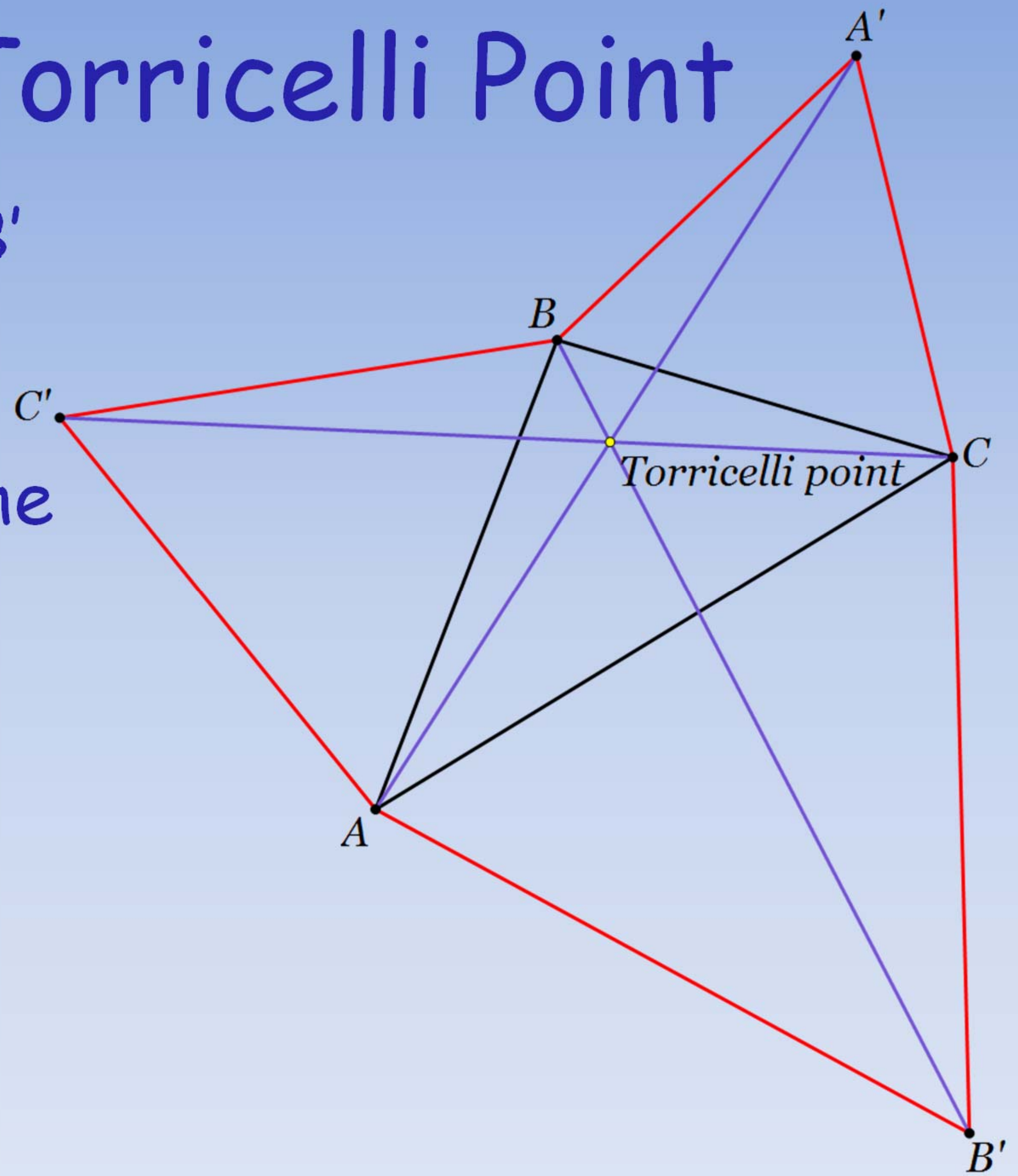
Torricelli Point

Construct an equilateral triangle on each side of ΔABC outside ΔABC .



