

The Divine Proportion

MA 341 - Topics in Geometry
Lecture 20



What is the Divine Proportion?

In mathematics and the arts, two quantities are in the **golden ratio** if the ratio of the sum of the quantities to the larger quantity is equal to the ratio of the larger quantity to the smaller one.

$$\frac{a+b}{a} = \frac{a}{b} \quad \text{for } a > b$$

What is the Divine Proportion?

Other names frequently used for the golden ratio are the

golden section

golden cut

golden proportion

mean ratio

divine proportion

mean of Phidias

golden mean

golden number

extreme ratio

medial section

divine section

Denoted by $\phi = \varphi$

Value?

Does this ratio have a number associated with it,
like π =

3.141592653589793238462643383279502884197
1693993751058209749445923078164062862089
9862803482534211706798214808651328230664
70938446095505822317253594081284811174502
84102701938521105559644622948954930381964
4288109756659334461284756482337867831652
71201909145648566923460348610454326648213
3936072602491412737245870066063155881748
81520920962829254091715364367892590360011
33053054882046652138414695194151160943305
727036575959195309218611738193261179310...

Value?

If $a/b = \varphi$, then

$$\frac{a}{b} = \frac{a+b}{a}$$

$$\frac{a}{b} = 1 + \frac{b}{a}$$

$$\varphi = 1 + \frac{1}{\varphi}$$

$$\varphi^2 = \varphi + 1$$

$$\varphi^2 - \varphi - 1 = 0$$

Value?

$$\varphi = \frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2}$$

$$\varphi = \frac{1 \pm \sqrt{5}}{2}$$

Which is it? Is it + or -?

What do we know about φ ?

$a > b$ so $\varphi > 1$ and $\varphi = \frac{1 + \sqrt{5}}{2}$

Phi

$$\varphi = \frac{1 + \sqrt{5}}{2} =$$

1.618033988749894848204586834365638117
7203091798057628621354486227052604628
1890244970720720418939113748475408807
5386891752126633862223536931793180060
766726354433389086595939582905638322
66131992829026788067520876689250171169
62070322210432162695486262963136144381
4975870122034080588795445474924618569
5364864449241044320771344947049565846
7885098743394422125448770664780915884
6074998871240076521705751797883416...

Properties of φ

From earlier we have that

$$\varphi = 1 + \frac{1}{\varphi}$$

Therefore,

$$\frac{1}{\varphi} = \varphi - 1$$

Properties of φ

$$\frac{1}{\varphi} = \varphi - 1$$

$$\begin{aligned}\frac{1}{\varphi} &= \frac{2}{1 + \sqrt{5}} = \frac{2(1 - \sqrt{5})}{(1 + \sqrt{5})(1 - \sqrt{5})} \\ &= \frac{2(1 - \sqrt{5})}{-4} \\ &= \frac{-1 + \sqrt{5}}{2}\end{aligned}$$

Φ and φ

Sometimes authors use:

$$\Phi = 1.6180\dots$$

and

$$\varphi = 0.6180\dots = 1/\Phi$$

History

Euclid's Elements provides first known written definition of golden mean:

"A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the less."

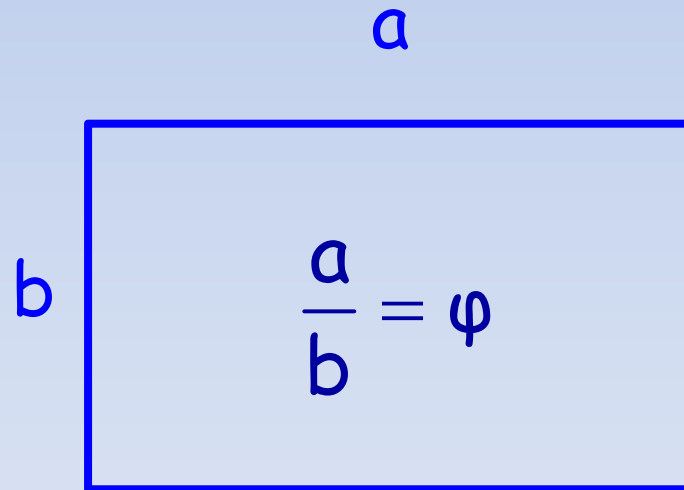
Euclid gives a construction for cutting a line "in extreme and mean ratio", i.e. the golden ratio.

Several propositions and their proofs employ the golden ratio.

Some of these propositions show that the golden ratio is an irrational number.

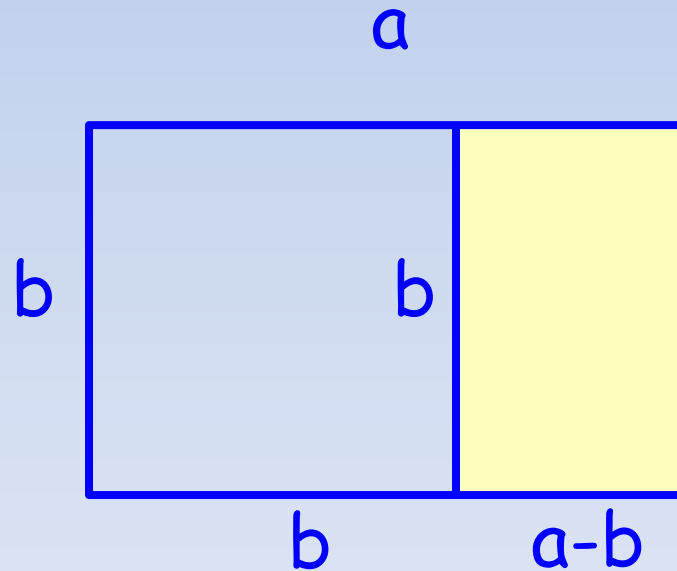
Golden Rectangle

- A rectangle is called a golden rectangle if its sides are in the ratio of the golden mean:



Golden Rectangle

Remove a square from the golden rectangle:

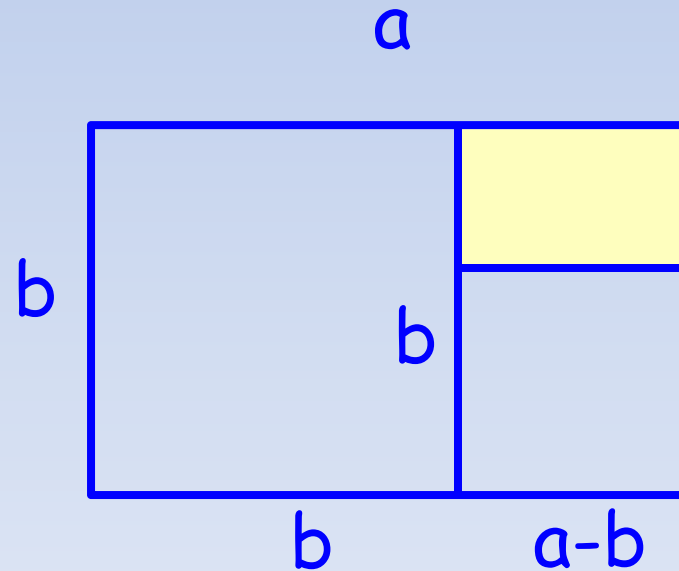


$$\begin{aligned}\frac{a}{b} &= \varphi \Rightarrow a = b\varphi \\ \frac{b}{a-b} &= \frac{b}{b\varphi - b} \\ &= \frac{1}{\varphi - 1} = \frac{1}{1/\varphi} \\ &= \varphi\end{aligned}$$

Golden Rectangle

The remaining rectangle is a golden rectangle!

Do it again!



Golden Rectangle

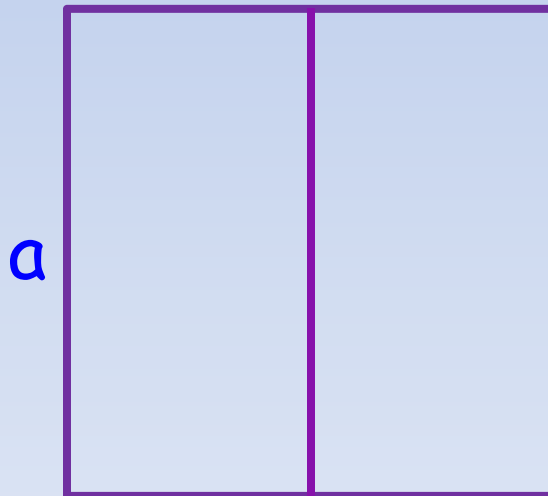
This actually gives us a construction for a golden rectangle using a compass and straightedge - thus GeoGebra or Sketchpad

Start with a square



Golden Rectangle

Find midpoint of the base and split square in two.

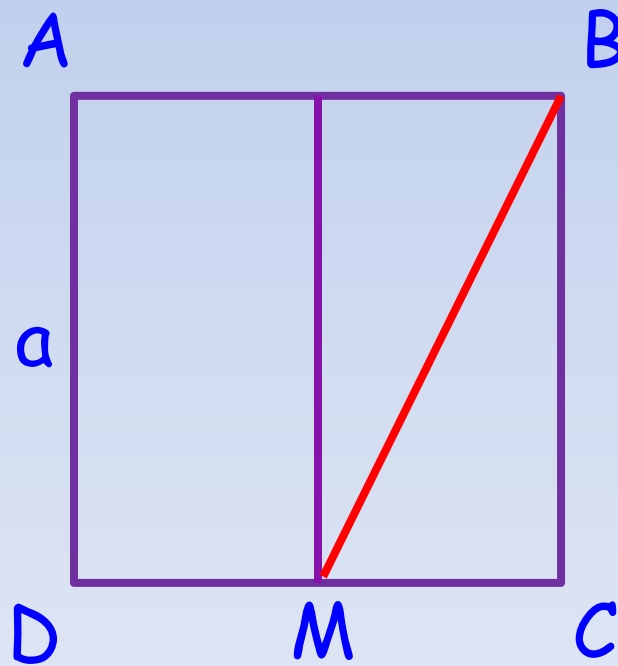


Golden Rectangle

Construct diagonal MB.

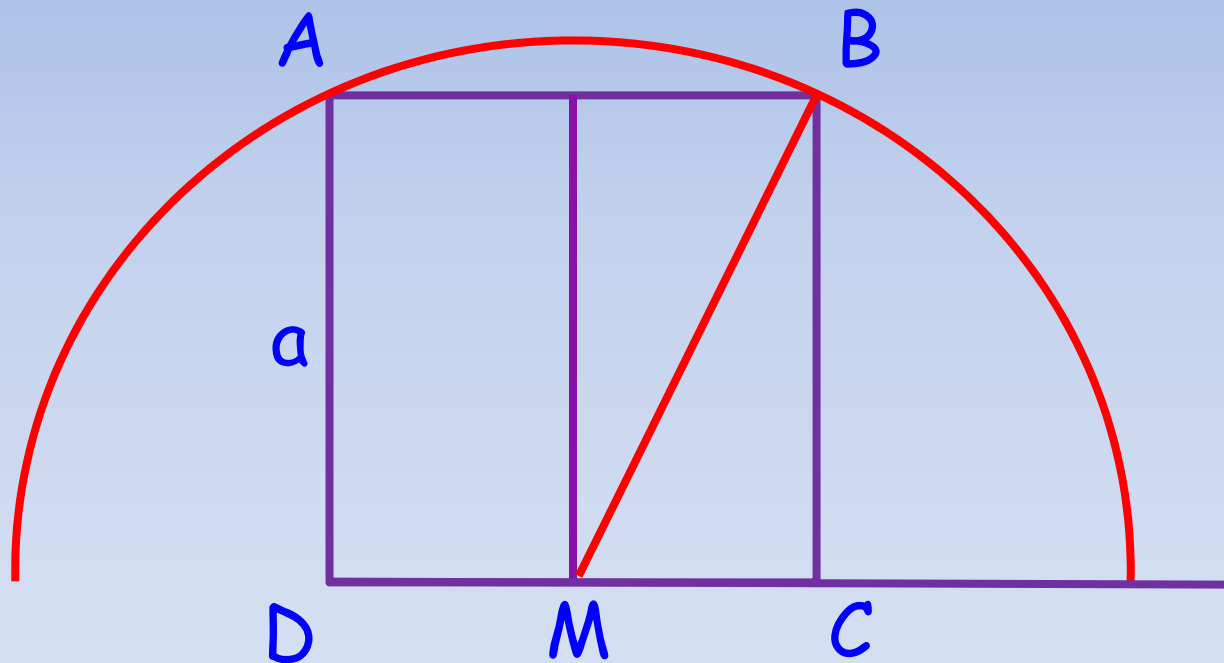
$$MC = a/2 \text{ and } BC = a$$

$$MB = ?$$



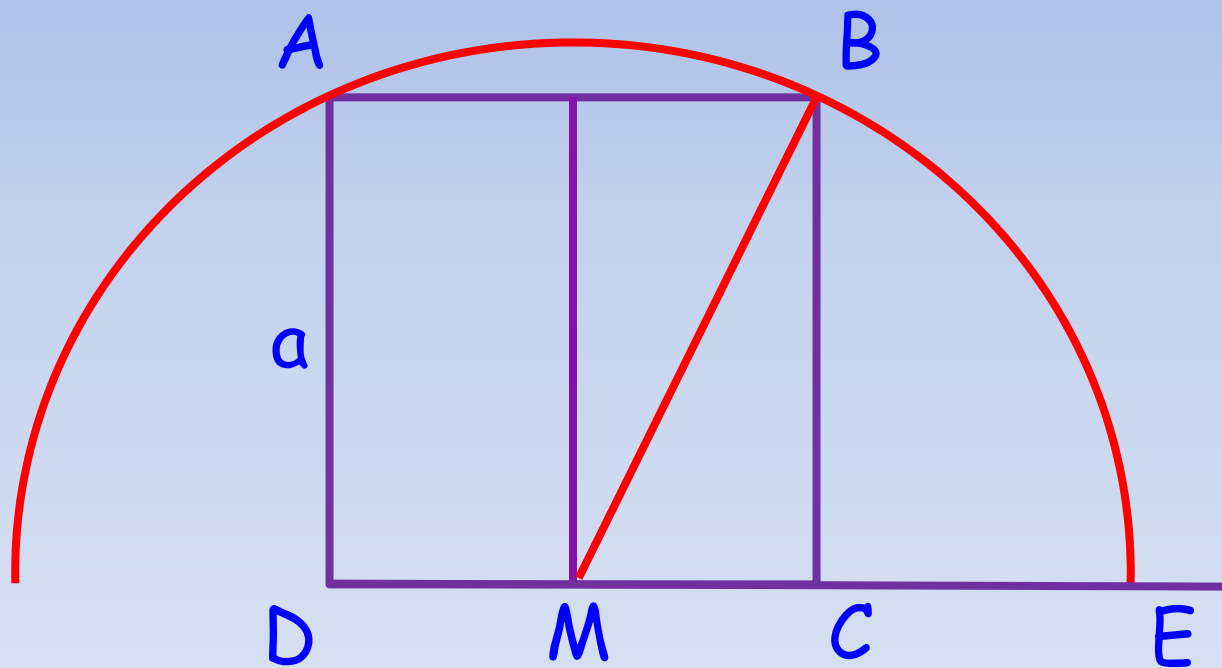
Golden Rectangle

Construct circle with radius MB centered at M .



