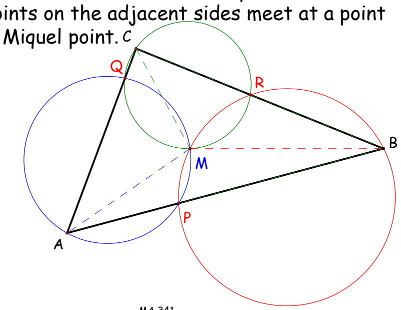
Pedal Triangles and the Simson Line

MA 341 – Topics in Geometry
Lecture 18

Miquel's Theorem

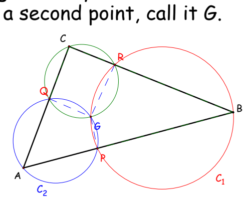
If P, Q, and R are on BC, AC, and AB respectively, then the three circles determined by a vertex and the two points on the adjacent sides meet at a point called the Miquel point.



Miquel's Theorem

Let △ABC be our triangle and let P,Q, and R be the points on the sides of the triangle. Construct the circles of the theorem. Consider two of the circles, C_1 and C_2, that pass through P. They intersect at P, so they must intersect at a second point, call it G.

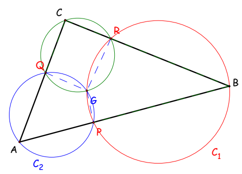
In circle C_2
$\angle QGP + \angle QAP = 180$
In circle C_1
$\angle RGP + \angle RBP = 180$



Miquel's Theorem

$$\measuredangle QGP + \measuredangle QGR + \measuredangle RGP = 360$$
$$(180 - \measuredangle A) + \measuredangle QGR + (180 - \measuredangle B) = 360$$
$$\measuredangle QGR = \measuredangle A + \measuredangle B$$
$$= 180 - \measuredangle C$$

Thus, $\angle QGR$ and $\angle C$ are supplementary and so Q, G, R, and C are concyclic. These circle then intersect in one point.



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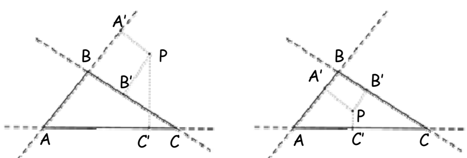
Pedal Triangle

For any triangle ΔABC and any point P, let A', B', C' be the feet of the perpendiculars from P to the (extended) sides of ΔABC.



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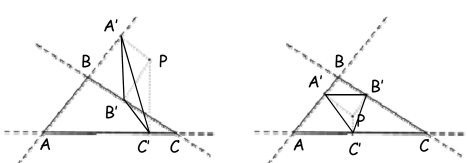
Pedal Triangle

Form the triangle ΔA'B'C'.



Do we always get a triangle?

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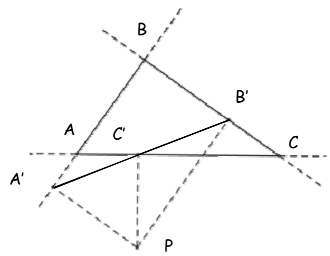
Pedal Triangle

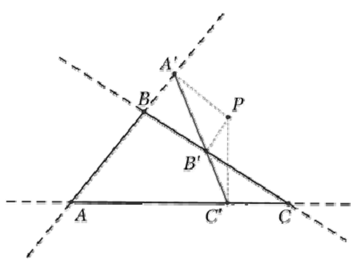
Form the triangle $\Delta A'B'C'$.



What is it with P?

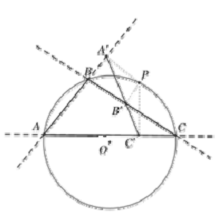
Pedal Triangle

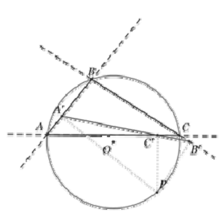
Can we characterize the points where the pedal triangle is a "degenerate triangle"?



Simson-Wallace Line

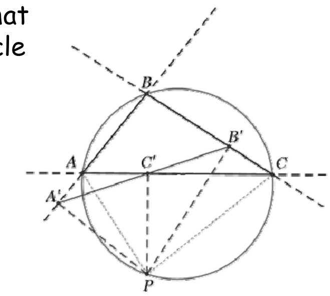
Theorem (Wallace, Simson): Given a reference triangle ΔABC, if P lies on the circumcircle of ΔABC then the pedal triangle is degenerate.