Torricelli Point

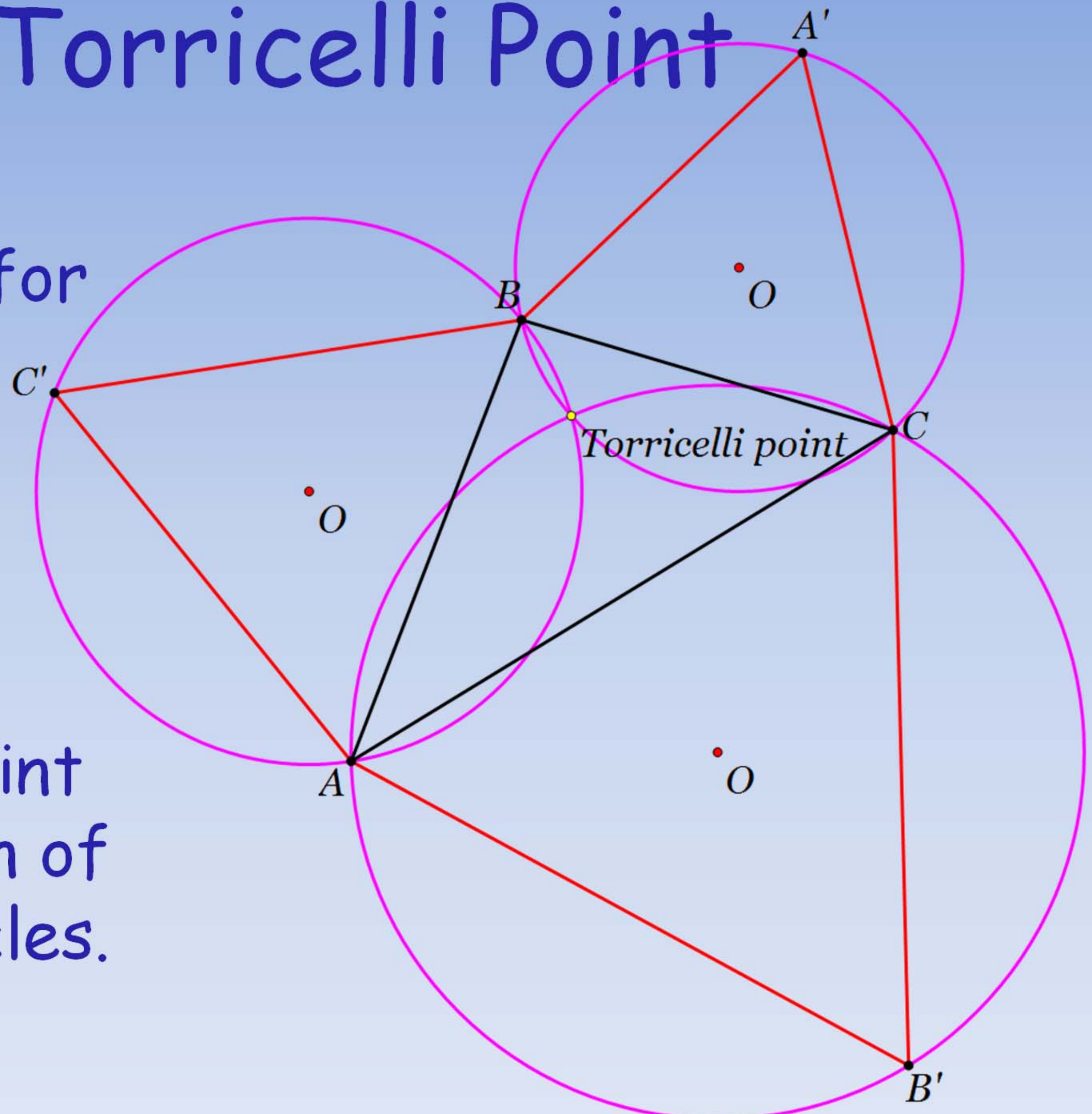
Connect AA′, BB′ and CC′. These lines are concurrent in the Torricelli point.



Torricelli Point

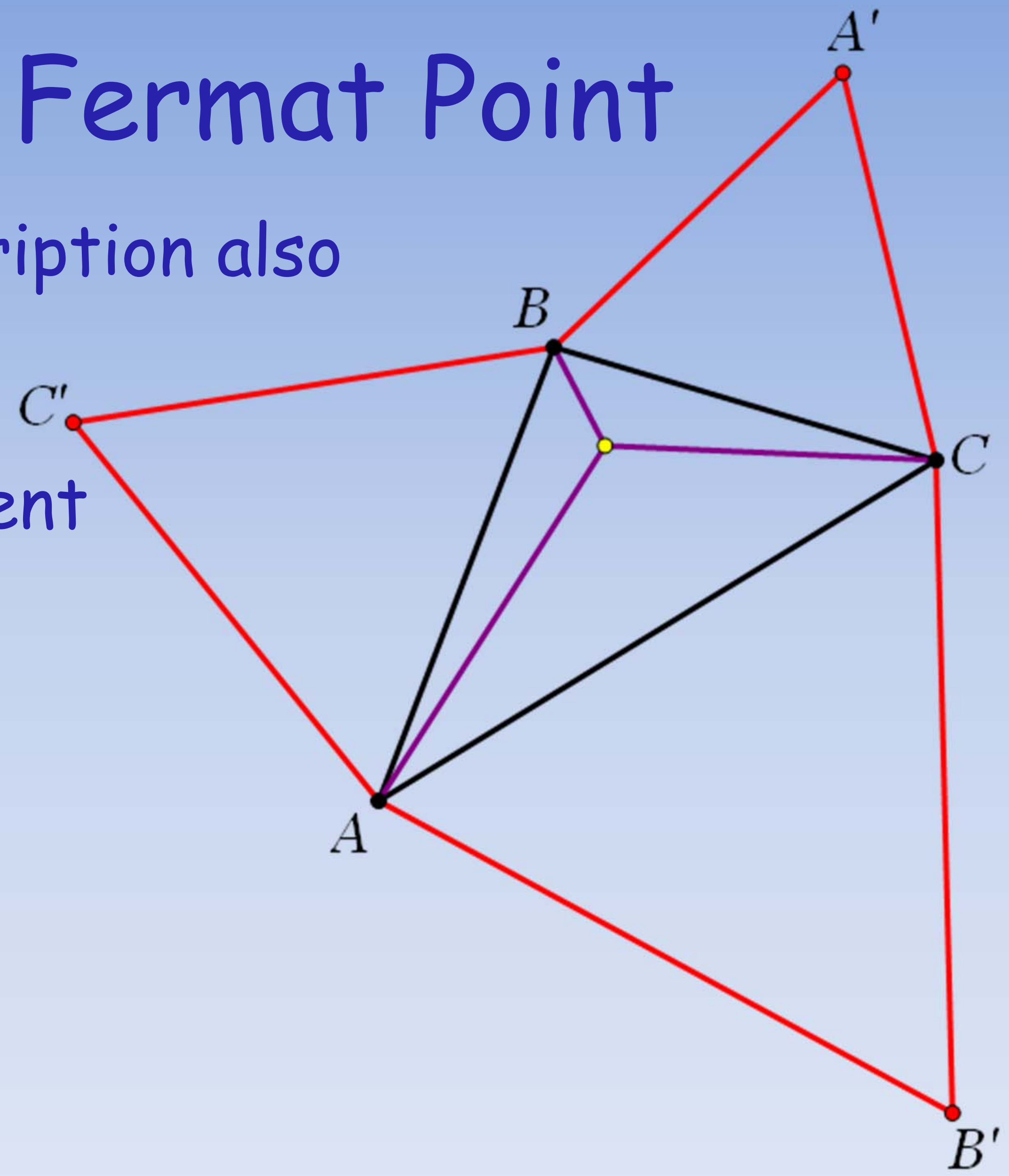
Construct the circumcircles for each of the equilateral triangles.