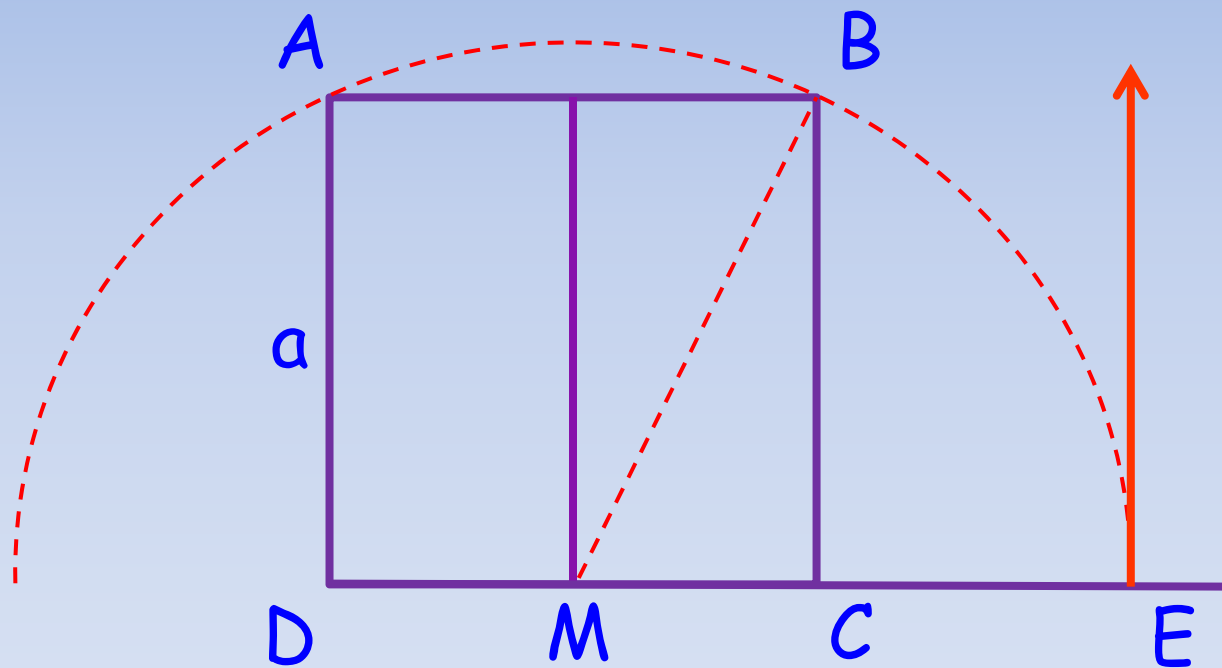
Golden Rectangle

Mark point of intersection E.



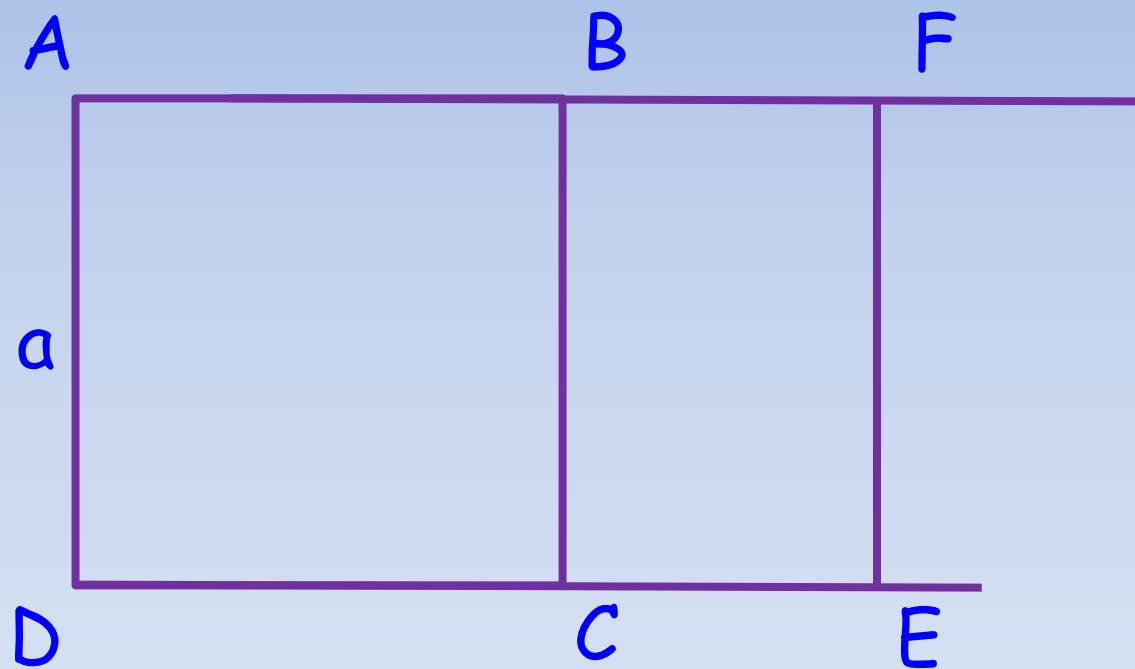
Golden Rectangle

Construct perpendicular at E.



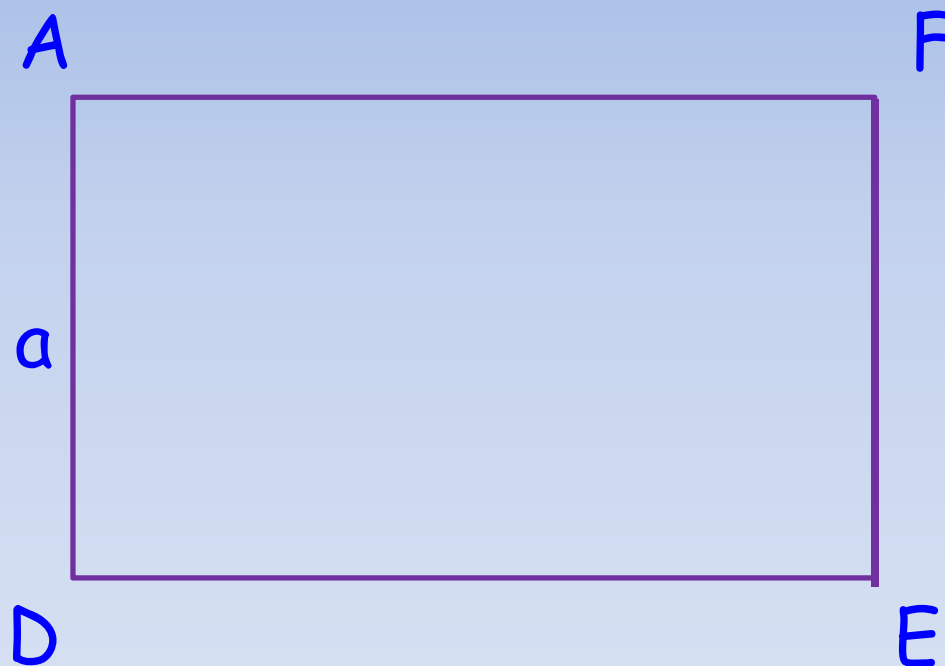
Golden Rectangle

Extend AB to meet this perpendicular.



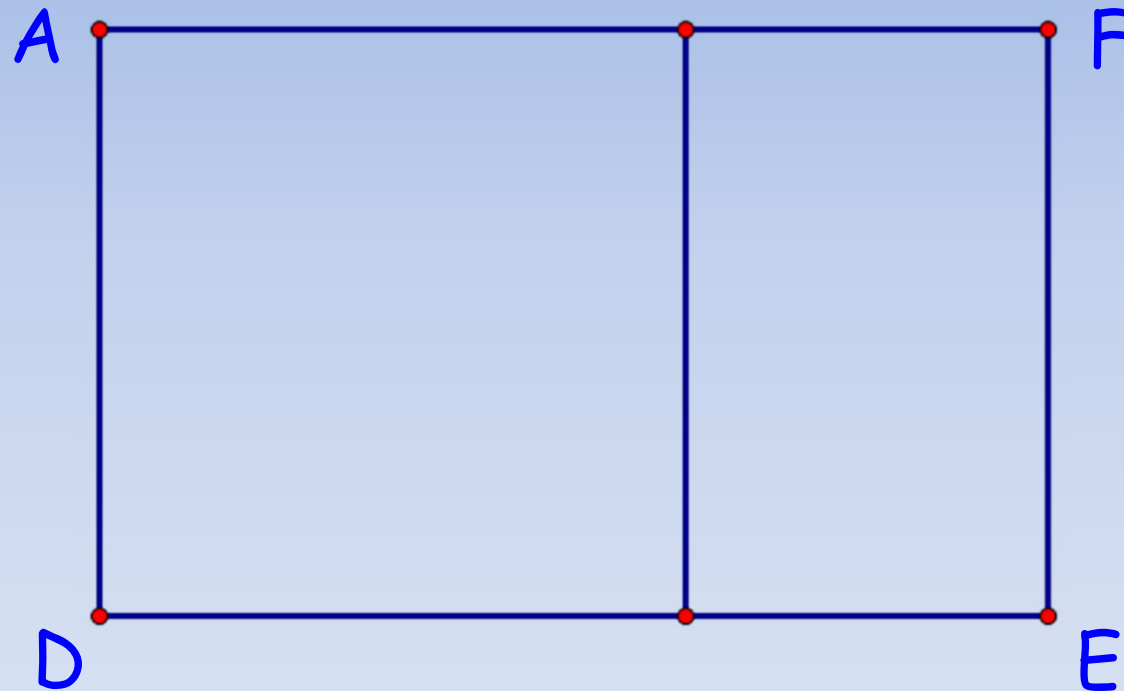
Golden Rectangle

AFED is a golden rectangle.



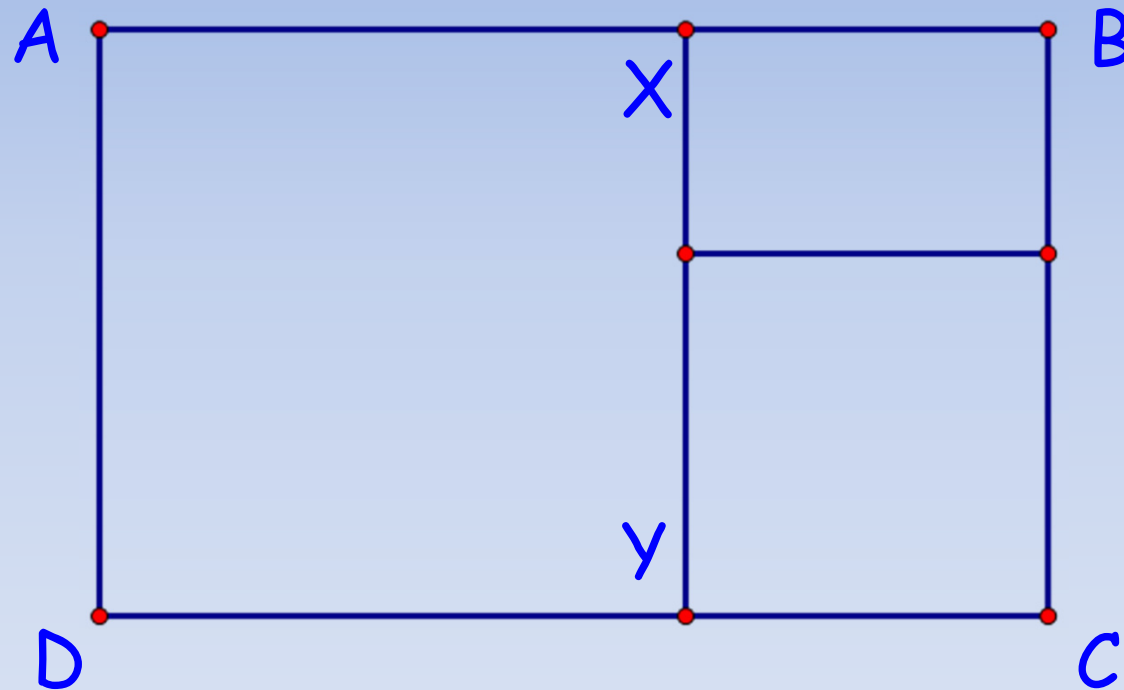
Golden Spiral

Construct a golden rectangle ABCD.



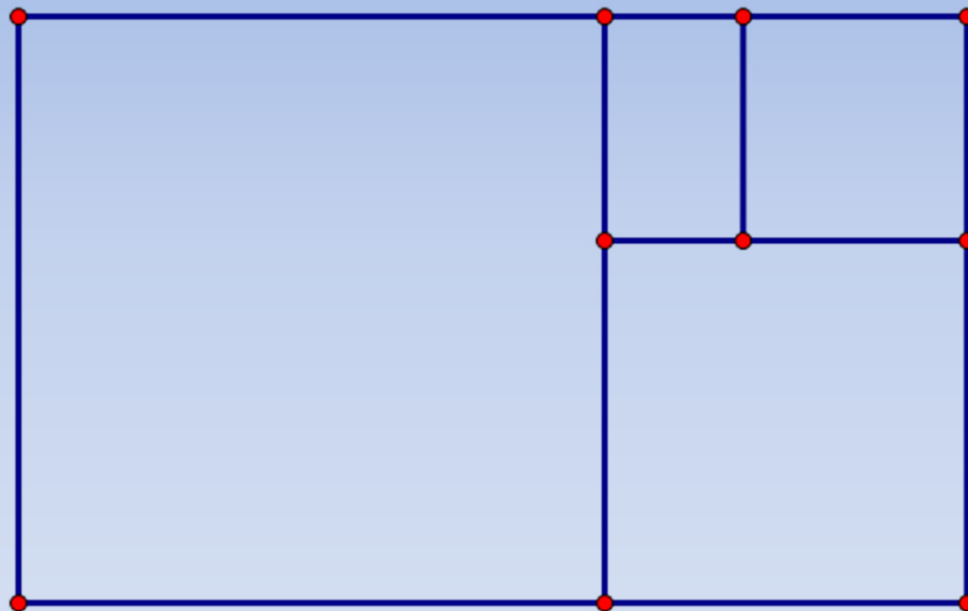
Golden Spiral

Construct a square inside $XBCY$.



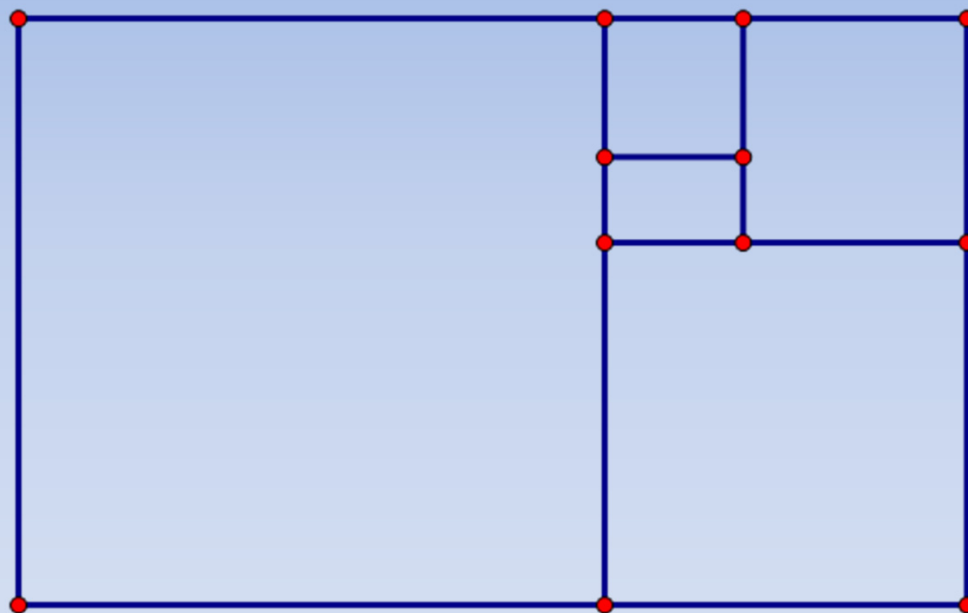
Golden Spiral

Construct another square inside the smaller golden rectangle..