The Simson Line

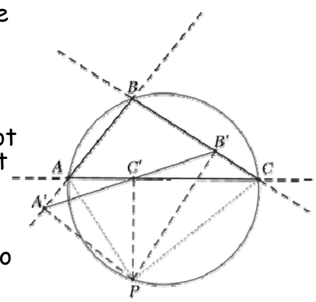
Proof: Assume that P is on circumcircle of ΔABC



The Simson Line

Proof: First, assume that P is on the circumcircle.

WLOG we can assume that P is on arc AC that does not contain B and P is at least as far from C as it is from A. If necessary you can relabel the points to make this so.

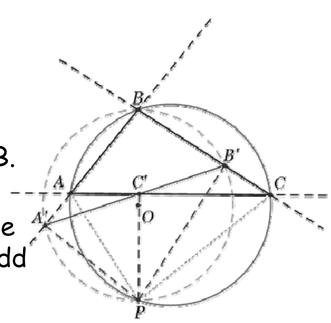


The Simson Line

P also lies on the circumcircle of triangle ΔB'BA'. Why?

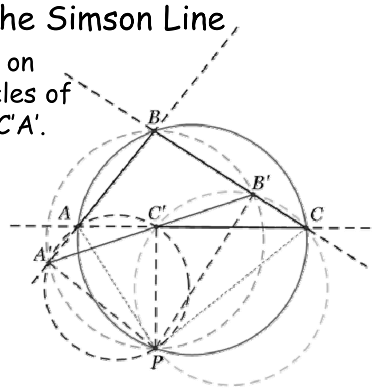
$\angle$PB'B =90= $\angle$PA'B.

$\Rightarrow$ PA'BB' cyclic quadrilateral since opposite angles add up to 180.



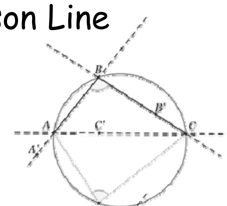
The Simson Line

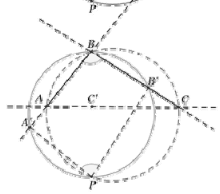
Likewise P lies on the circumcircles of ΔB′C′C and ΔAC′A′.



The Simson Line

∠APC + ∠B = 180
and
∠A′PB′ + ∠B = 180
So
∠APC = ∠A′PB′





The Simson Line

∠APC -∠APB′ = ∠A′PB′- ∠APB′

∠B′PC = ∠A′PA.

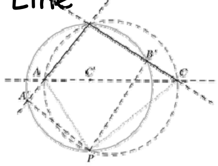
Now, B′, C, P and C′ are concyclic so by Star Trek Lemma

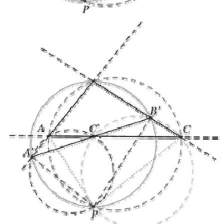
∠B′PC = ∠B′C′C.

Similarly,

∠A′PA = ∠A′C′A.

making A′, B′, C′ collinear.





The Simson Line

The converse of this theorem is also true.

That is if ΔA'B'C' is degenerate then P must lie on the circumcircle of ΔABC.

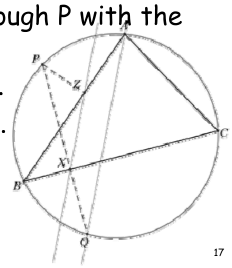
Lemma 1

Choose P on the circumcircle of ΔABC. Let Q be the intersection of the perpendicular to BC through P with the circumcircle (Q≠P).

Let X be foot of P in BC.

Let Z be foot of P in AB.

If Q≠A, then ZX||QA.



Proof

Assume X≠Z. If P=B, then P=B=X=Z, so P≠B. So, consider the unique circle with diameter PB.

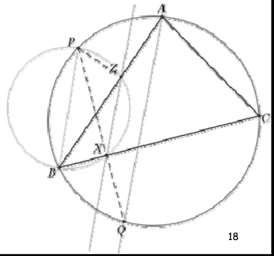
$\angle PXB = 90 = \angle PZB$

$\Rightarrow$ X,Z are concyclic with P & B.

$\Rightarrow \angle PXZ = \angle PBZ$

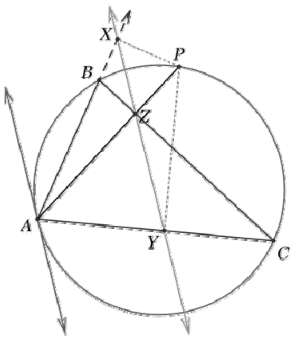
$\angle PBZ = \angle PBA = \angle PQA$

$\Rightarrow XZ || QA$



Lemma 2

If the altitude AD of ΔABC meets the circumcircle at P, then the Simson line of P is parallel to the line tangent to the circle at A.



Proof

XYZ is the Simson line of P.