The Torricelli point is the point of intersection of the circumcircles.

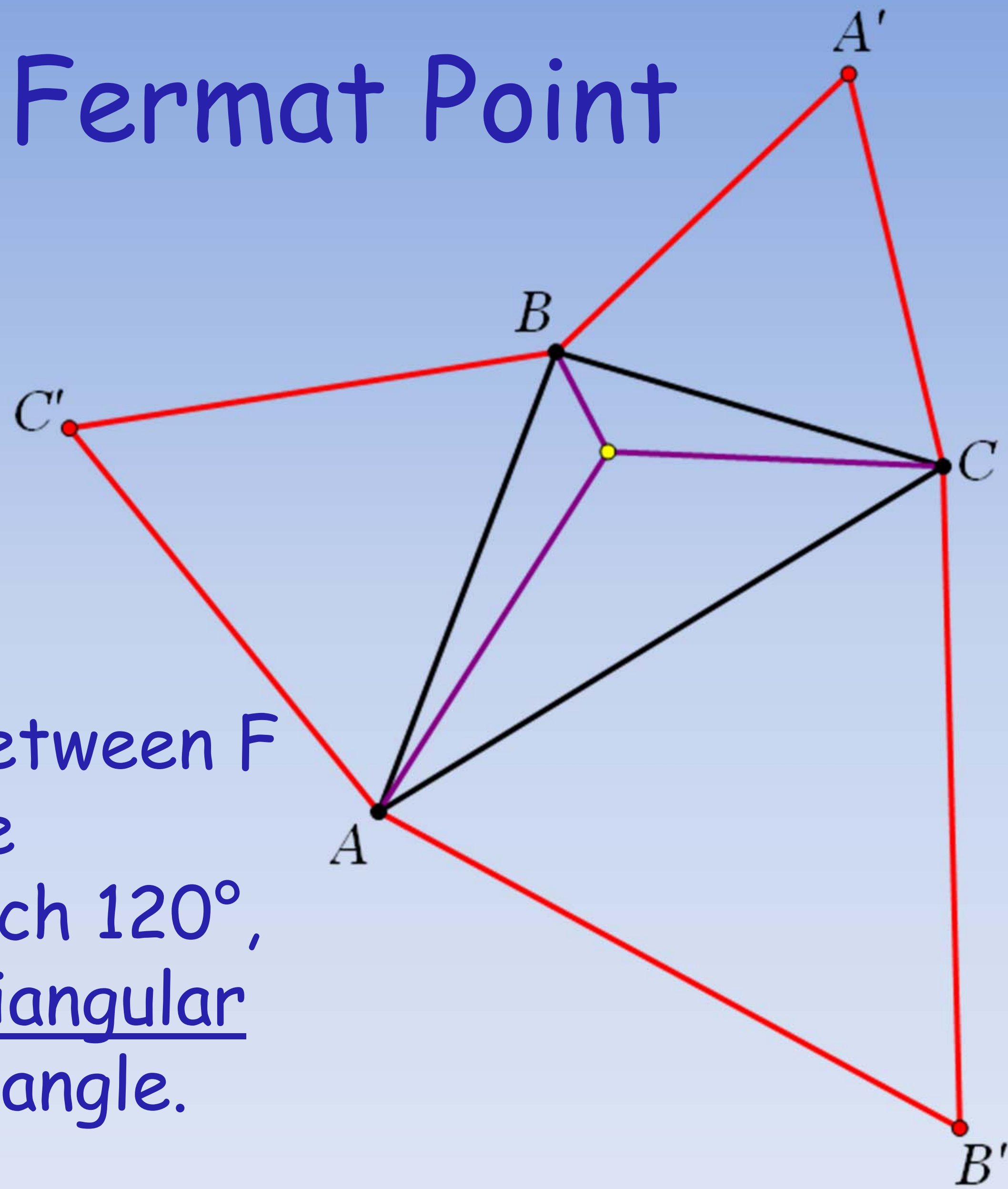


Fermat Point

The first description also due to Fermat. His interest was in a different direction.