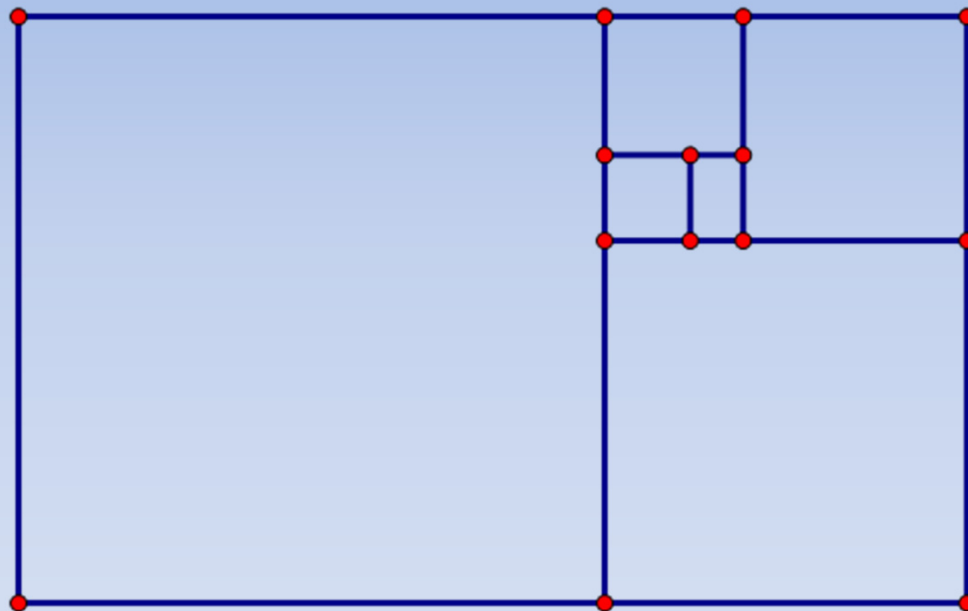
Golden Spiral

Again



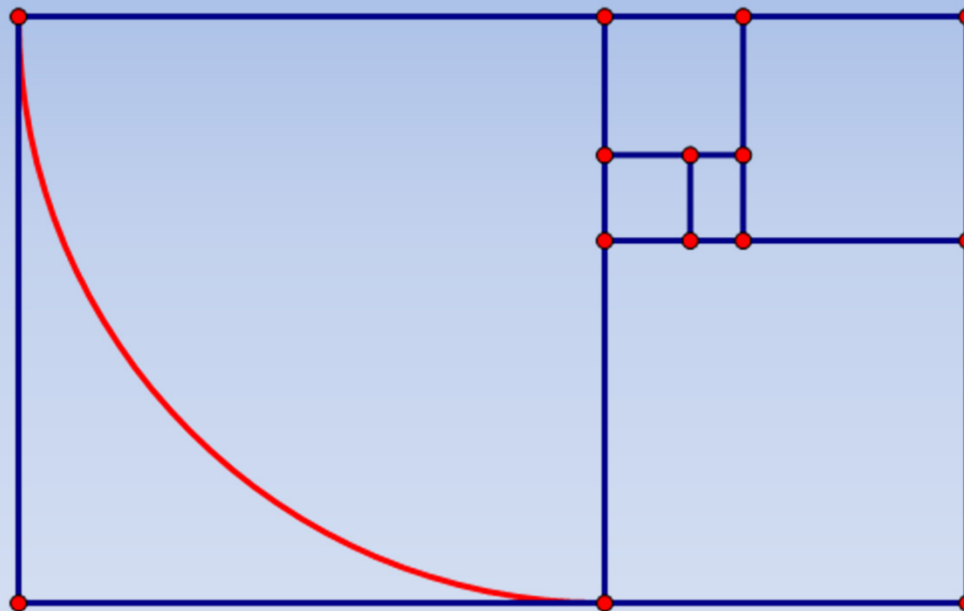
Golden Spiral

Again



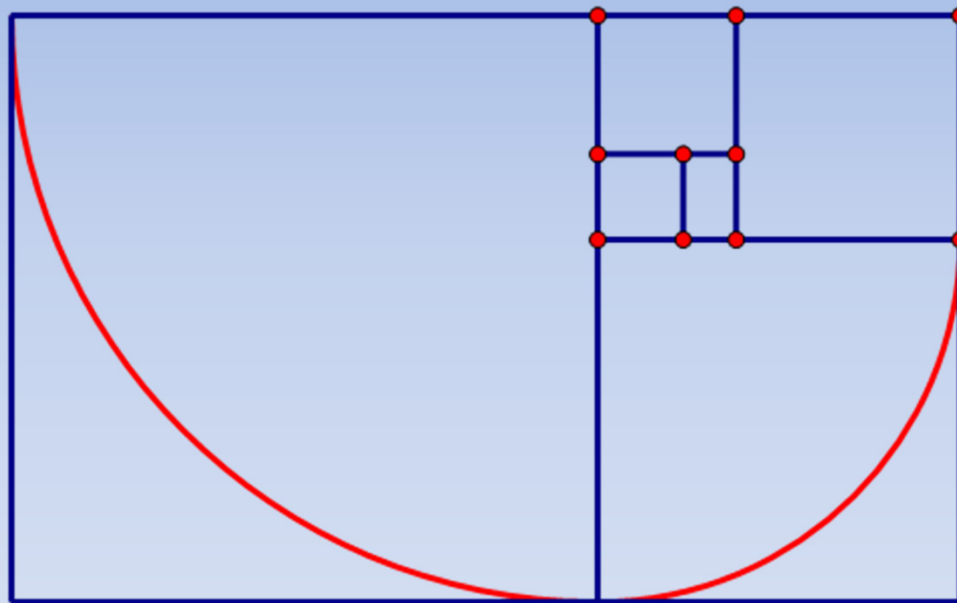
Golden Spiral

In each square construct a quarter circle:



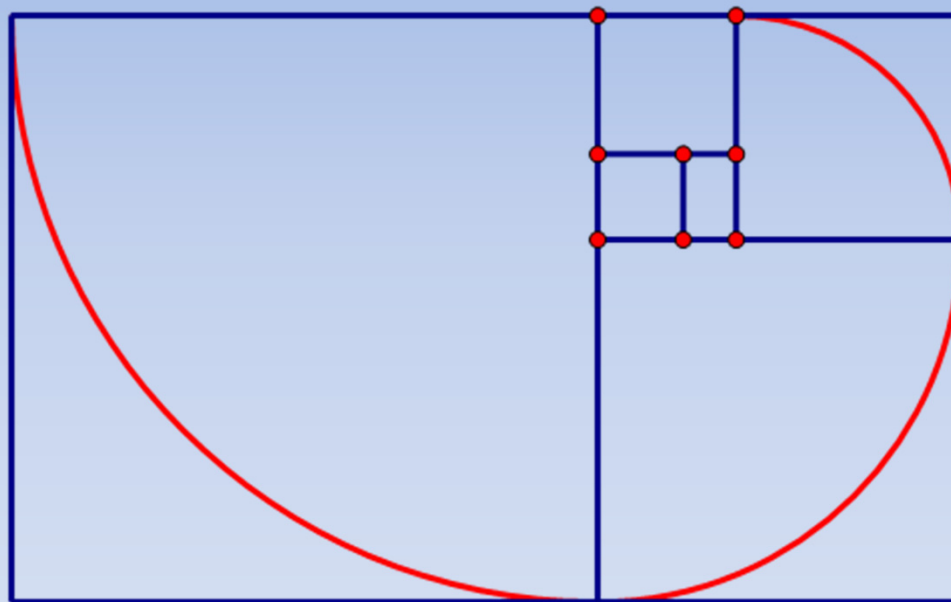
Golden Spiral

In each square construct a quarter circle:



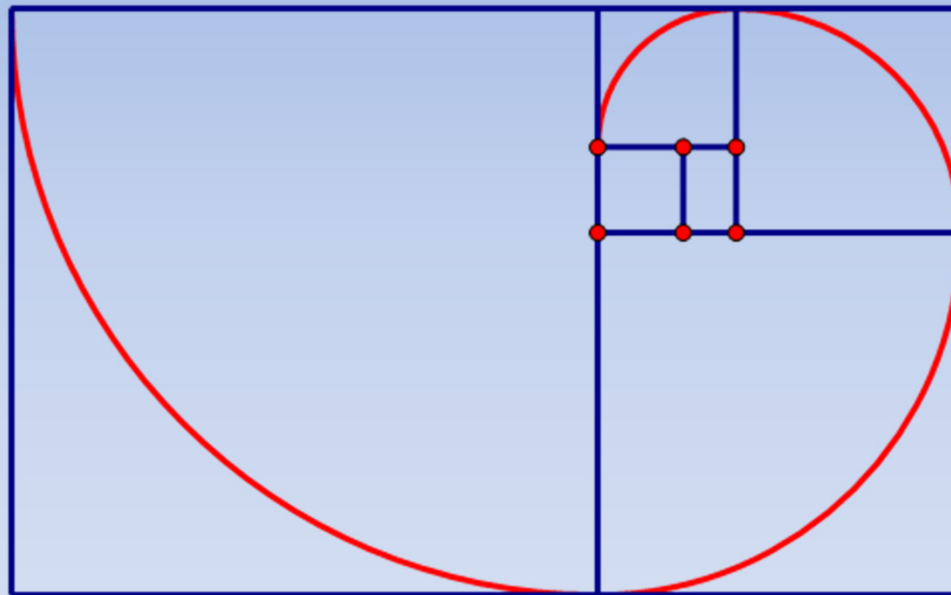
Golden Spiral

In each square construct a quarter circle:



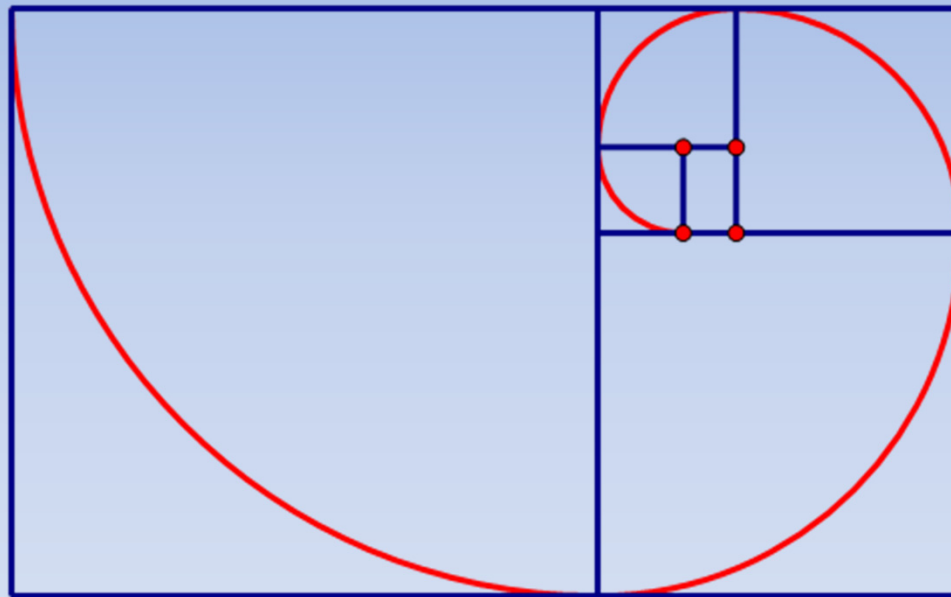
Golden Spiral

In each square construct a quarter circle:



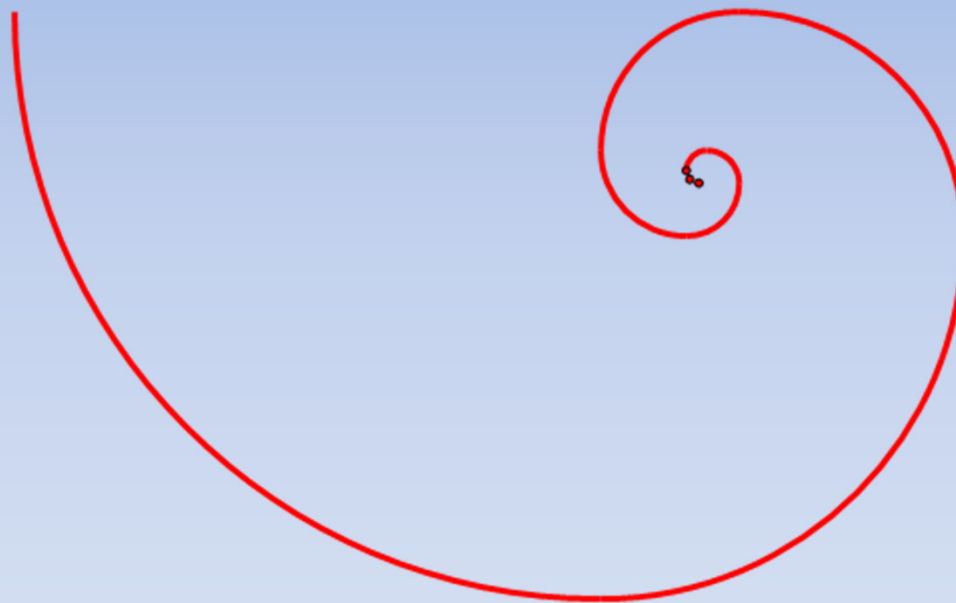
Golden Spiral

In each square construct a quarter circle:



Golden Spiral

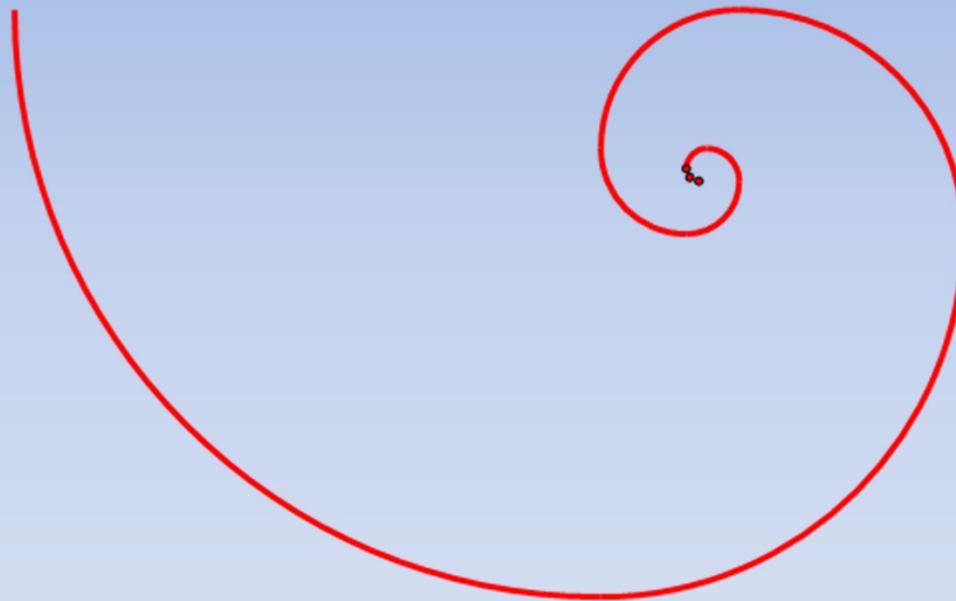
In each square construct a quarter circle:



Golden Spiral

This does give a logarithmic spiral:

$\Theta = a \ln(b \cdot r)$, in polar coordinates



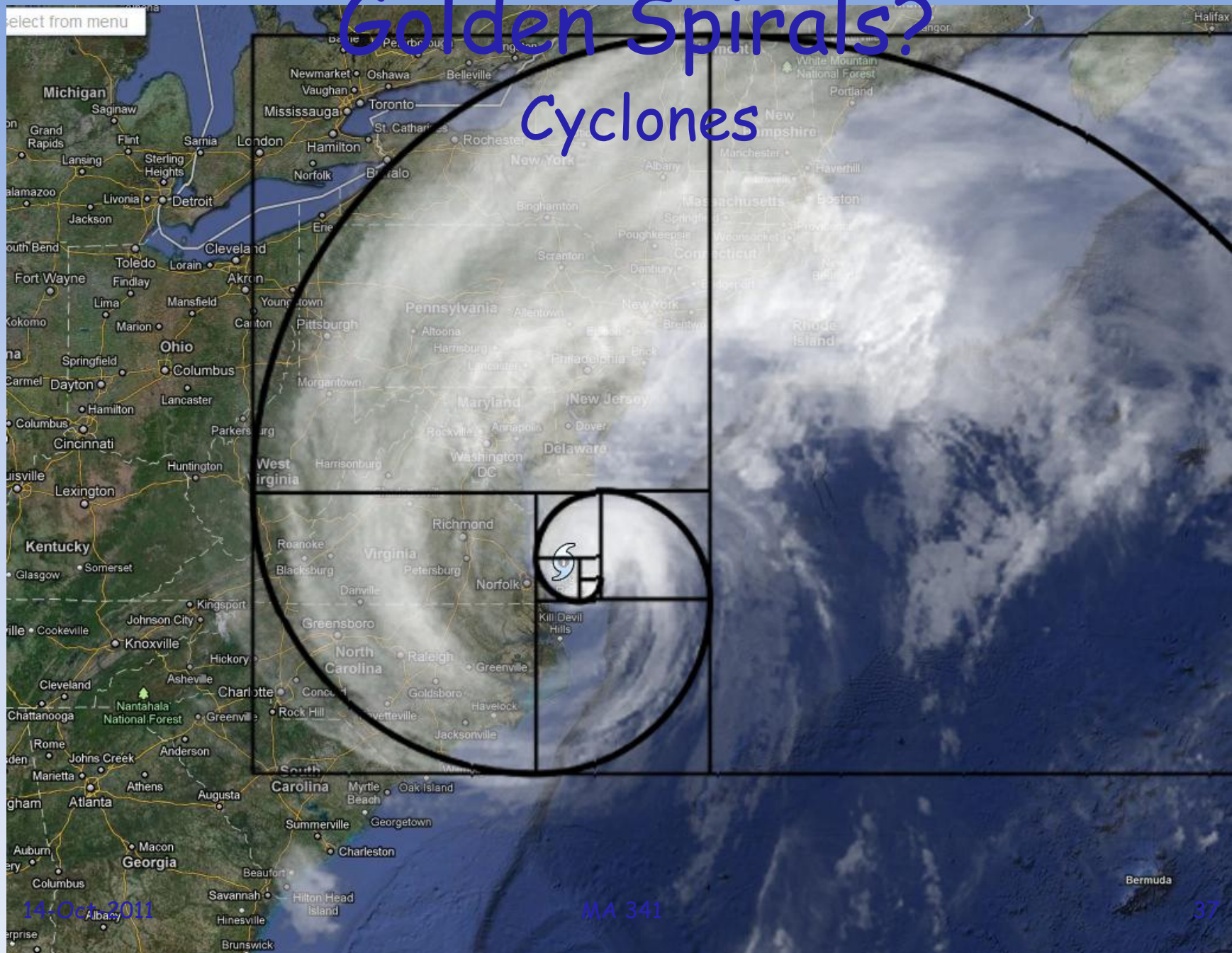
Golden Spirals?



Golden Spirals?

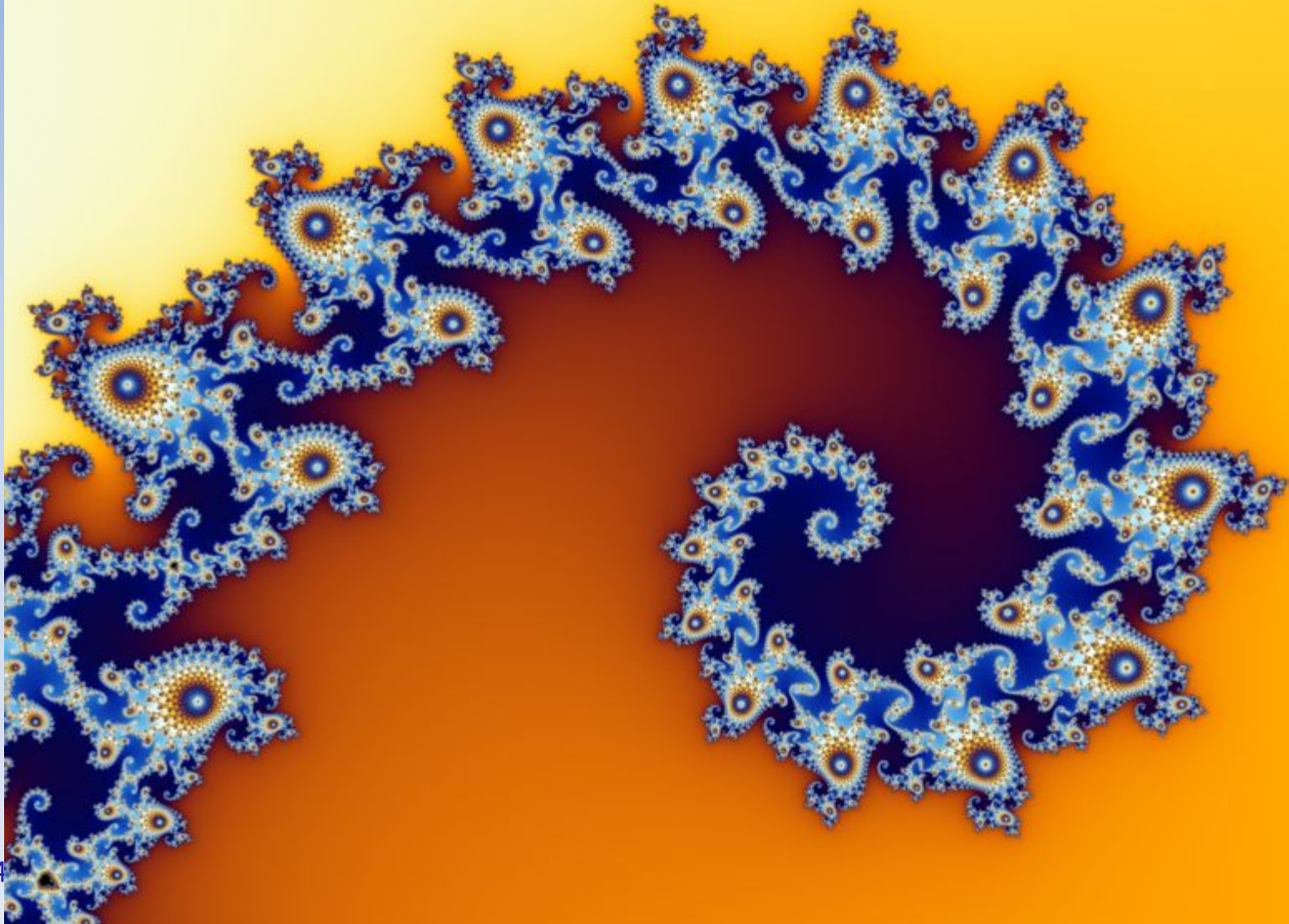


Golden Spirals? Cyclones



Golden Spirals?

Mandelbrot Set



Golden Spirals?

Flower heads



Golden Spirals?

Phyllotaxis
(<http://www.math.smith.edu/phylo/>)



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How?

Consider the following list of numbers:

1,1,2,3,5,8,13,21,34,55,89,144,...

This is the Fibonacci sequence $\{F_n\}$. We are interested in the quotients

$$F_{n+1}/F_n$$

Fibonacci Connection

$F_0=1$	$F_9=55$	$F_5/F_4 = 1.6$
$F_1=1$	$F_{10}=89$	$F_6/F_5 = 1.625$
$F_2=2$	$F_{11}=144$	$F_7/F_6 = 1.6153.$
$F_3=3$	$F_{12}=233$	$F_8/F_7 = 1.6190$
$F_4=5$	$F_{13}=377$	$F_9/F_8 = 1.6176$
$F_5=8$	$F_1/F_0 = 1$	$F_{10}/F_9 = 1.618181$
$F_6=13$	$F_2/F_1 = 2$	$F_{11}/F_{10} = 1.617977$
$F_7=21$	$F_3/F_2 = 1.5$	$F_{12}/F_{11} = 1.618055$
$F_8=34$	$F_4/F_3 = 1.6667$	$F_{13}/F_{12} = 1.618025$

Fibonacci Connection

Is it true that:

$$\lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n} = \varphi ?$$

Note: Since $\varphi^2 = \varphi + 1$, multiplying by φ^{n-1} gives a Fibonacci type relationship:

$$\varphi^{n+1} = \varphi^n + \varphi^{n-1}$$

$$(F_{n+1} = F_n + F_{n-1})$$

And it so happens that

$$F_n = \frac{\varphi^n - (1 - \varphi)^n}{\sqrt{5}} = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}$$

Other representations?

Consider the following sequence:

$$a_0 = 1, a_1 = \sqrt{1 + \sqrt{a_0}}, \dots, a_n = \sqrt{1 + \sqrt{a_{n-1}}}$$

$$a_n = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}}}$$

What is $\lim a_n$?

First, we need to know that $\{a_n\}$ has a limit.
This can be shown with calculus.

Let $L = \lim a_n$

Then,

Other representations?

$$L = \lim_{n \rightarrow \infty} \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}}}$$

$$L = \sqrt{1 + \lim_{n \rightarrow \infty} \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}}}}$$

$$L = \sqrt{1 + L}$$

$$L^2 = L + 1$$

Hey!!! $L = \varphi$, so

$$\varphi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}}}$$

Another representations

Consider the following sequence:

$$b_0 = 1, b_1 = 1 + \frac{1}{1 + b_0}, \dots, b_n = 1 + \frac{1}{1 + b_{n-1}}$$

$$b_n = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\ddots}}}}$$

Other representations?

What is $\lim b_n$?

First, we need to know that $\{b_n\}$ has a limit. This can be shown with calculus.

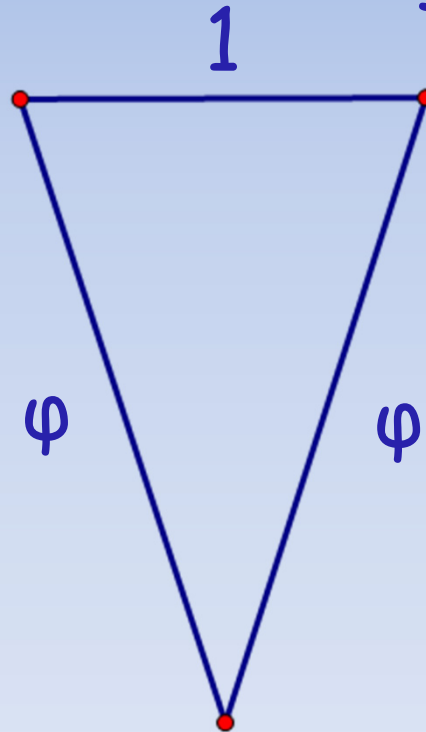
Let $L = \lim b_n$

$$\text{Then, } L = 1 + \frac{1}{L}$$

Again, $L = \varphi$.

Golden Triangle

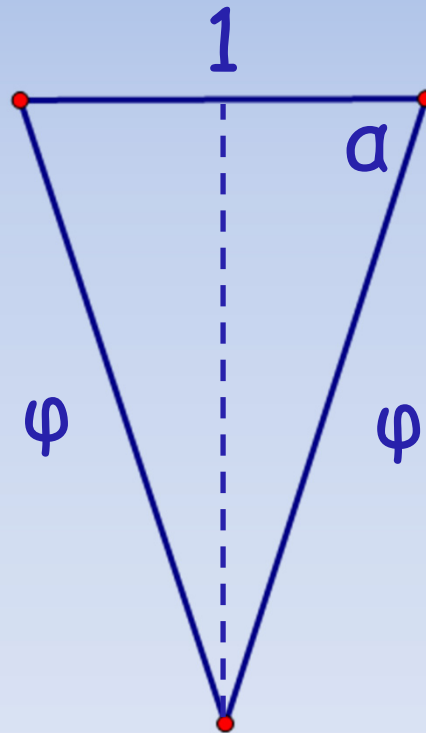
A golden triangle is an isosceles triangle where the ratio of the longer side to the base is φ .



Golden Triangle

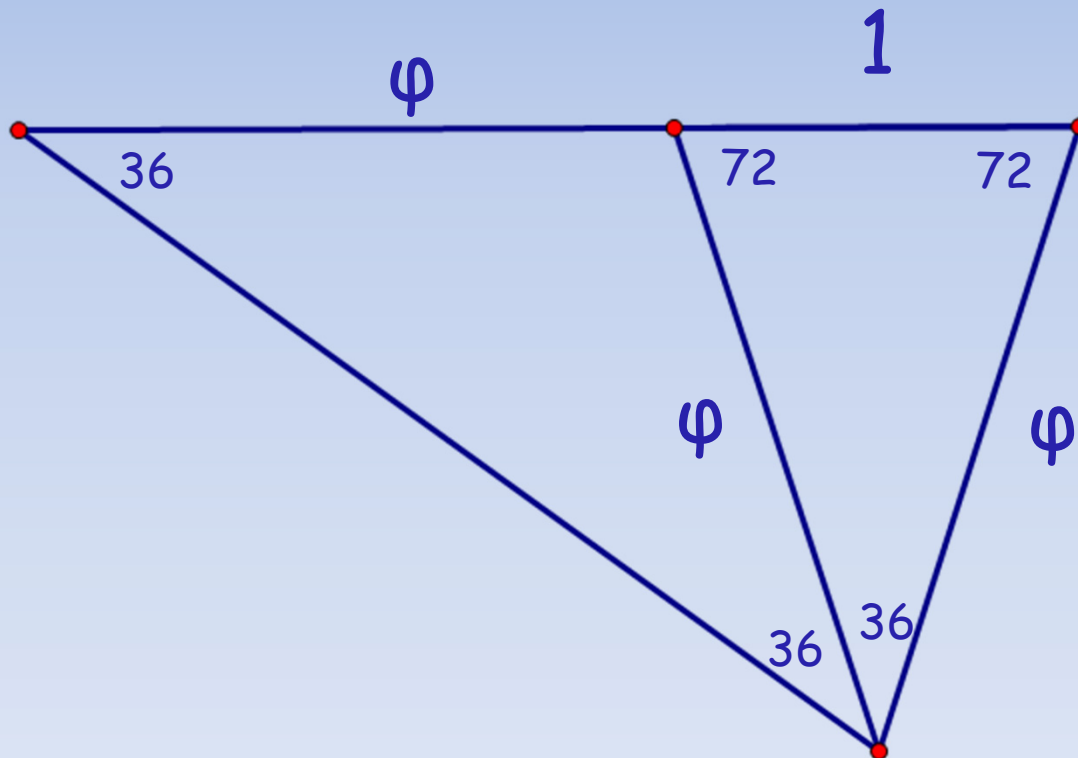
What are the angles?

