The Radical Axis

MA 341 – Topics in Geometry
Lecture 24

What is the Radical Axis?

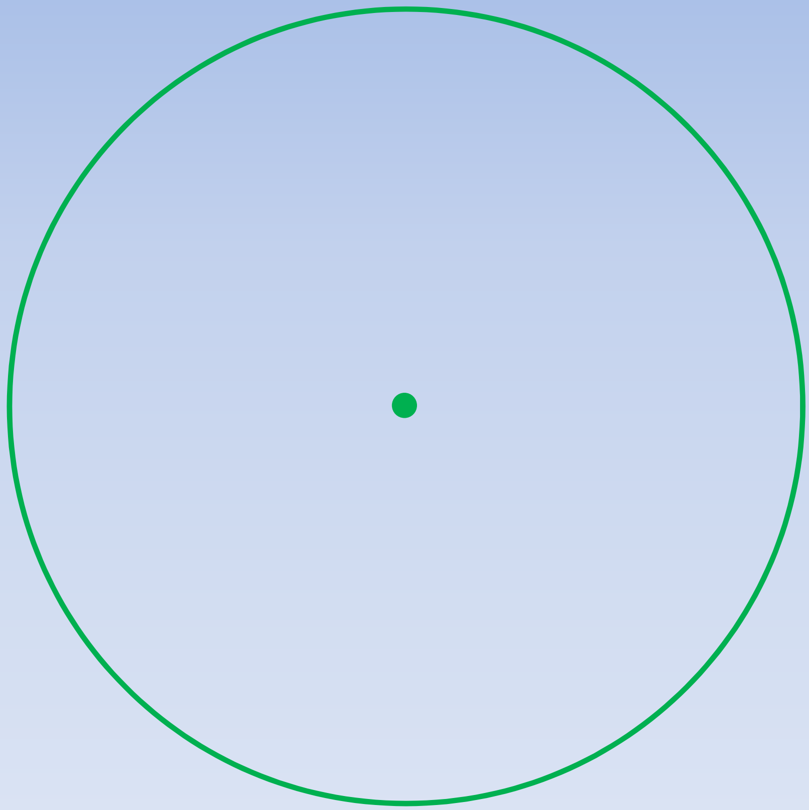
Is it $\sqrt{x}$? $\sqrt{y}$?

Is it one of the Axis powers gone rogue?

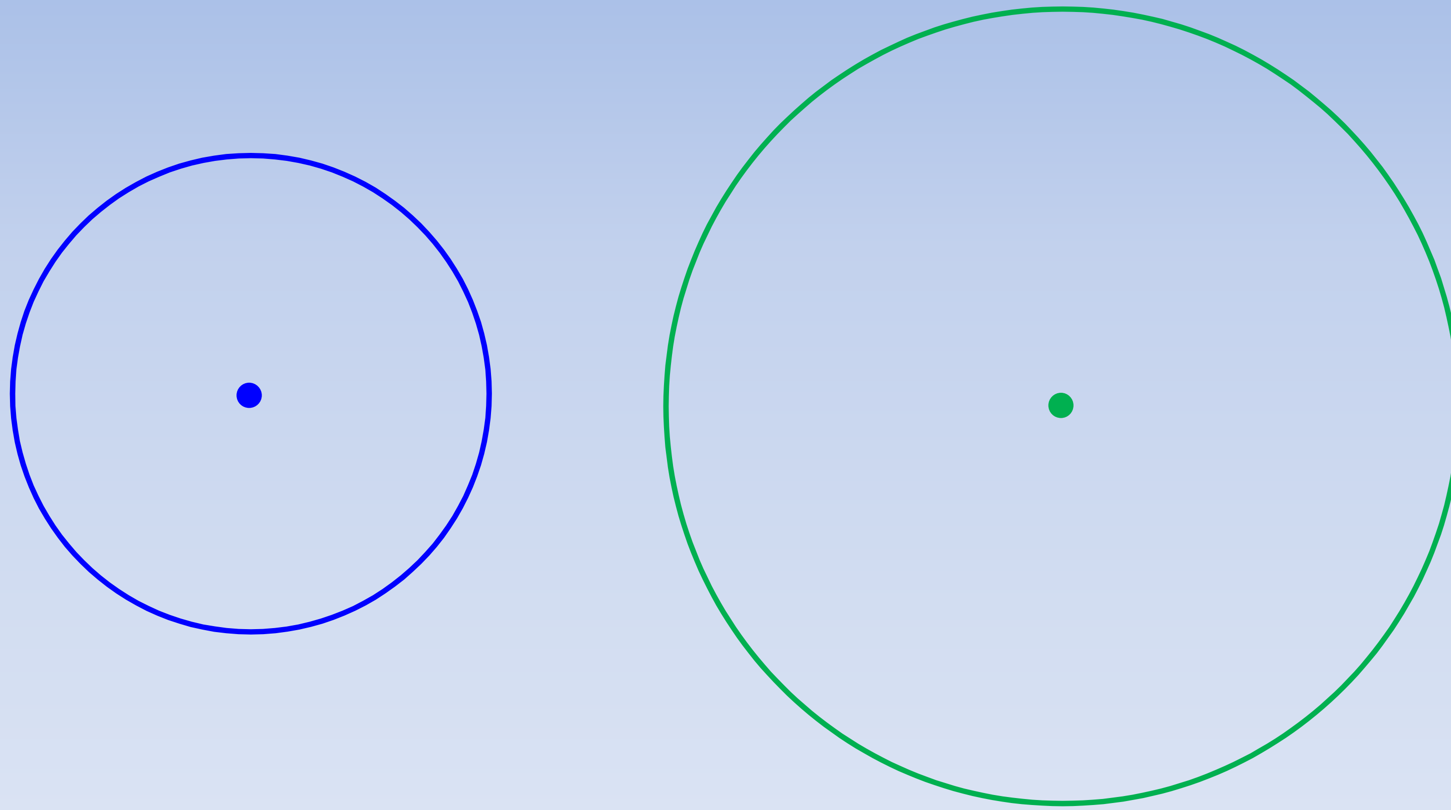
No, it is the following:

The radical axis of two circles is the locus of points at which tangents drawn to both circles have the same length.

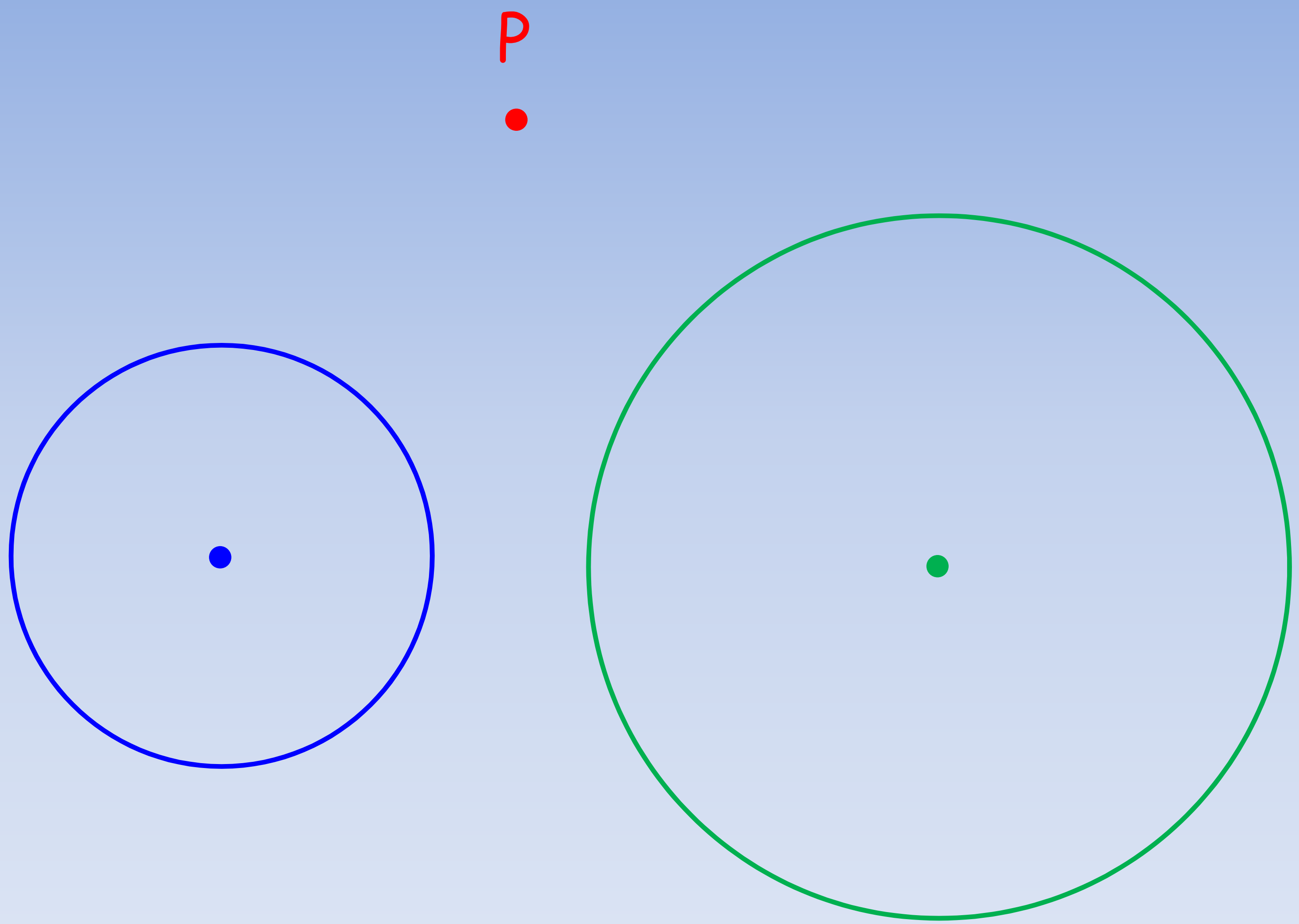
What is the Radical Axis?