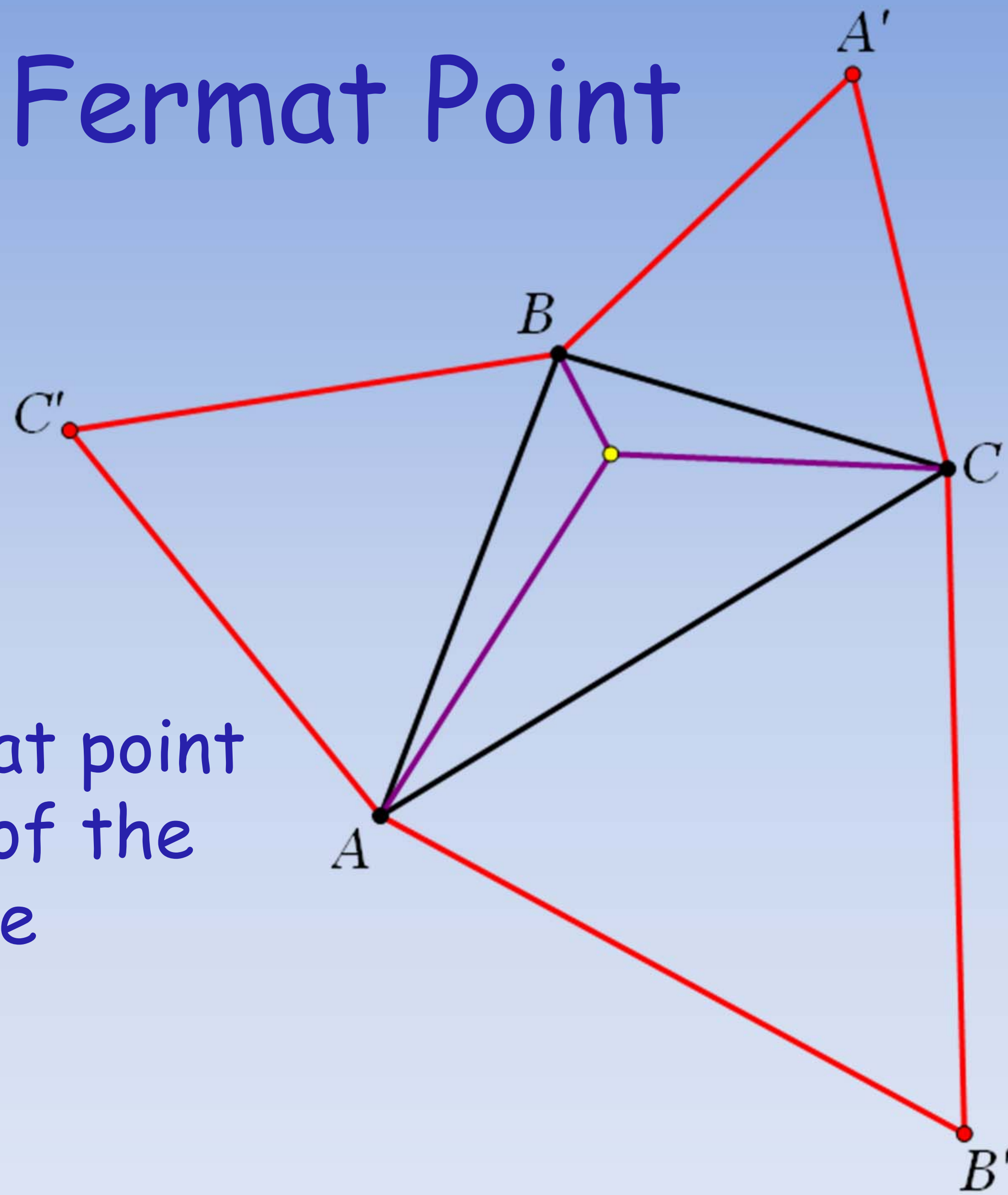


Fermat Point

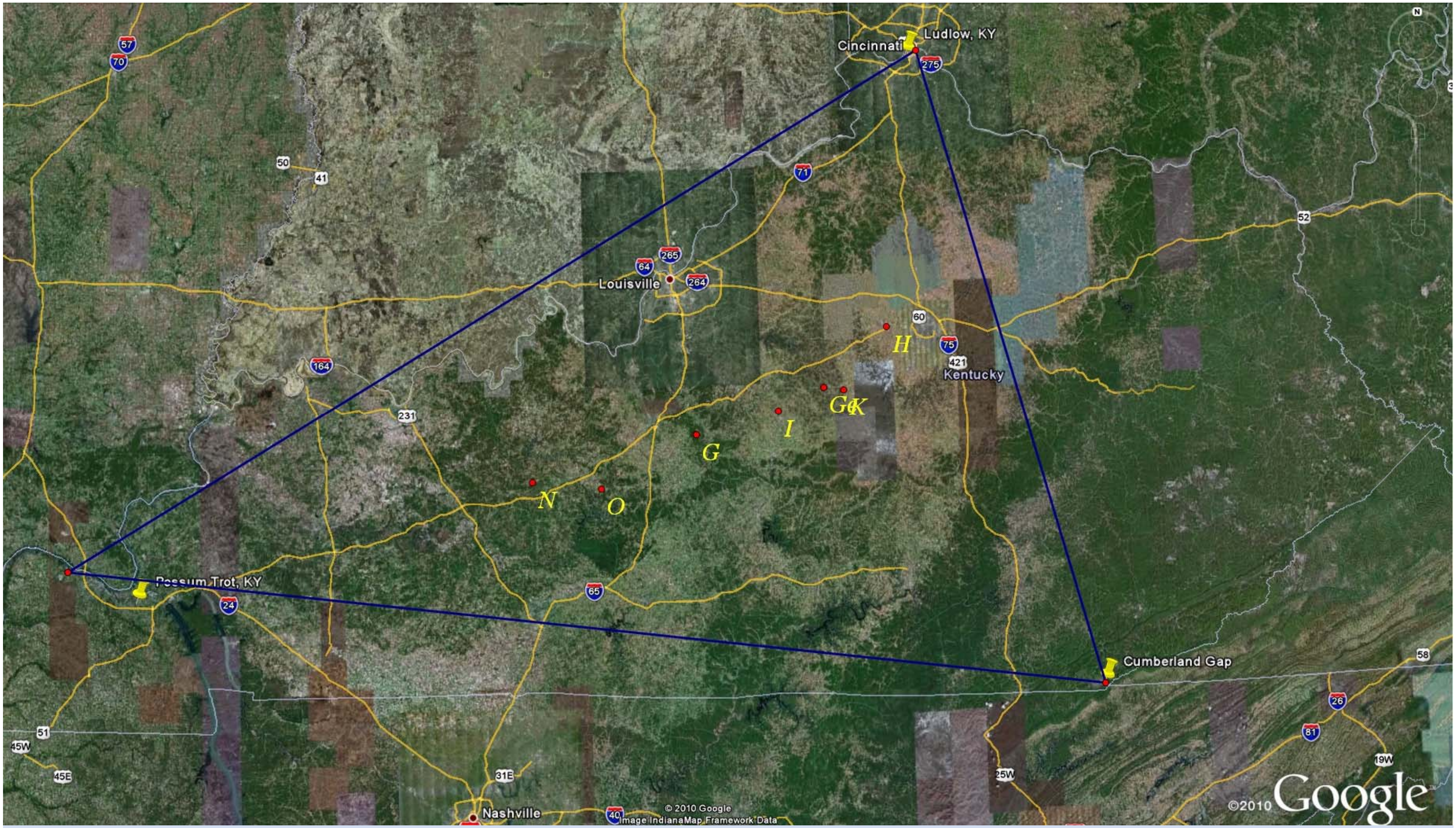


The 3 angles between F and each of the vertices are each 120°, so it is the <u>equiangular point</u> of the triangle.

Fermat Point



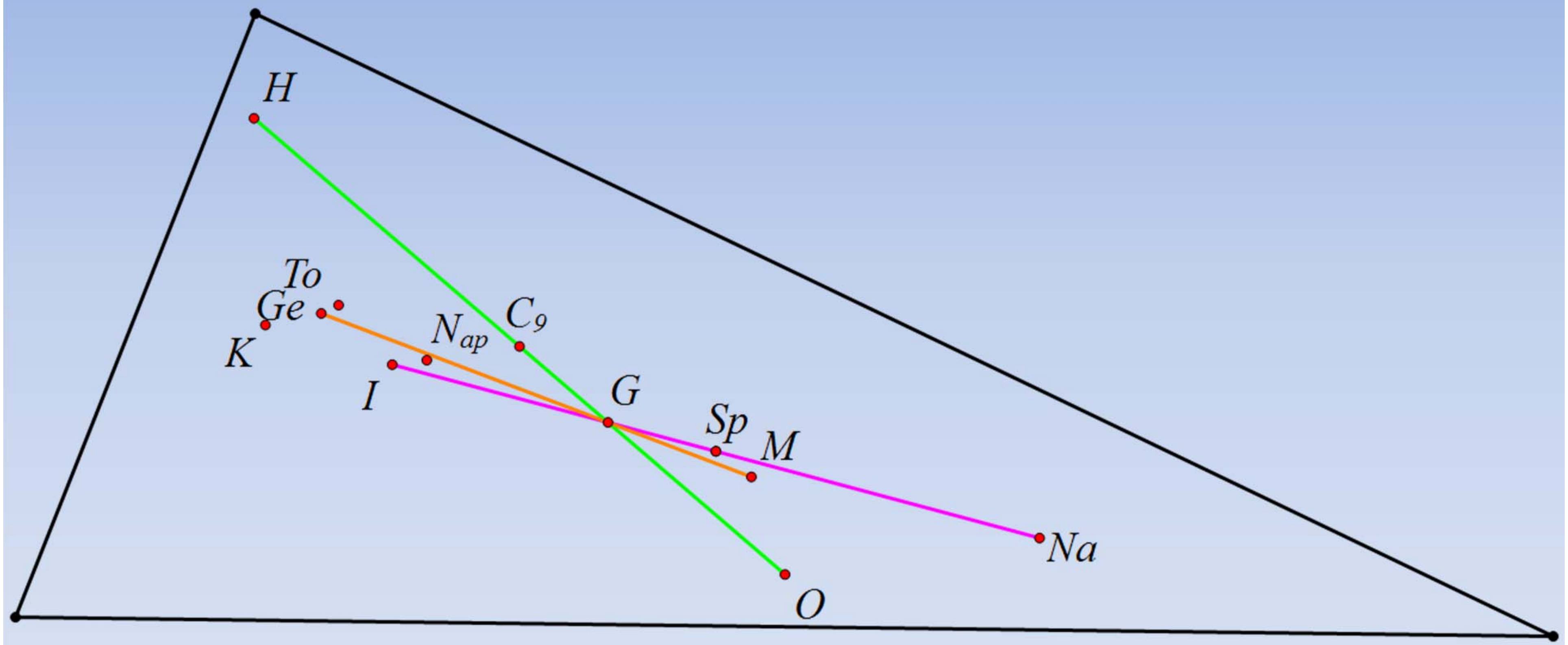
Also, the Fermat point minimizes sum of the distances to the vertices.