$\cos(a) = \frac{1}{2}/\varphi \Rightarrow a = 72^\circ$, making summit angle 36°



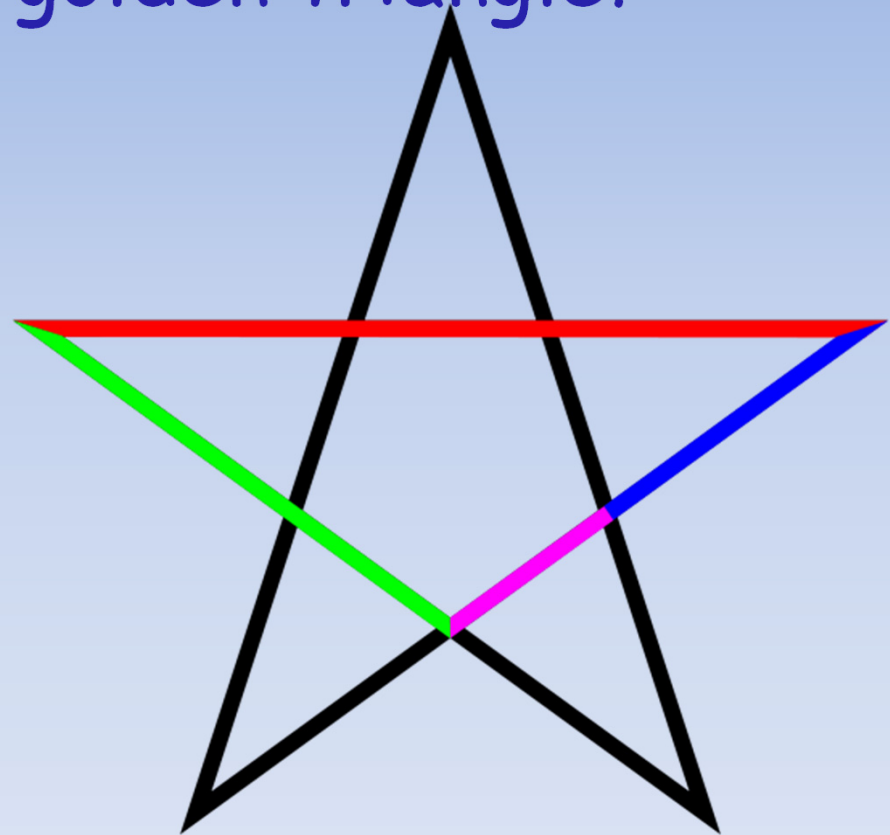
Golden Triangle

- $1 + \varphi = \varphi^2$, so the larger triangle is similar to the smaller with similarity φ .



Golden Triangle

Where do we find a golden triangle?



Golden Triangle

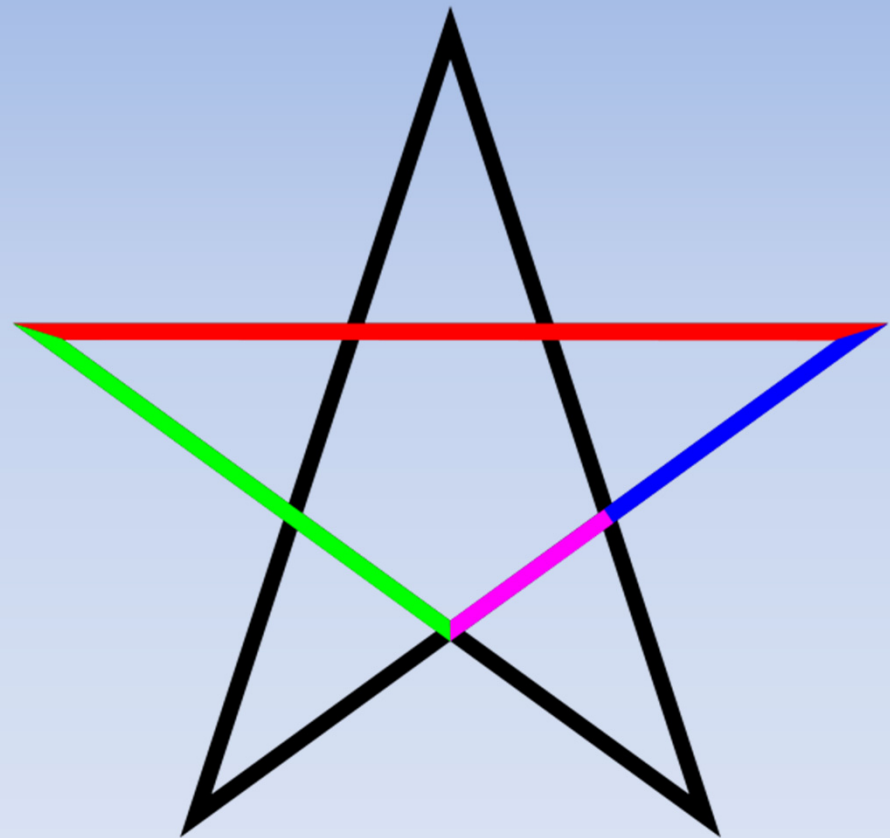
In fact:

Red/green

= green/blue

= blue/pink

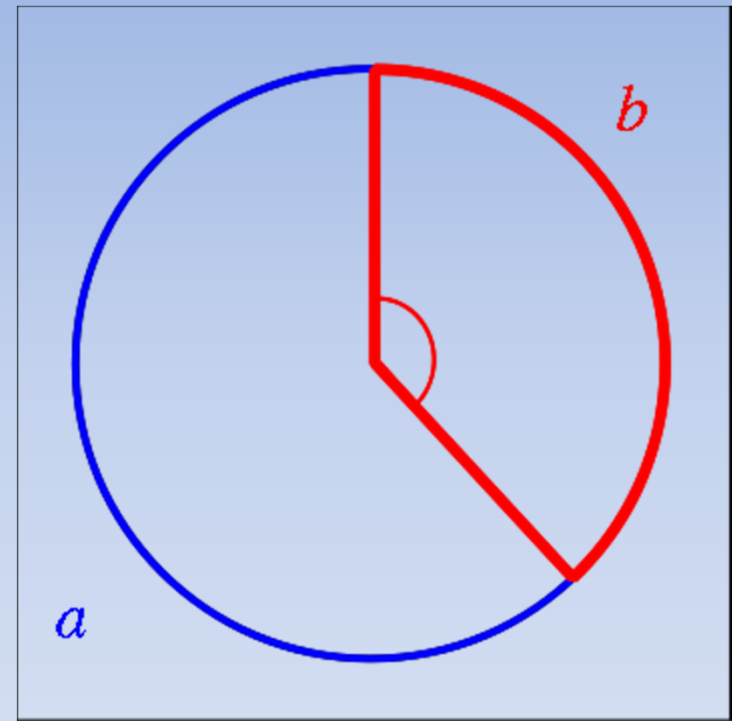
= φ



Golden Angle

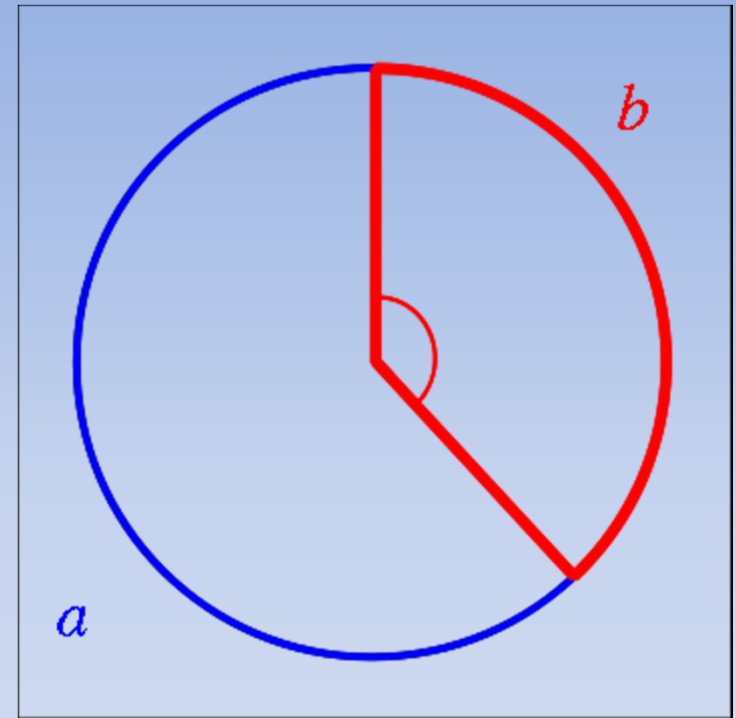
If ratio of arcs
 $a/b = \varphi$, then angle
subtended by smaller
arc is called **golden
angle**.

It measures
approximately 137.51° ,
or about 2.399963
radians.



Golden Angle

It is exactly

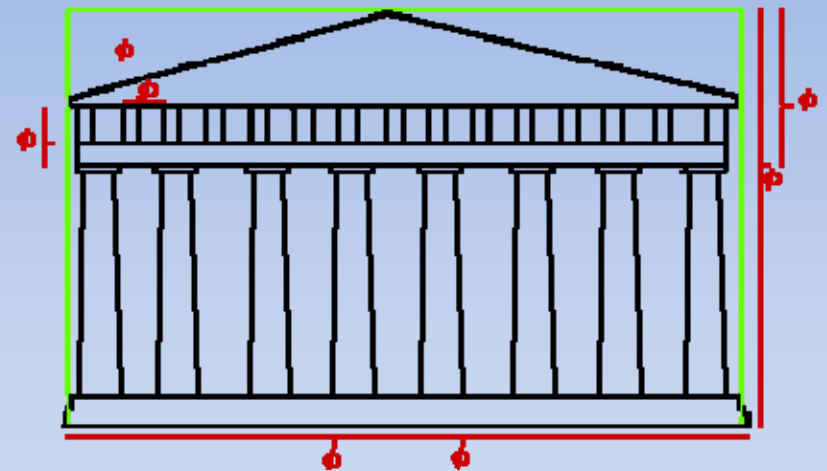


$$360\left(1 - \frac{1}{\varphi}\right) = 360(2 - \varphi) = \frac{360}{\varphi^2} = 180(3 - \sqrt{5})$$

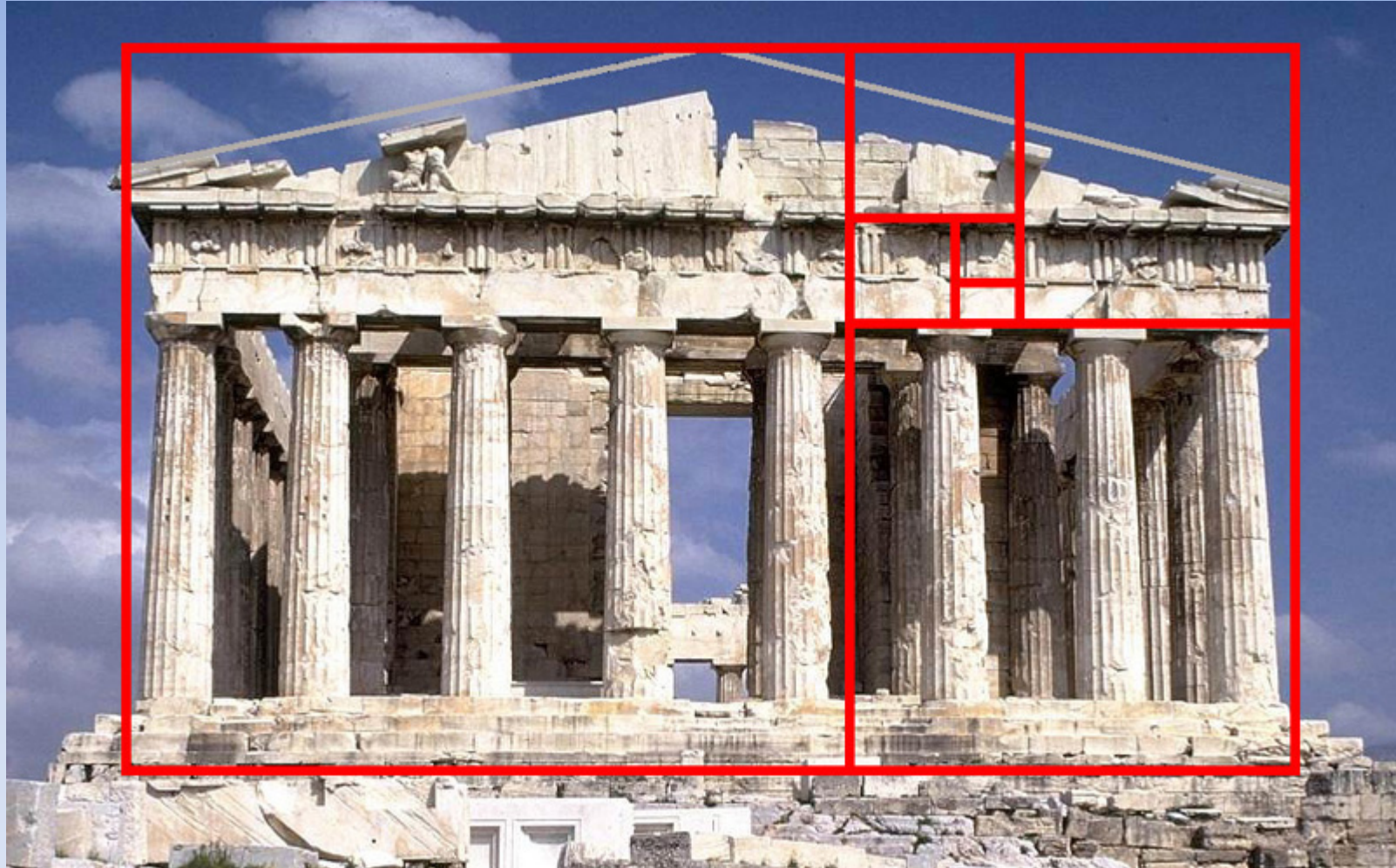
Golden Ratio and Art

- <http://www.goldennumber.net/>
- <http://www.keplersdiscovery.com/DivineProportion.html>
- <http://spiralzoom.com/>