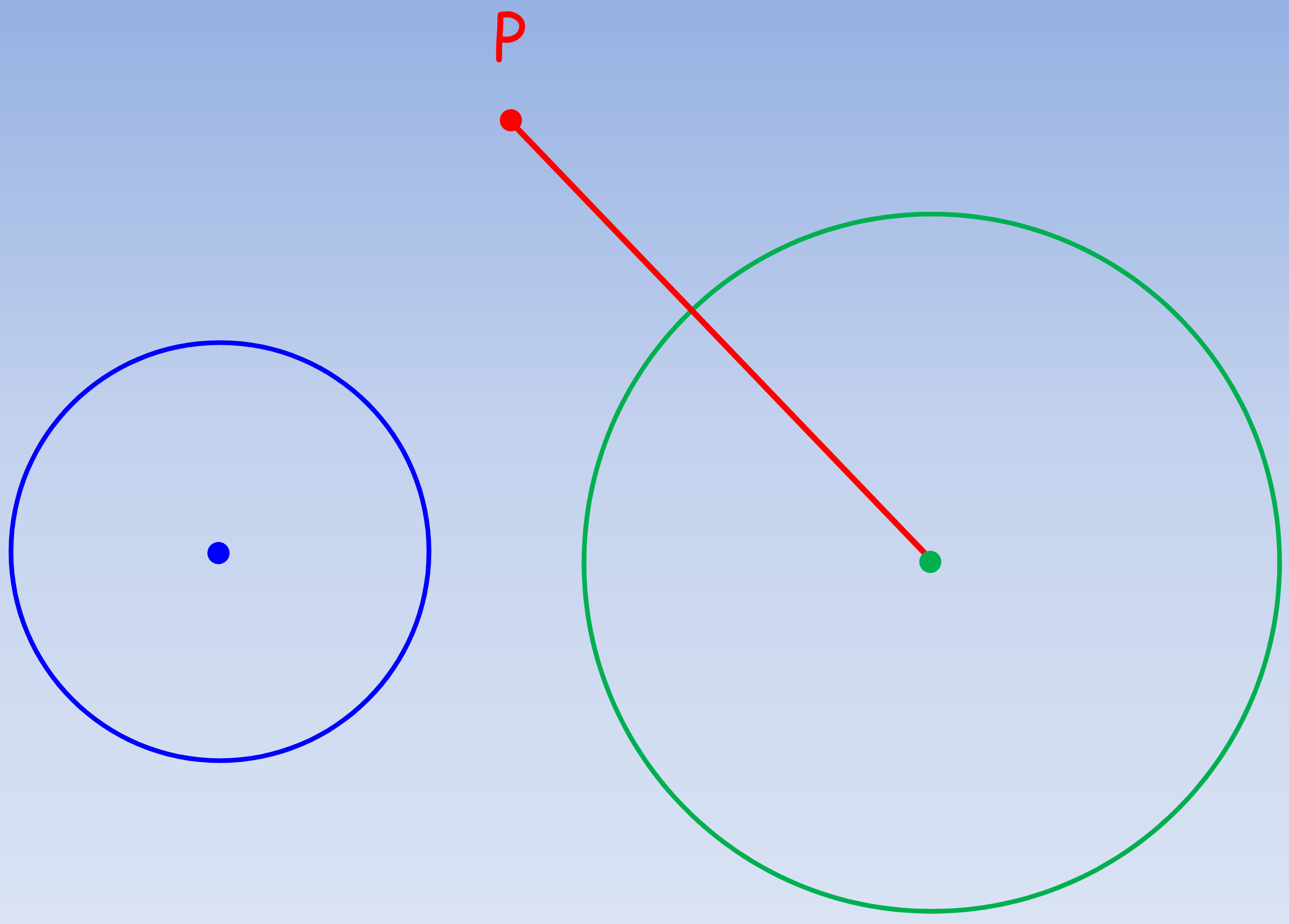
What is the Radical Axis?