Nagel point=Caneyville   Circumcenter=Rock Creek   Centroid=New Haven
Incenter = Springfield   Gergonne = Cornishville   Symmedian = Calvary & Hopewell Rd, Mercer Co

Orthocenter = Bluegrass Parkway & US 60, Versailles

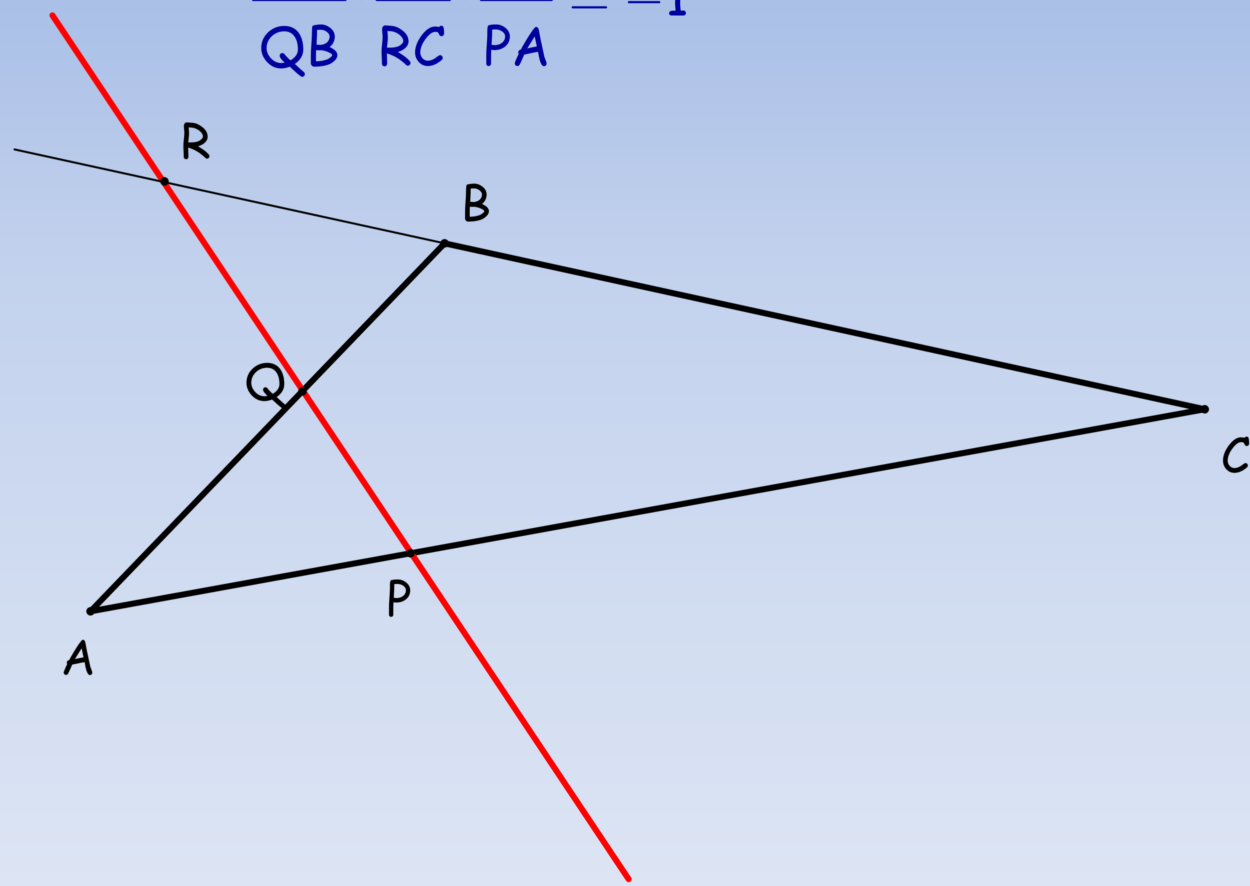
Various Centers



Menelaus's Theorem

The three points P, Q, and R one the sides AC, AB, and BC, respectively, of ΔABC are collinear if and only if

$$\frac{AQ}{QB} \cdot \frac{BR}{RC} \cdot \frac{CP}{PA} = -1$$



Menelaus's Theorem

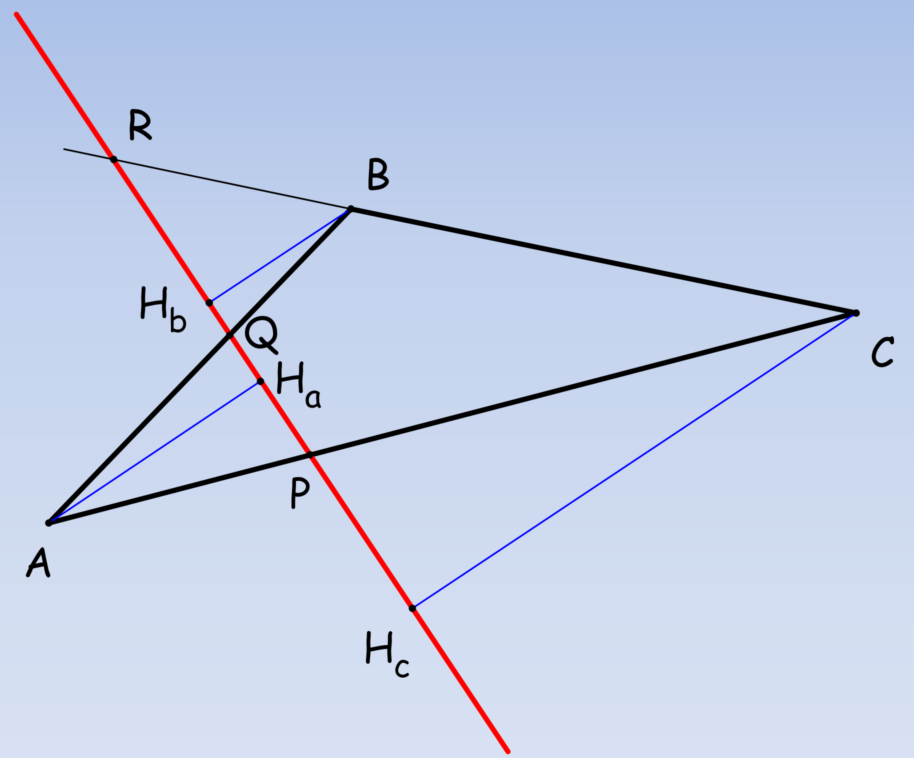
Assume P, Q, and R are collinear.