The Parthenon



The Parthenon



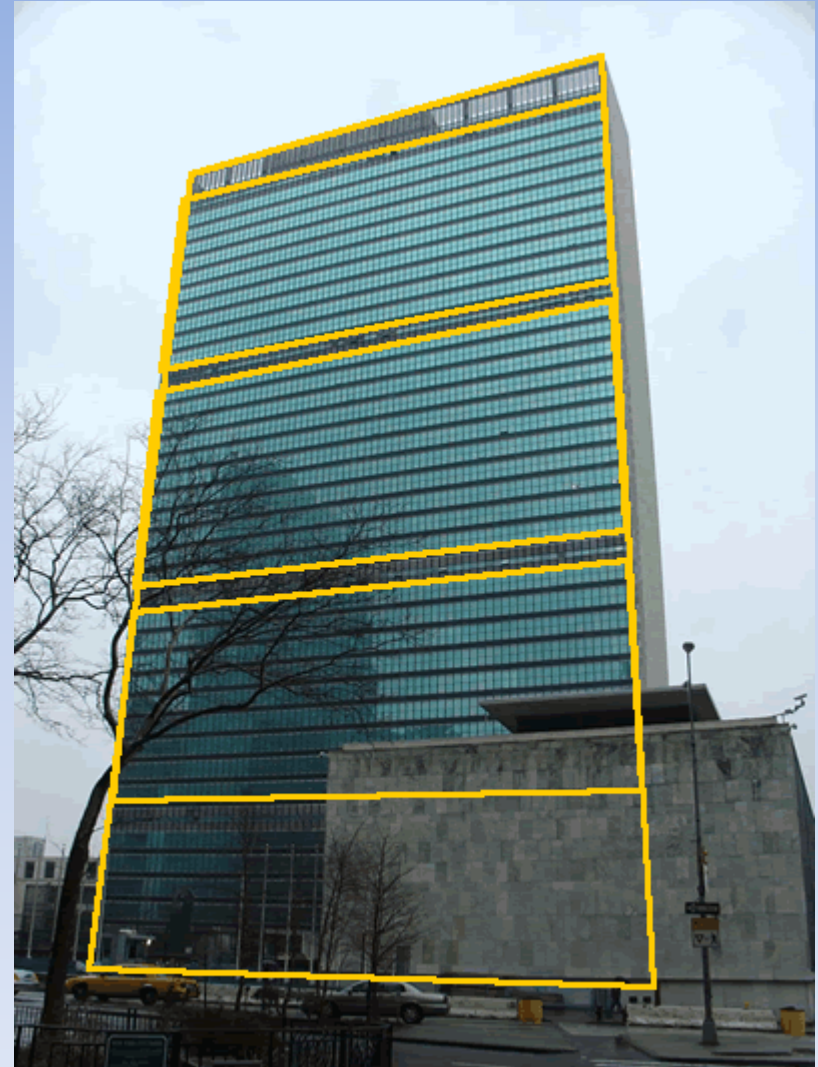
The Acropolis, Porch of the Maidens



Chartes Cathedral & UN Building



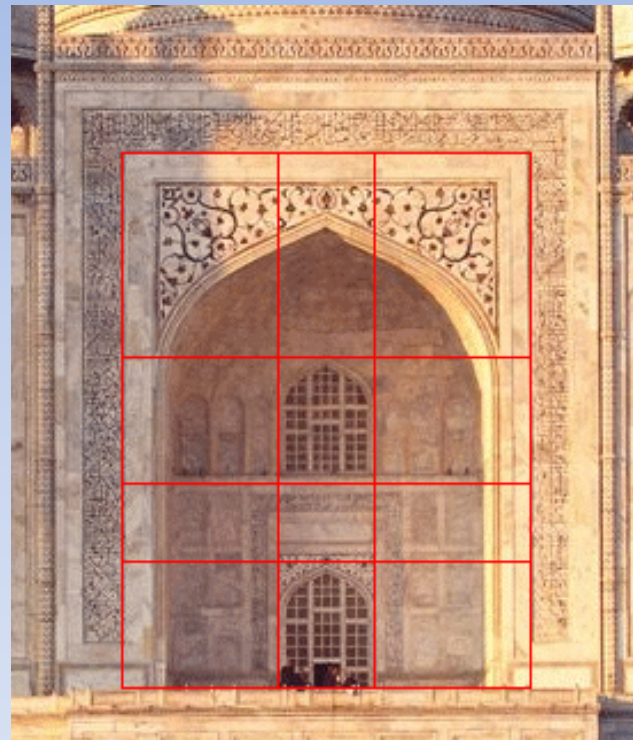
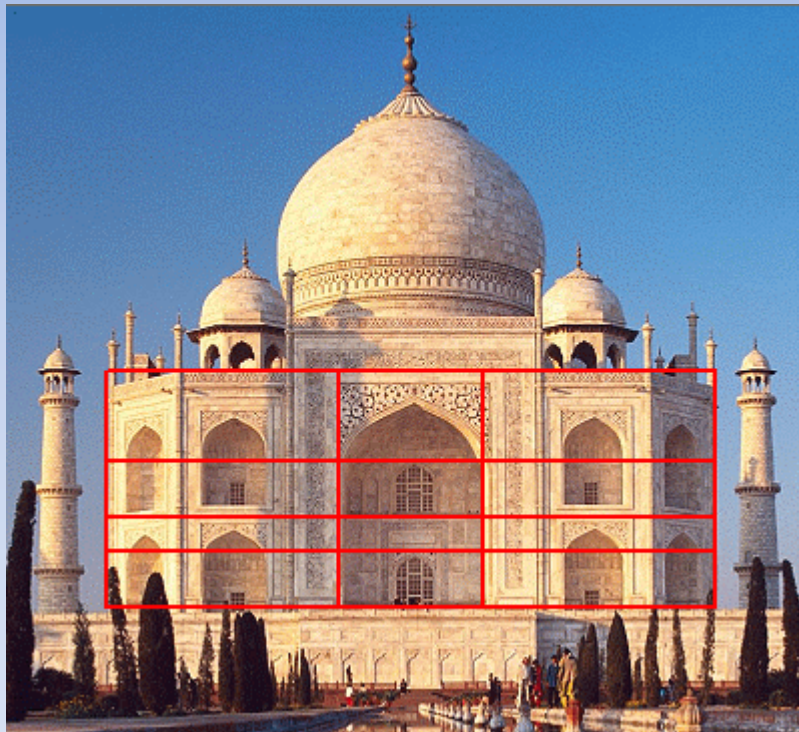
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Taj Mahal



Fra Luca Pacioli



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Pacioli's De divina proportione

Written in Milan in 1496-98,

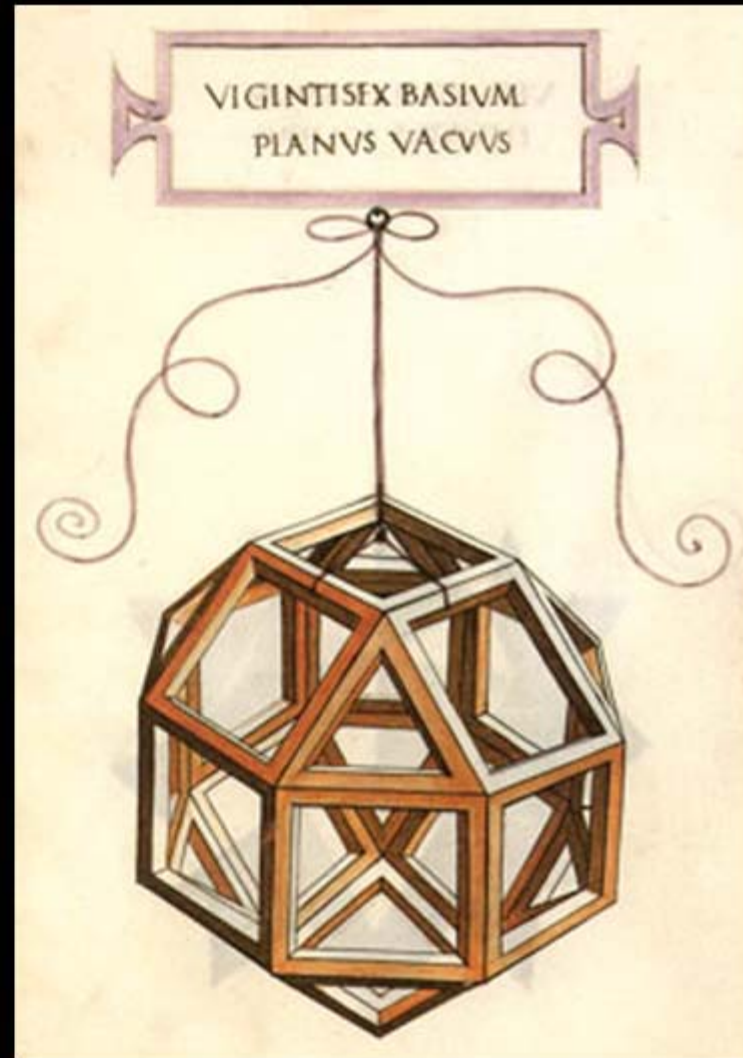
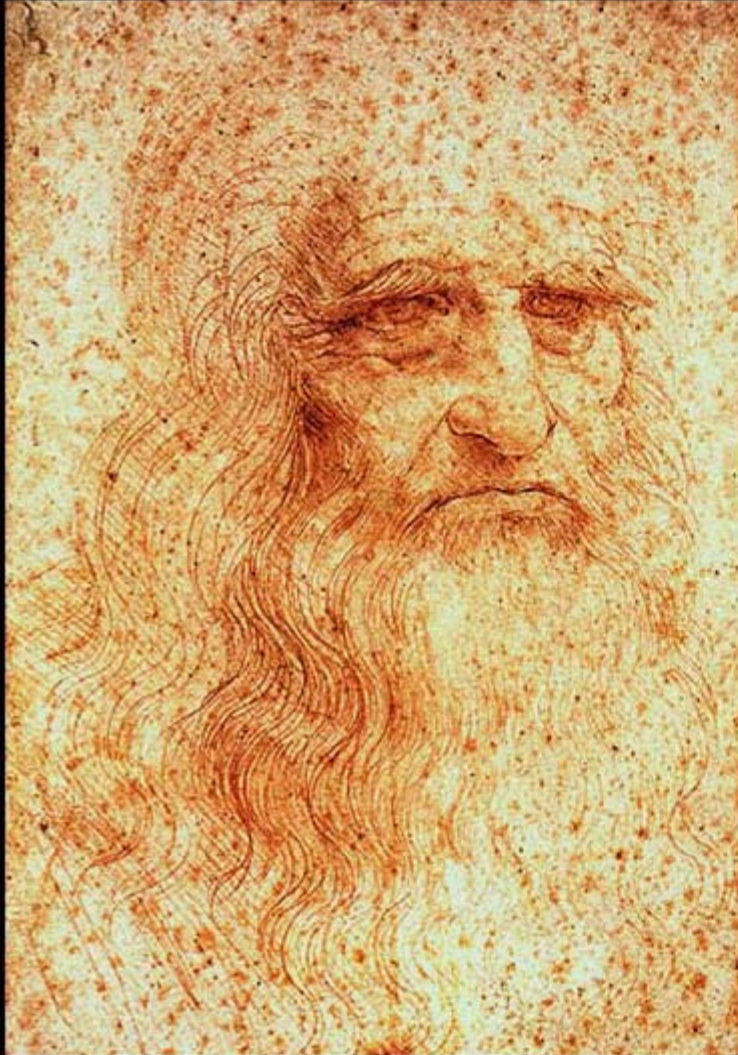
Published in Venice in 1509

The subject - mathematical and artistic proportion, especially mathematics of golden ratio and application in architecture.

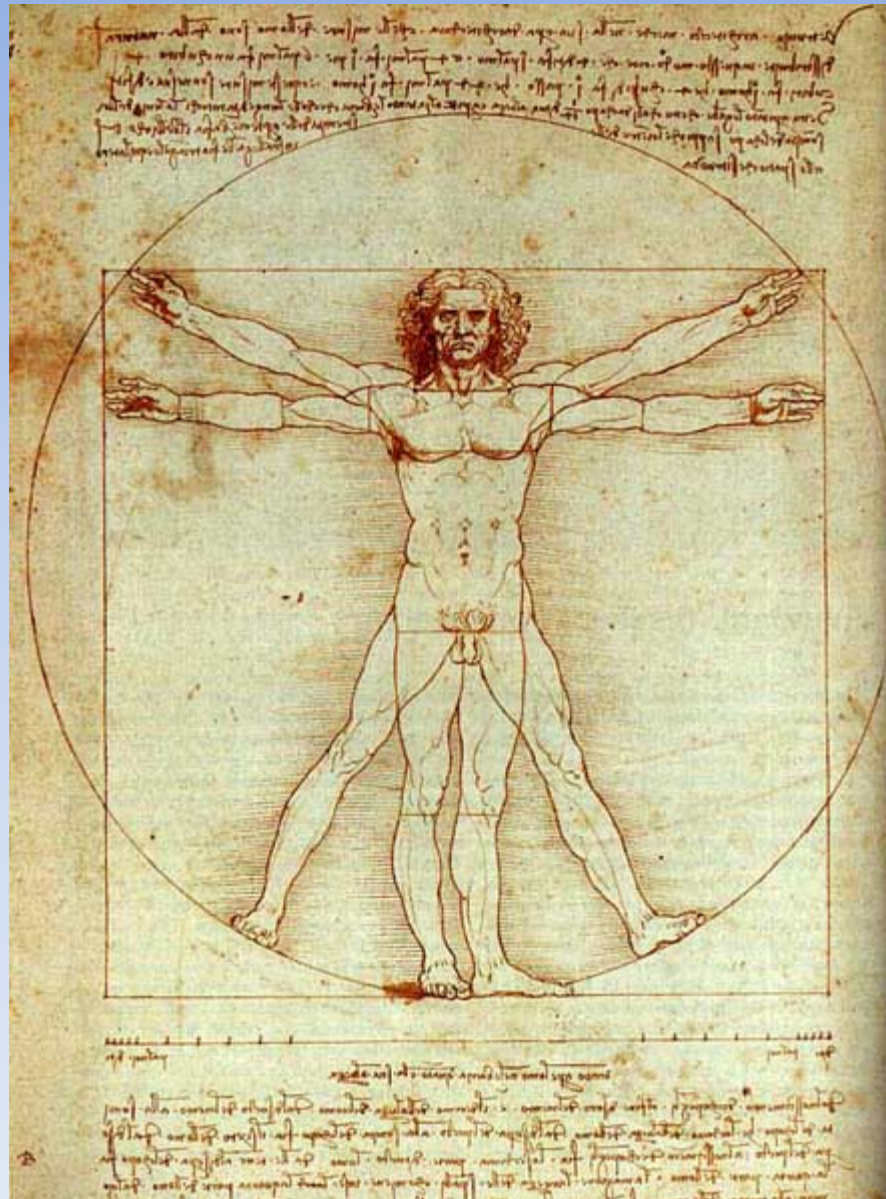
Leonardo da Vinci drew illustrations of regular solids in *De divina proportione* while living with and taking mathematics lessons from Pacioli.

Discusses use of perspective by painters such as Piero della Francesca, Melozzo da Forlì, and Marco Palmezzano