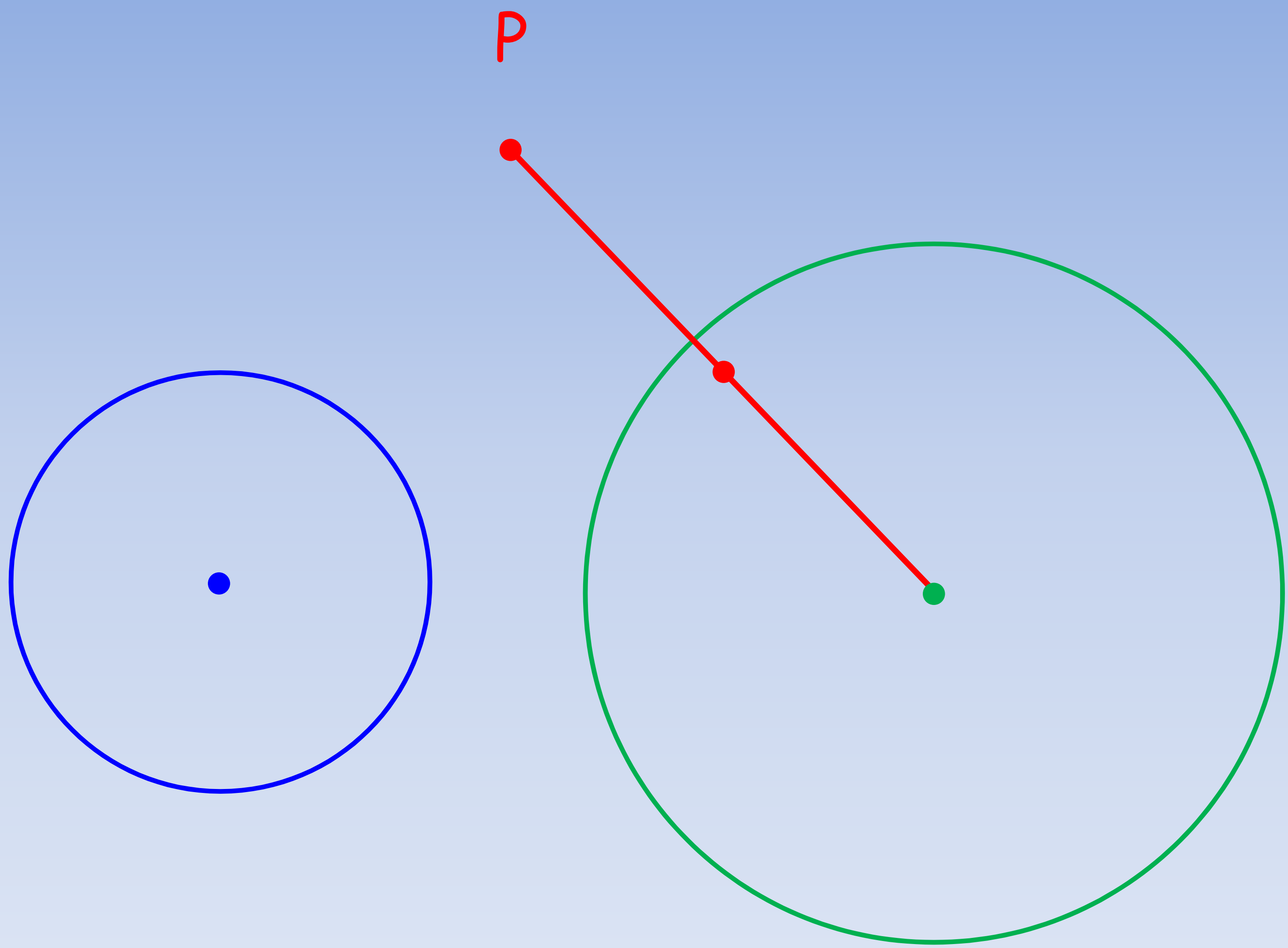
What is the Radical Axis?