From the vertices drop perpendiculars to the line.

$\Delta CH_cR \sim \Delta BH_bR$

$\Delta CH_cP \sim \Delta AH_aP$

$\Delta AH_aQ \sim \Delta BH_bQ.$



Menelaus' Theorem

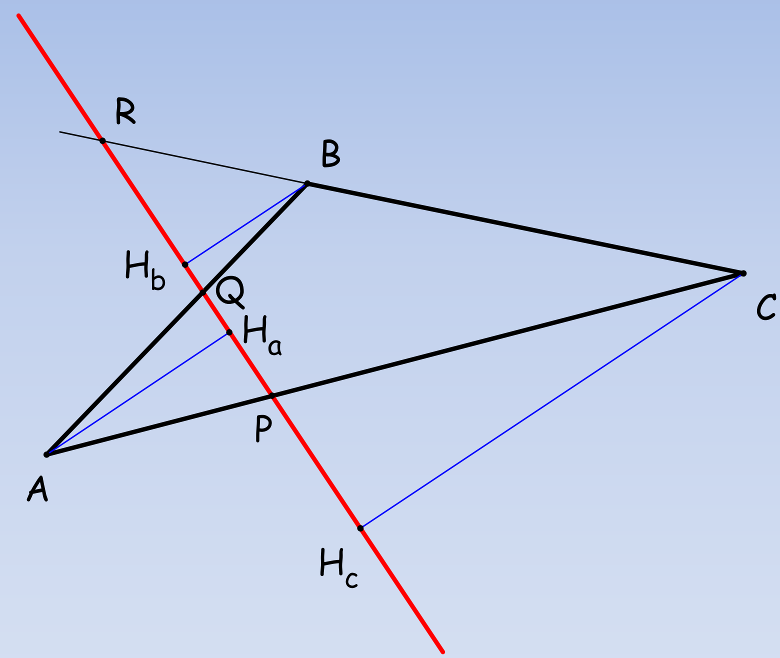
Therefore

$BR/CR = BH_b/CH_c$,

$CP/AP = CH_c/AH_a$,

$AQ/BQ = AH_a/BH_b$.

$$\frac{AQ}{QB}\cdot\frac{BR}{RC}\cdot\frac{CP}{PA}=\frac{AH_a}{BH_b}\cdot\frac{BH_b}{CH_c}\cdot\frac{CH_c}{AH_a}=1$$



BR/RC is a negative ratio if we take direction into account. This gives us our negative.

Menelaus's Theorem

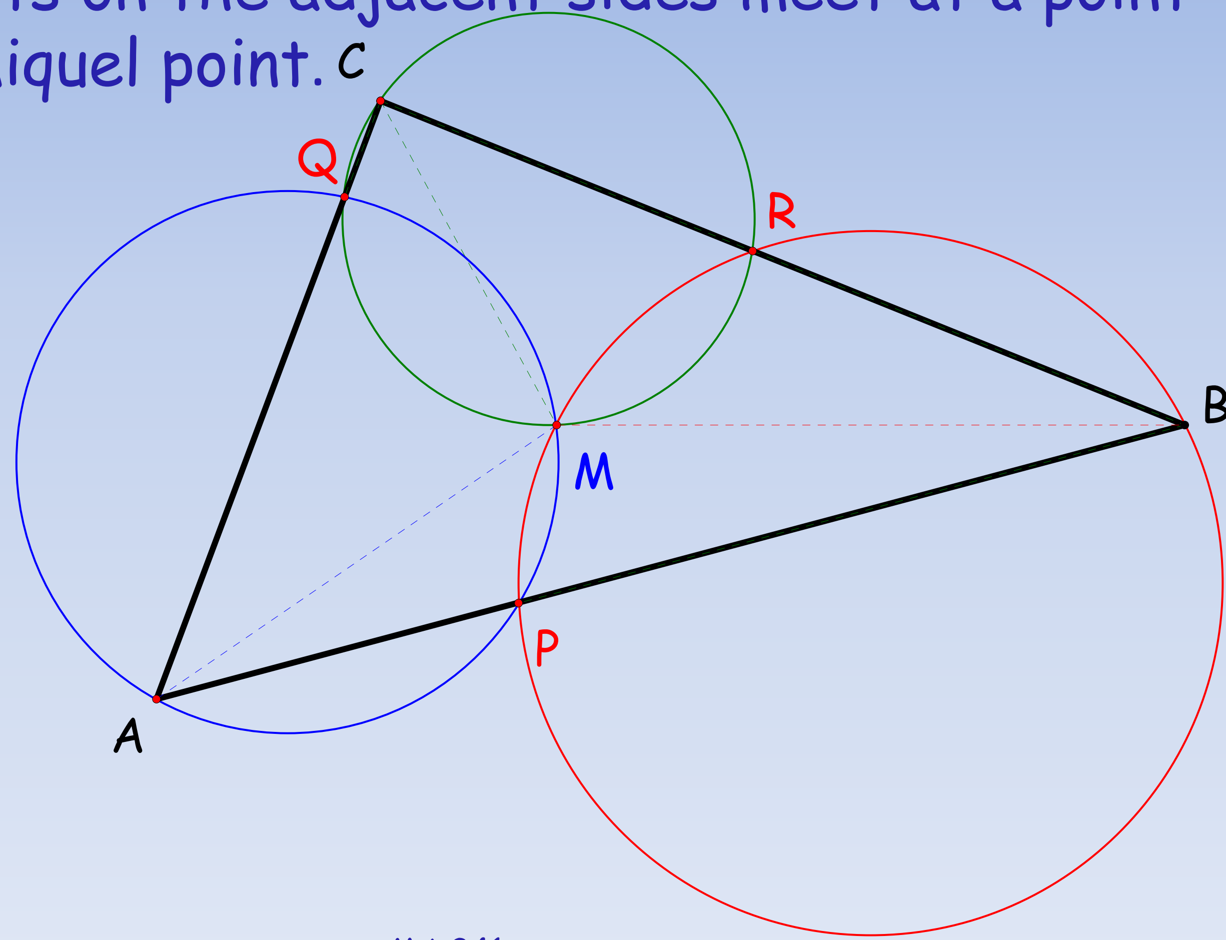
For the reverse implication, assume that we have three points such that $AQ/QB \cdot BR/RC \cdot CP/PA = 1$. Assume that the points are not collinear. Pick up any two. Say P and Q. Draw the line PQ and find its intersection R' with BC. Then