$\angle PXB = 90 = \angle PZB$

$\Rightarrow$ P,Z,B,X concyclic

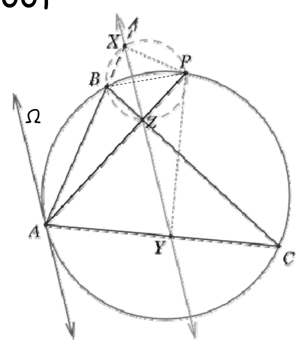
$\angle BXZ = \angle BPZ$

$\angle BPZ = \frac{1}{2}\overset{\frown}{AB}$

$\frac{1}{2}\overset{\frown}{AB} = \angle \Omega AB$

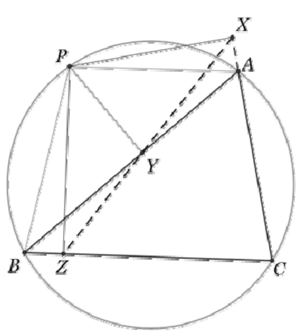
$\angle \Omega AB = \angle AXZ$

$\Rightarrow \Omega A || XY$



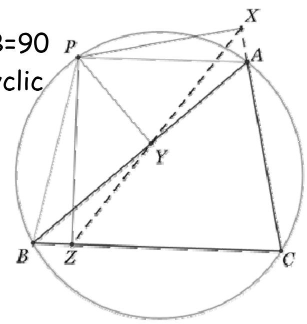
Lemma 3

From P on the circumcircle of ΔABC if perpen-diculars PX, PY, PZ are drawn to AC, AB, and AC, then (PA)(PZ)=(PB)(PX).



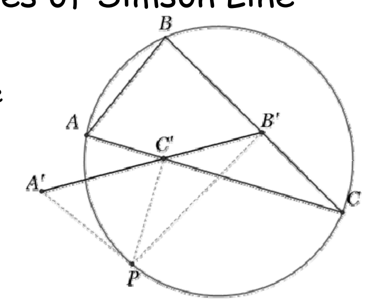
Proof

$\angle$PYB=90 and $\angle$PZB=90
Thus, P,Y,Z,B concyclic
Thus, $\angle$PBY=$\angle$PZY
Likewise P,X,A,Y concyclic
Thus, $\angle$PXY=$\angle$PAY
ΔPAB~ΔPXZ
(PA)(PZ)=(PB)(PX)



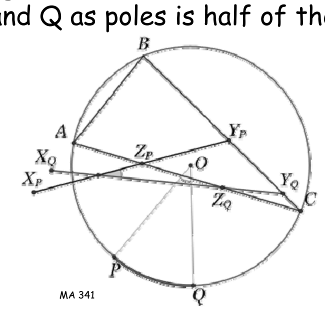
Properties of Simson Line

P is called the pole of the line A'B'.



Lemma 4

Let P and Q be points on the circumcircle of ABC. The angle between the Simson lines having P and Q as poles is half of the arc, $\overset{\frown}{PQ}$.



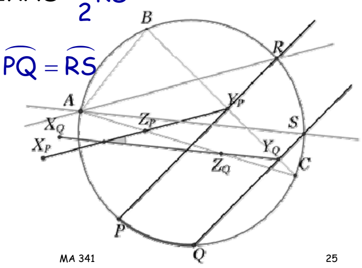
Proof

Proof: Extend PY_P to R and QY_Q to S.

AS || X_QY_Q and AR|| X_PY_P

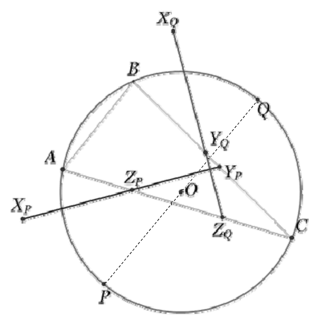
$\Rightarrow \angle Y_P\Omega Y_Q = \angle RAS = \frac{1}{2}\widehat{RS}$

Since PR||QS, $\widehat{PQ} = \widehat{RS}$



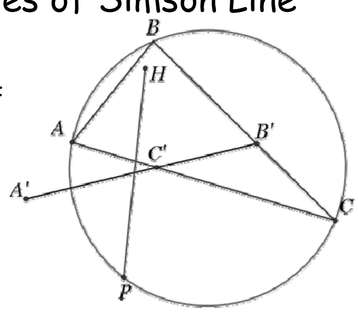
Lemma 5

Two Simson lines are perpendicular iff their poles are on opposite ends of a diameter.