Leonardo da Vinci



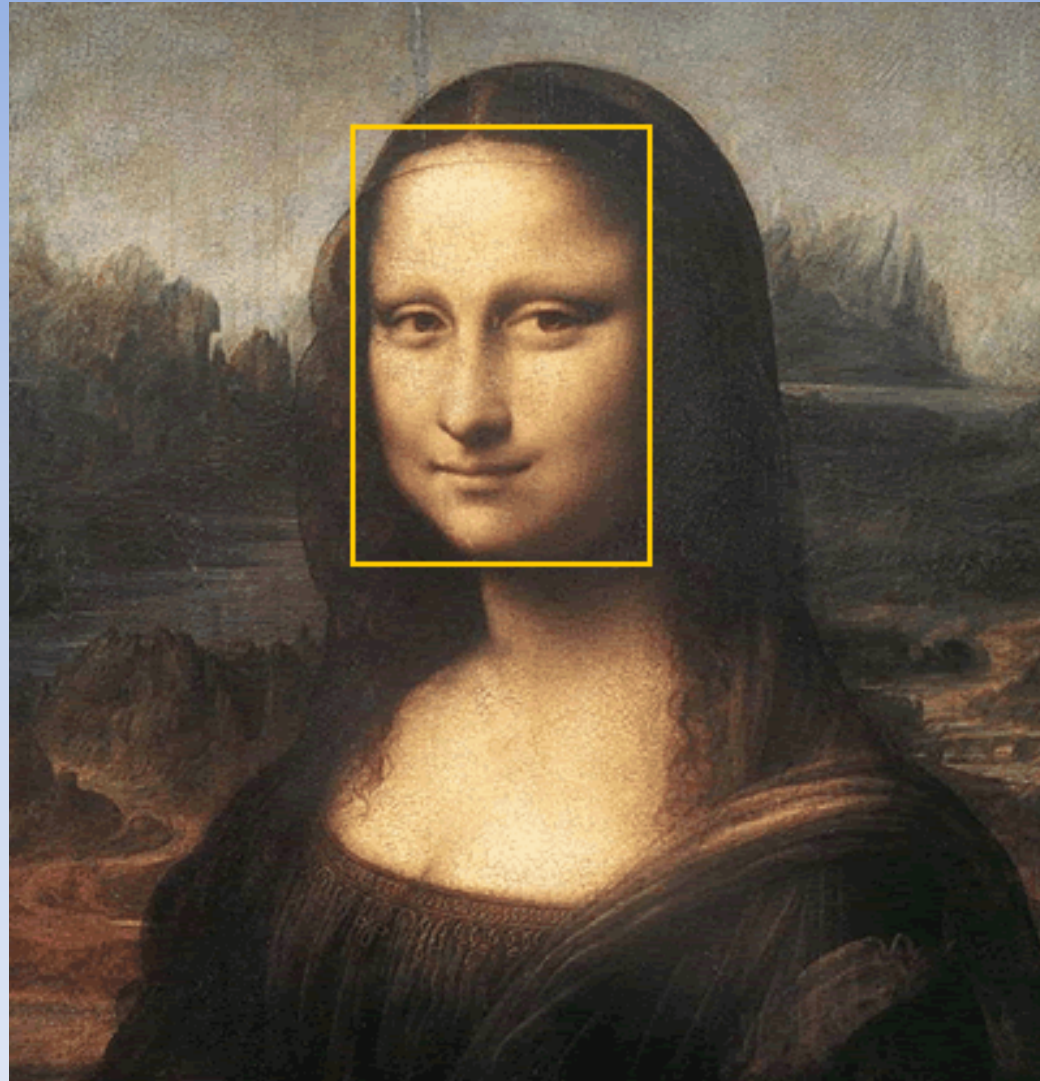
Leonardo da Vinci

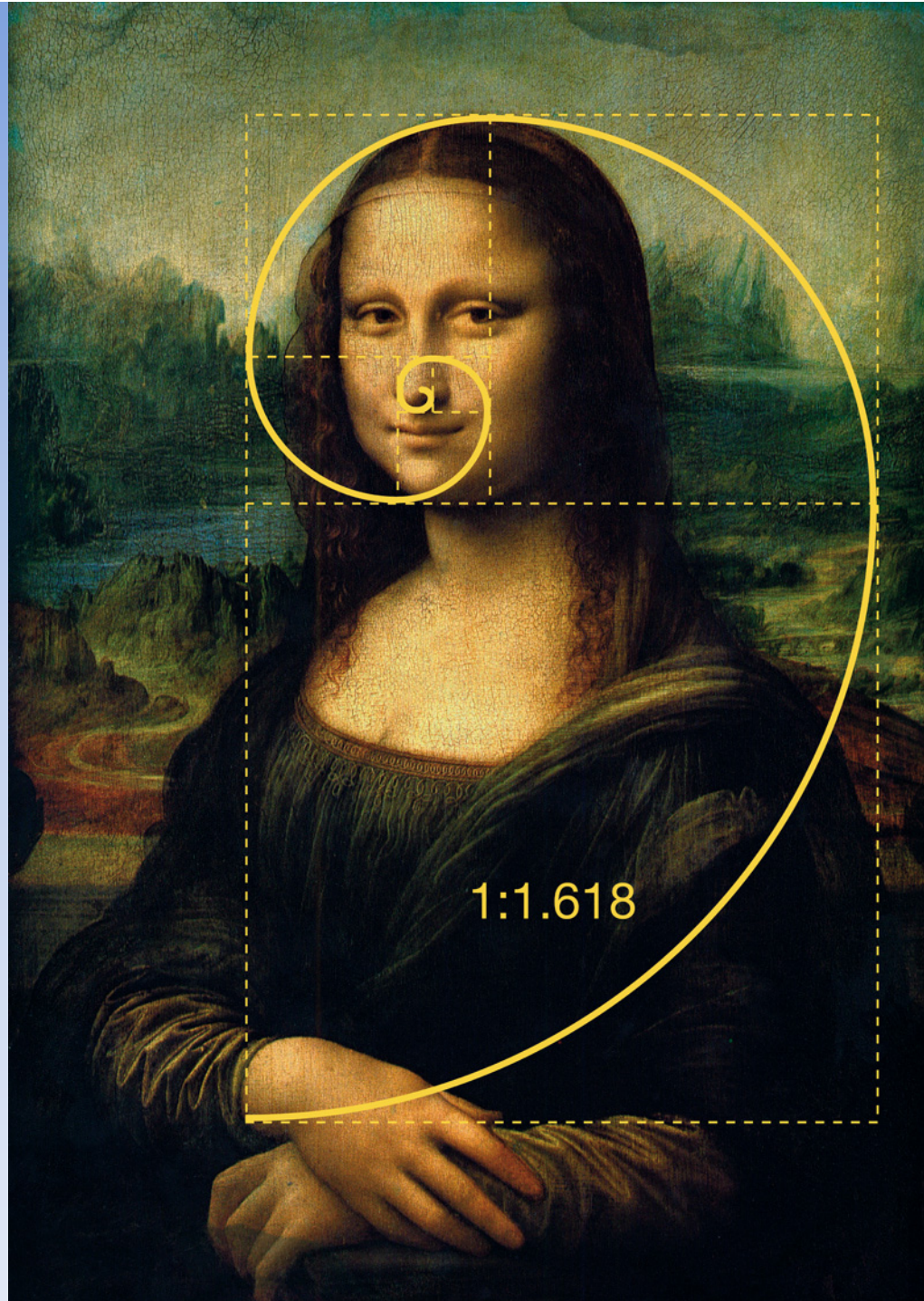


Facial Study

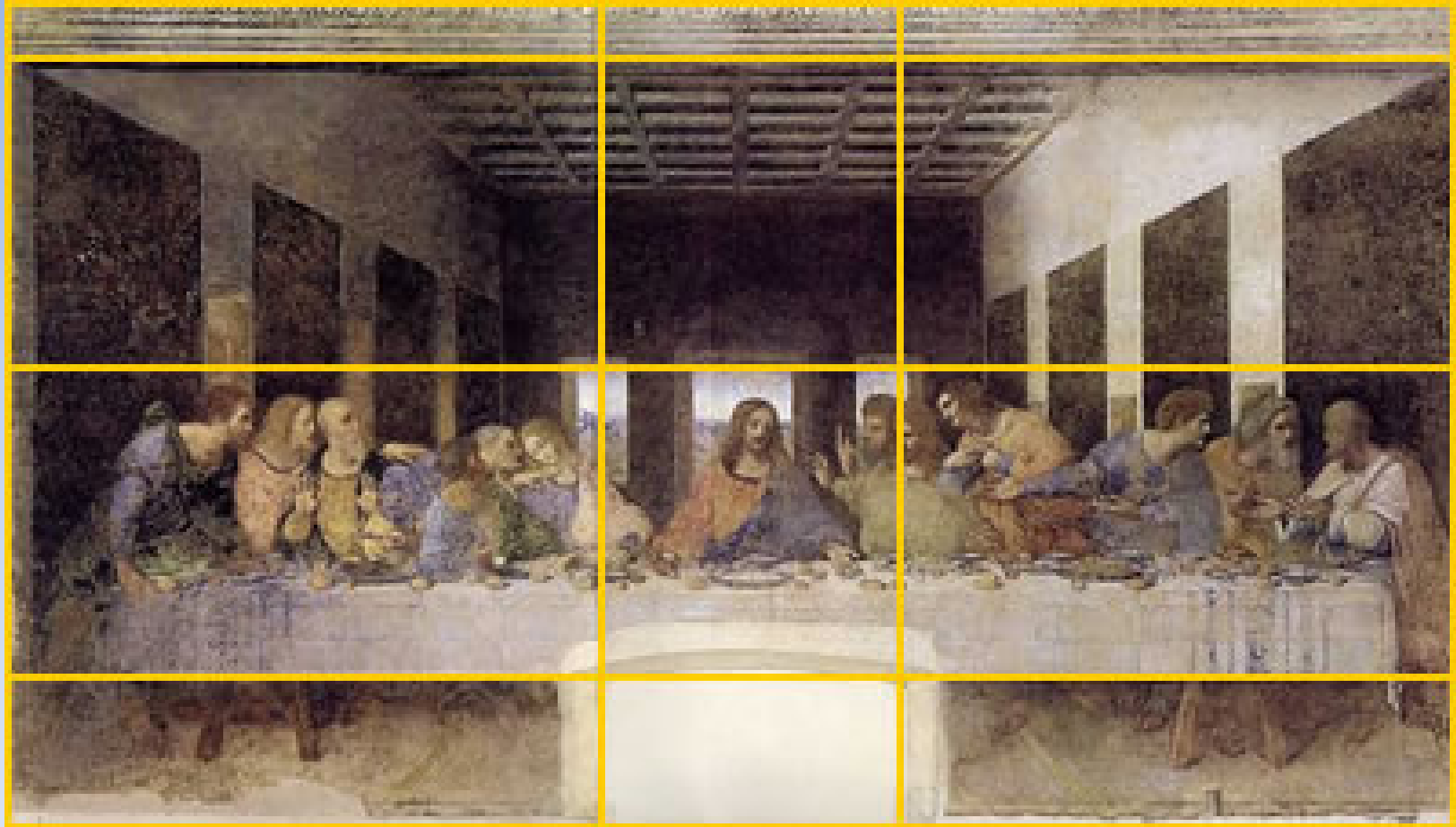


Mona Lisa

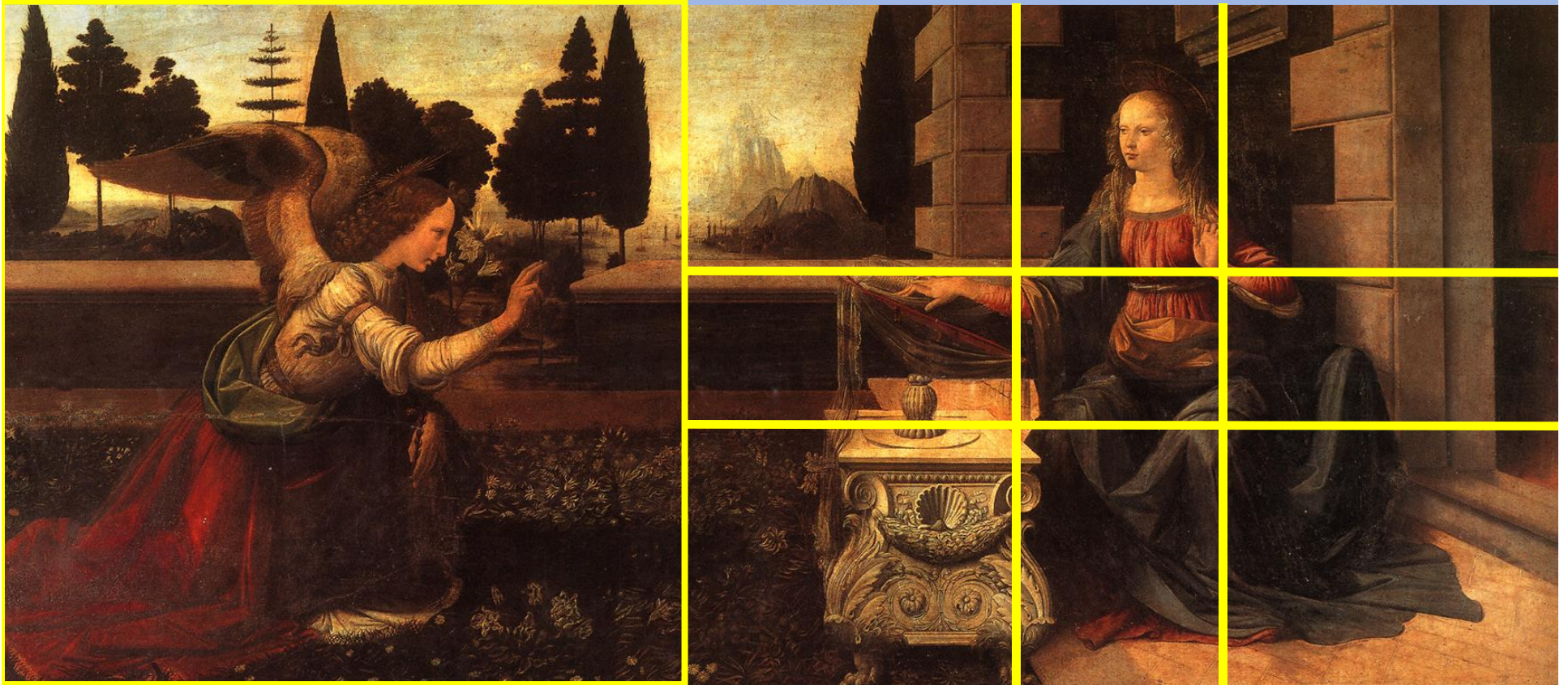




The Last Supper

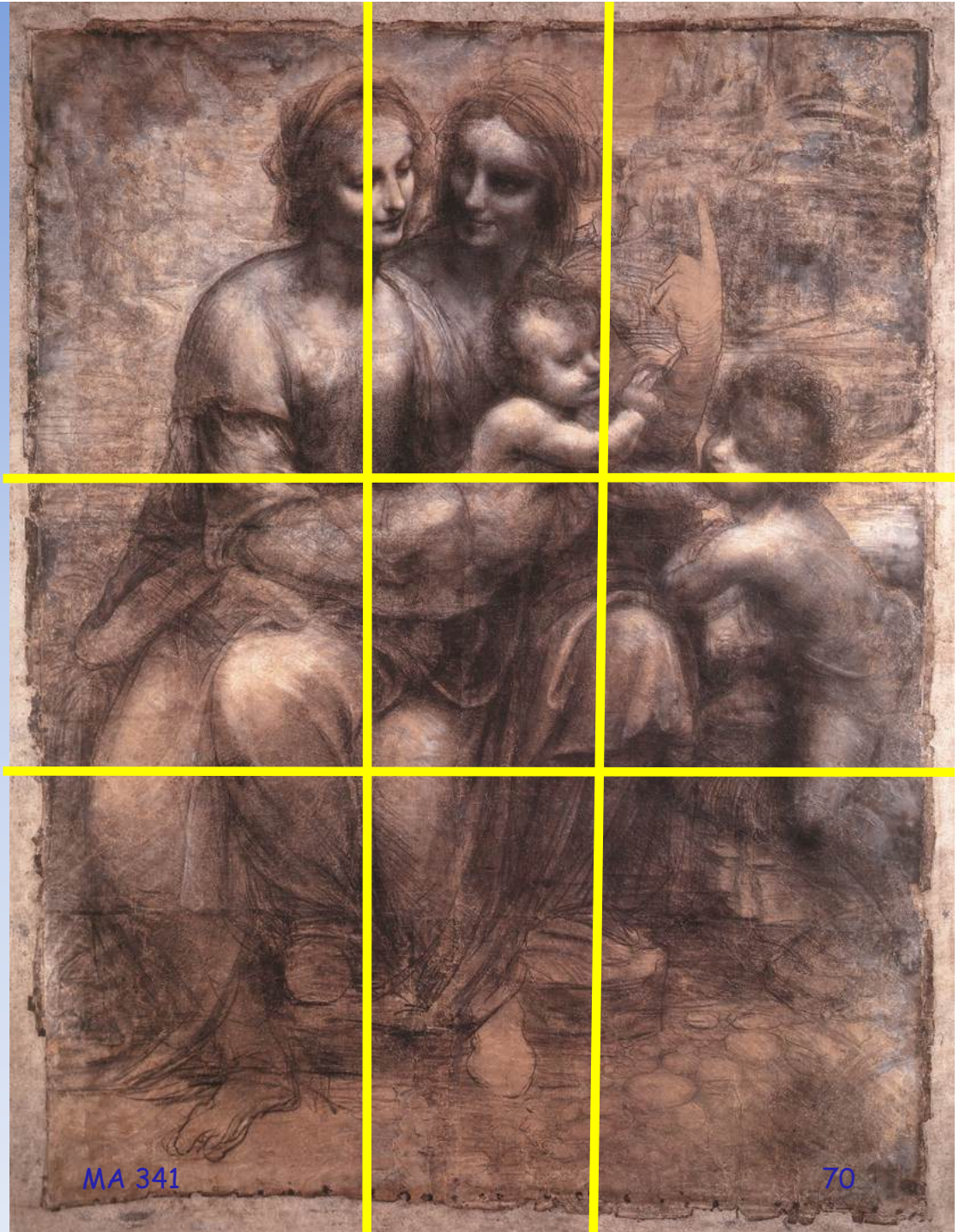


The Annunciation



Madonna and Child with St. Anne and St. John

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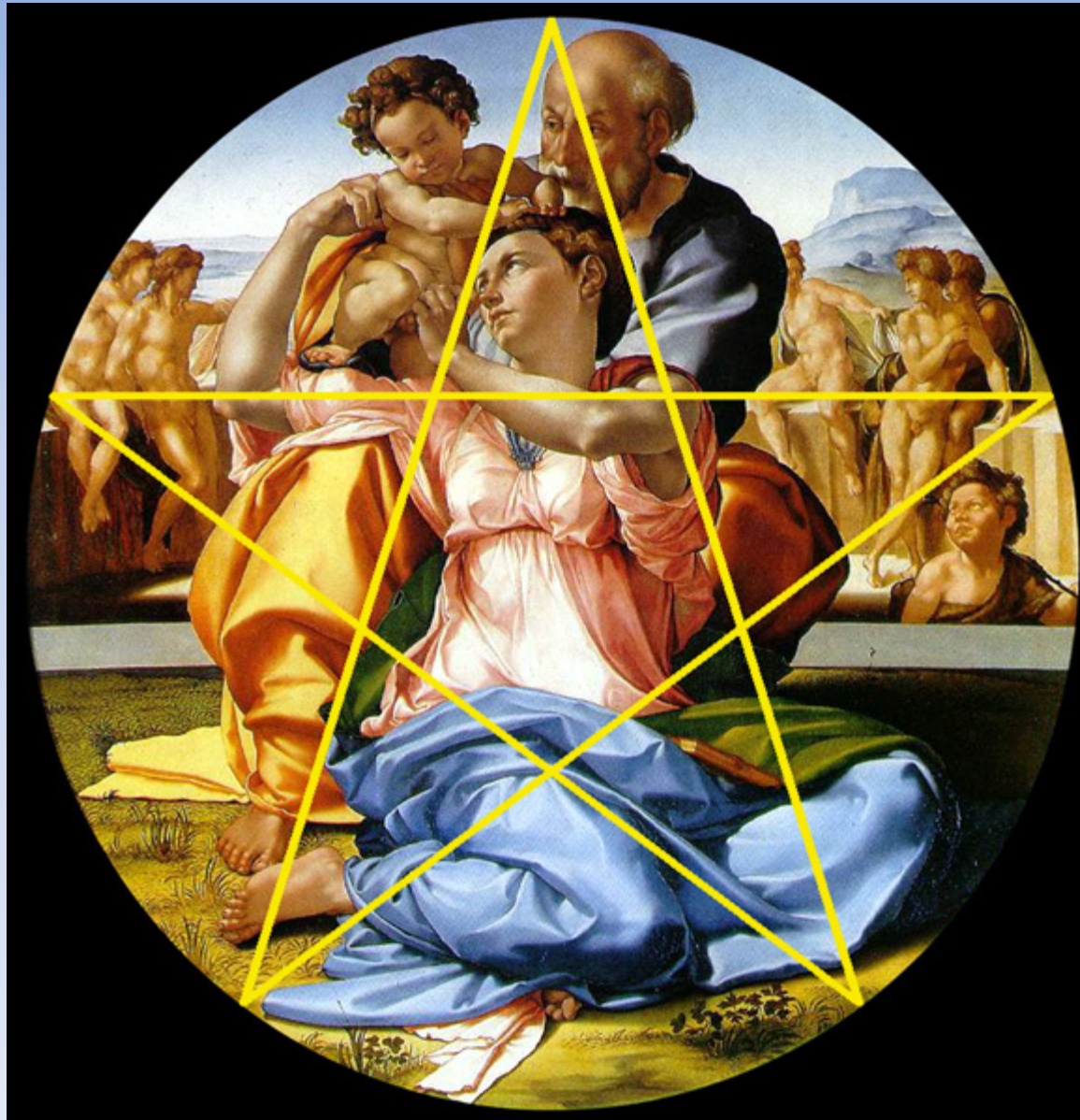


Michaelangelo & Raphael

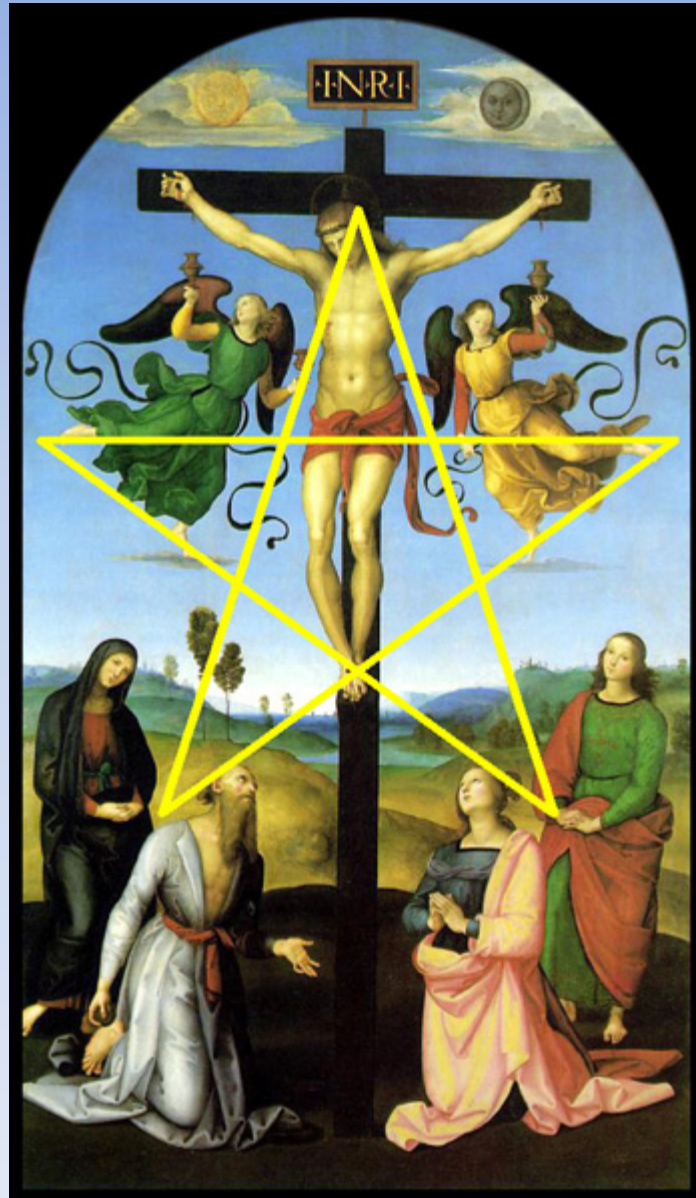
<http://web.me.com/paulscott.info/place/pm10/pm10.html>



The Holy Family - Michelangelo

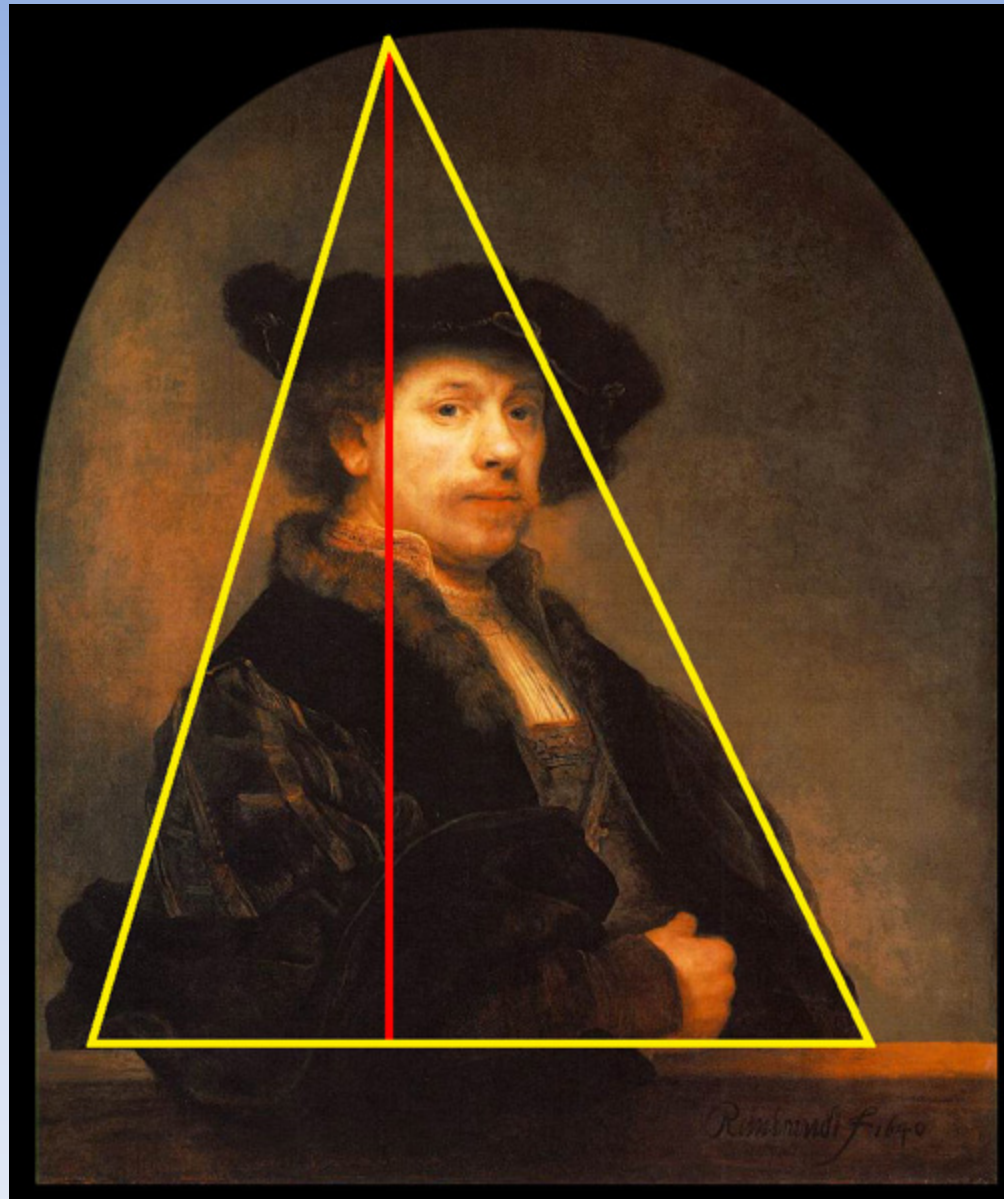


The Crucifixion - Raphael

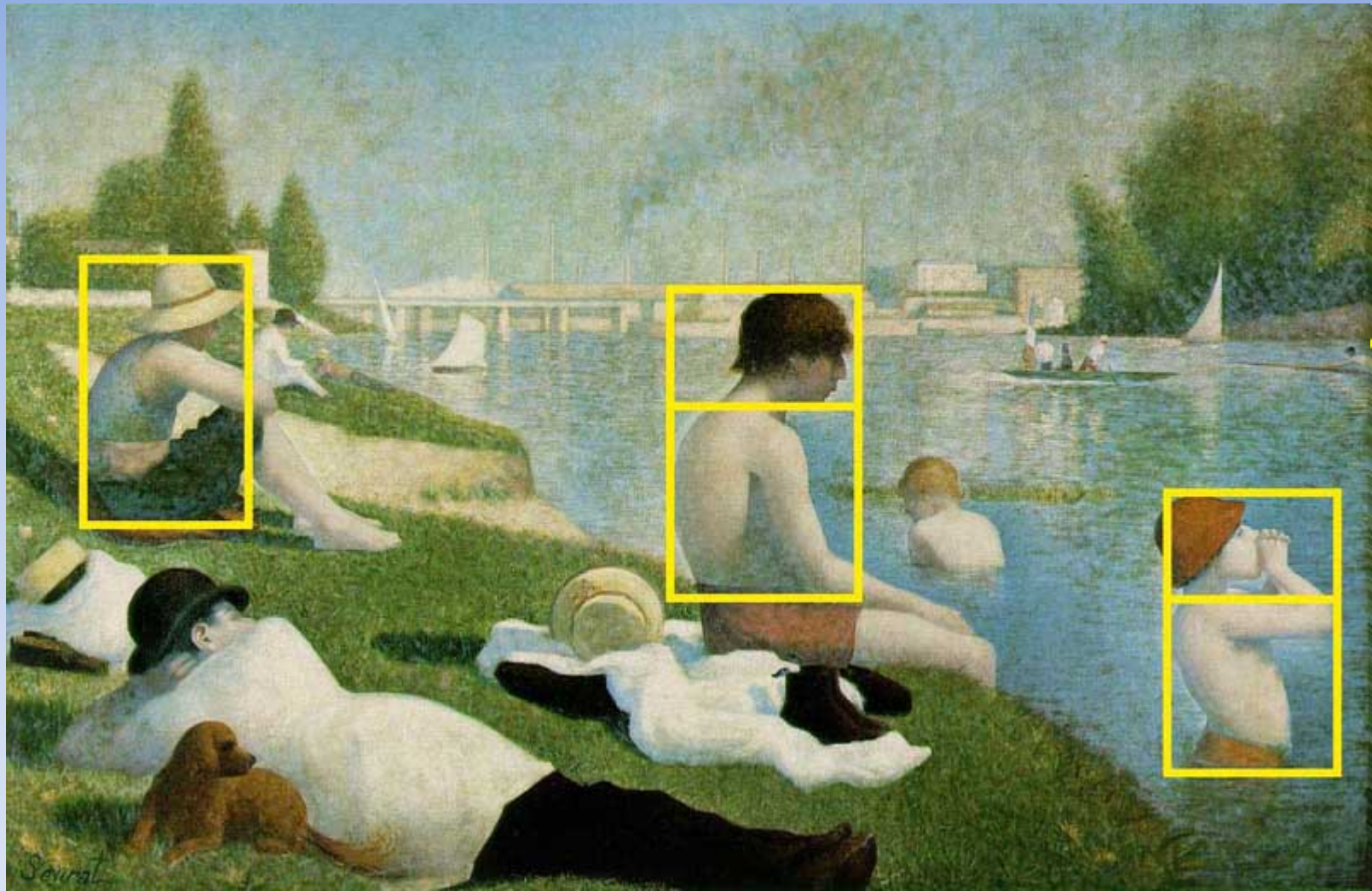


Self Portrait - Rembrandt

Red line
divides base
into golden
mean.



The Bathers - Seurat



The Perfect Face



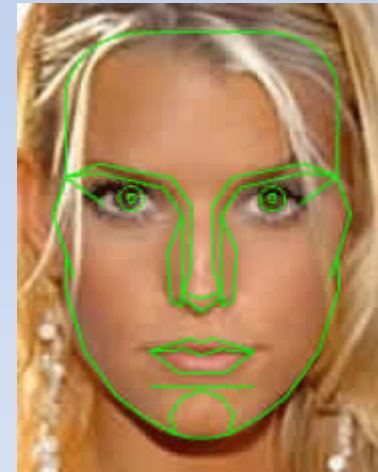
Dr. Stephen Marquardt