$AQ/QB \cdot BR'/R'C \cdot CP/PA = 1$.

Therefore $BR'/R'C = BR/RC$, from which $R' = R$.

Miquel's Theorem

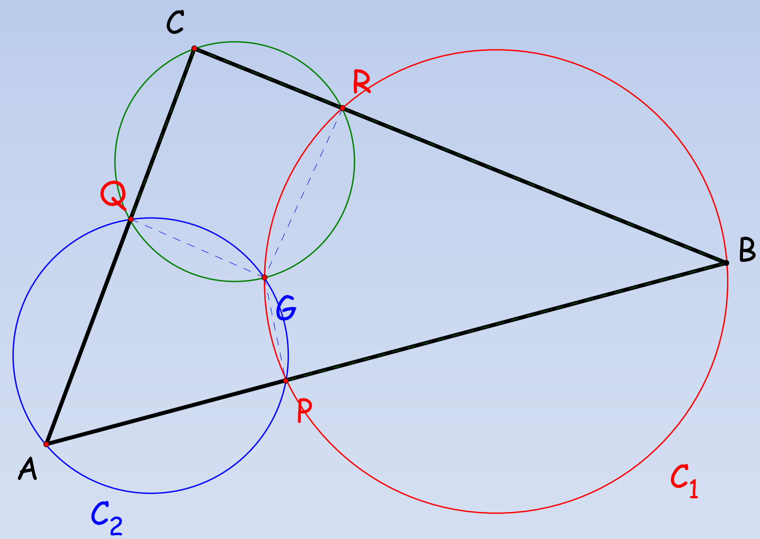
If P, Q, and R are on BC, AC, and AB respectively, then the three circles determined by a vertex and the two points on the adjacent sides meet at a point called the Miquel point.



Miquel's Theorem

Let $\triangle ABC$ be our triangle and let P,Q, and R be the points on the sides of the triangle. Construct the circles of the theorem. Consider two of the circles, C_1 and C_2, that pass through P. They intersect at P, so they must intersect at a second point, call it G.

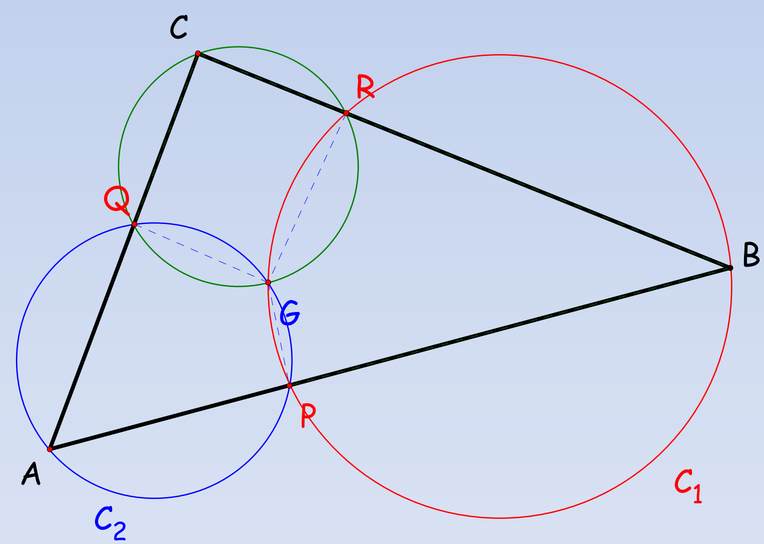
In circle C_2
$\angle QGP + \angle QAP = 180$
In circle C_1
$\angle RGP + \angle RBP = 180$



Miquel's Theorem

$$
\begin{aligned}
\measuredangle QGP + \measuredangle QGR + \measuredangle RGP &= 360 \\
(180 - \measuredangle A) + \measuredangle QGR + (180 - \measuredangle B) &= 360 \\
\measuredangle QGR &= \measuredangle A + \measuredangle B \\
&= 180 - \measuredangle C
\end{aligned}
$$

Thus, $\angle QGR$ and $\angle C$ are supplementary and so Q, G, R, and C are concyclic. These circle then intersect in one point.



Morley's Theorem

The adjacent trisectors of the angles of a triangle are concurrent by pairs at the vertices of an equilateral triangle.