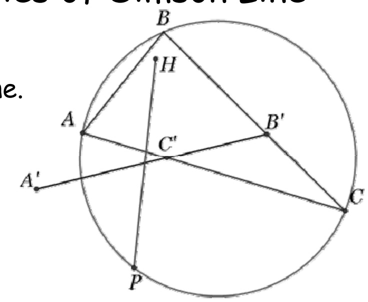
Properties of Simson Line

Find the orthocenter of ΔABC and construct HP.



Properties of Simson Line

HP intersects the Simson line.



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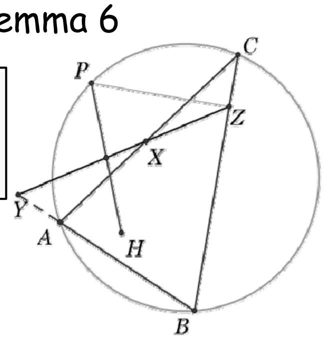
Lemma 6

The point of intersection is the midpoint of HP.



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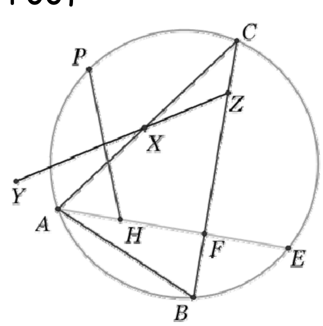
Proof

Construct AF.
Extend to E.
Mark H on AF.
Construct PH.



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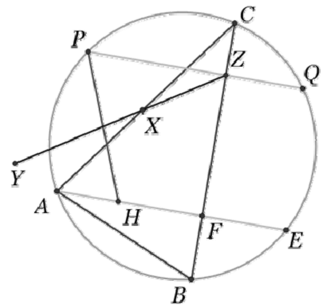
Proof

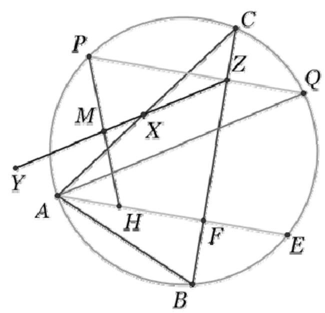
Construct PZ⊥BC.
Extend to Q.



Properties of Simson Line

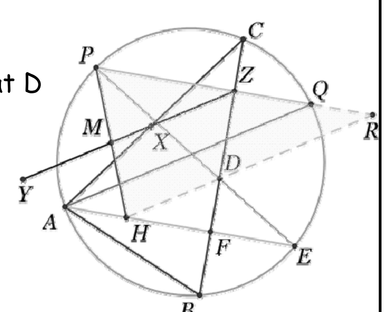
Construct AQ.

YZ||AQ by Lemma 1



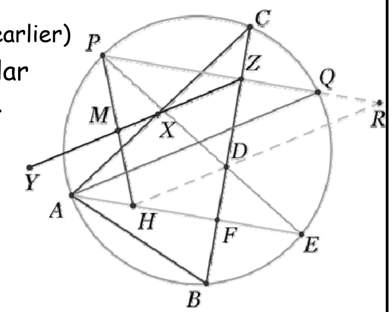
Properties of Simson Line

Construct PE.
Intersects BC at D
Construct HD
Extend to meet PQ at R
Consider ΔPHR.



Properties of Simson Line

HF=FE (proven earlier)
DF=perpendicular bisector of HE.
⇒DH=DE
∠PQA= ∠PEA
= ∠RHE
= ∠PRH



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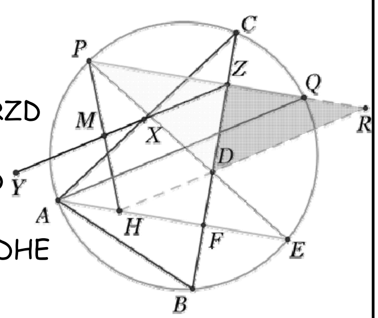
Properties of Simson Line

∠PQA = ∠PRH
⇒HR||AQ
⇒HR||YZ
Show: ΔPZD=ΔRZD
DZ=DZ
∠PZD=90=∠RZD
PR||AE ⇒
∠ZPD=∠DEH=∠DHE
=∠ZRD



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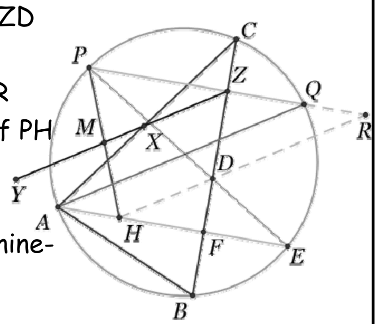
Properties of Simson Line

Thus, ΔPZD=ΔRZD
⇒ PZ=ZR
⇒ Z=midpoint PR
⇒ M=midpoint of PH

Note: M lies on nine-point circle



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