(http://www.beautyanalysis.com/index2_mba.htm)

Claims that this gives the most beautiful shape of human face

Used decagons and pentagons and embodies ϕ in all their dimensions.

The Perfect Face

This mask of the human face is based on the Golden Ratio. The proportions of the length of the nose, the position of the eyes and the length of the chin, all conform to some aspect of the Golden Ratio.



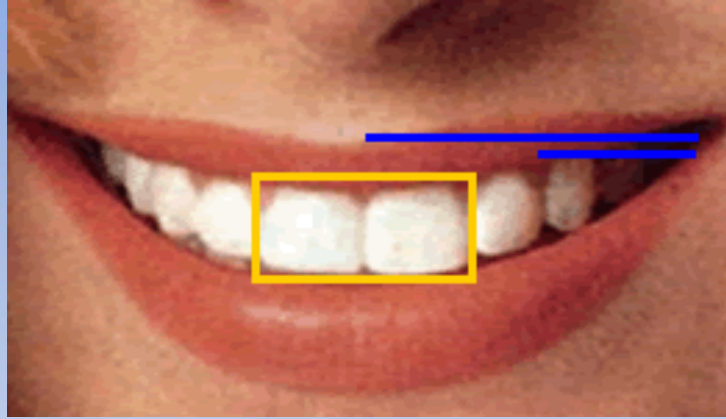


The Perfect Face?



The Perfect Face?

The Perfect Smile



- Front two teeth form a golden rectangle
- Also a golden ratio in height to width of center two teeth.
- Ratio of the width of the 2 center teeth to those next to them is φ .
- Ratio of width of smile to 3rd tooth from center is φ

See Donald Duck in Mathemagic Land by
Disney Studios

http://www.youtube.com/watch?v=oT_Bxgah9zc

Without mathematics, there is no art.

- Fra. Luca Pacioli