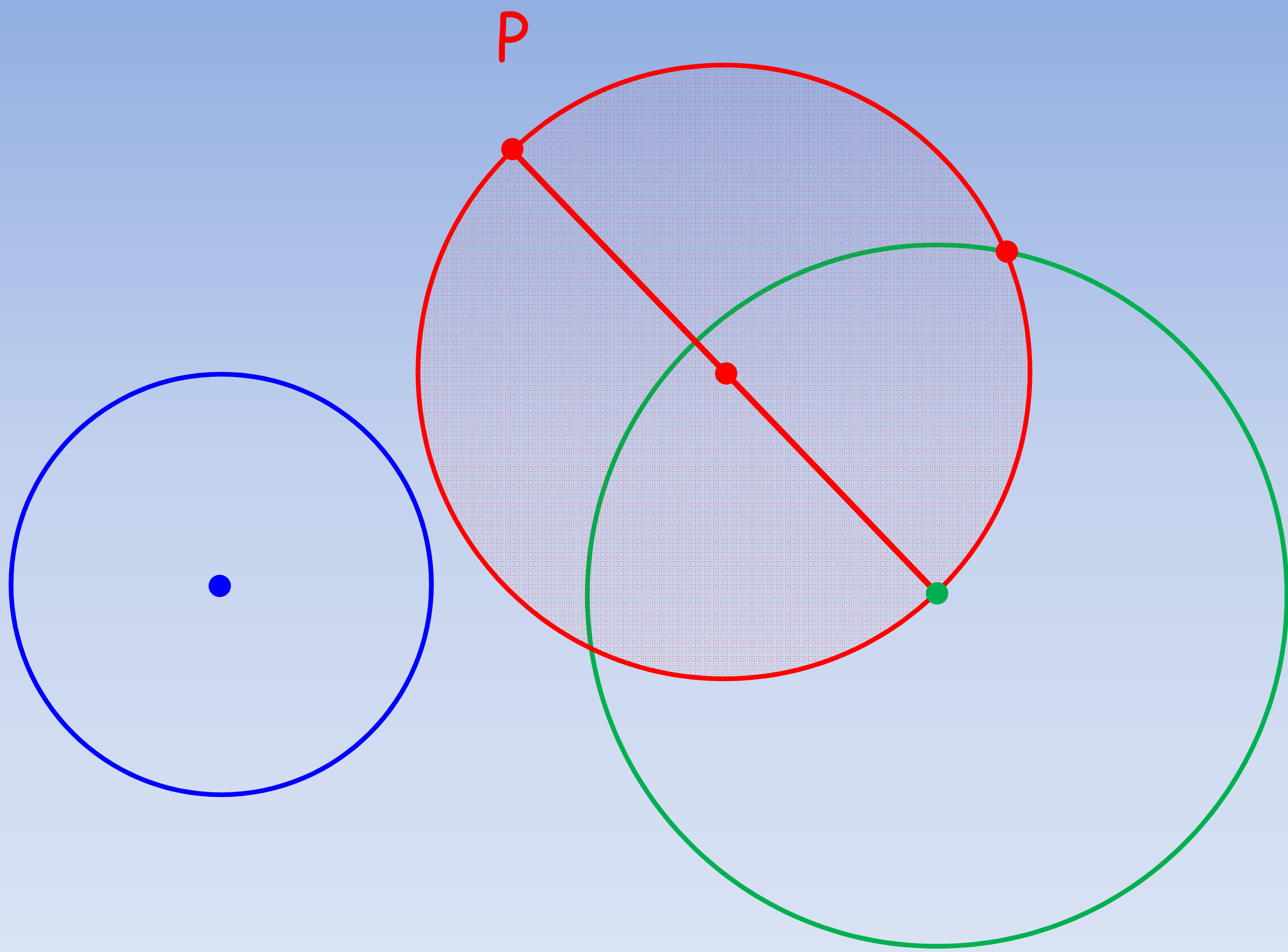
What is the Radical Axis?



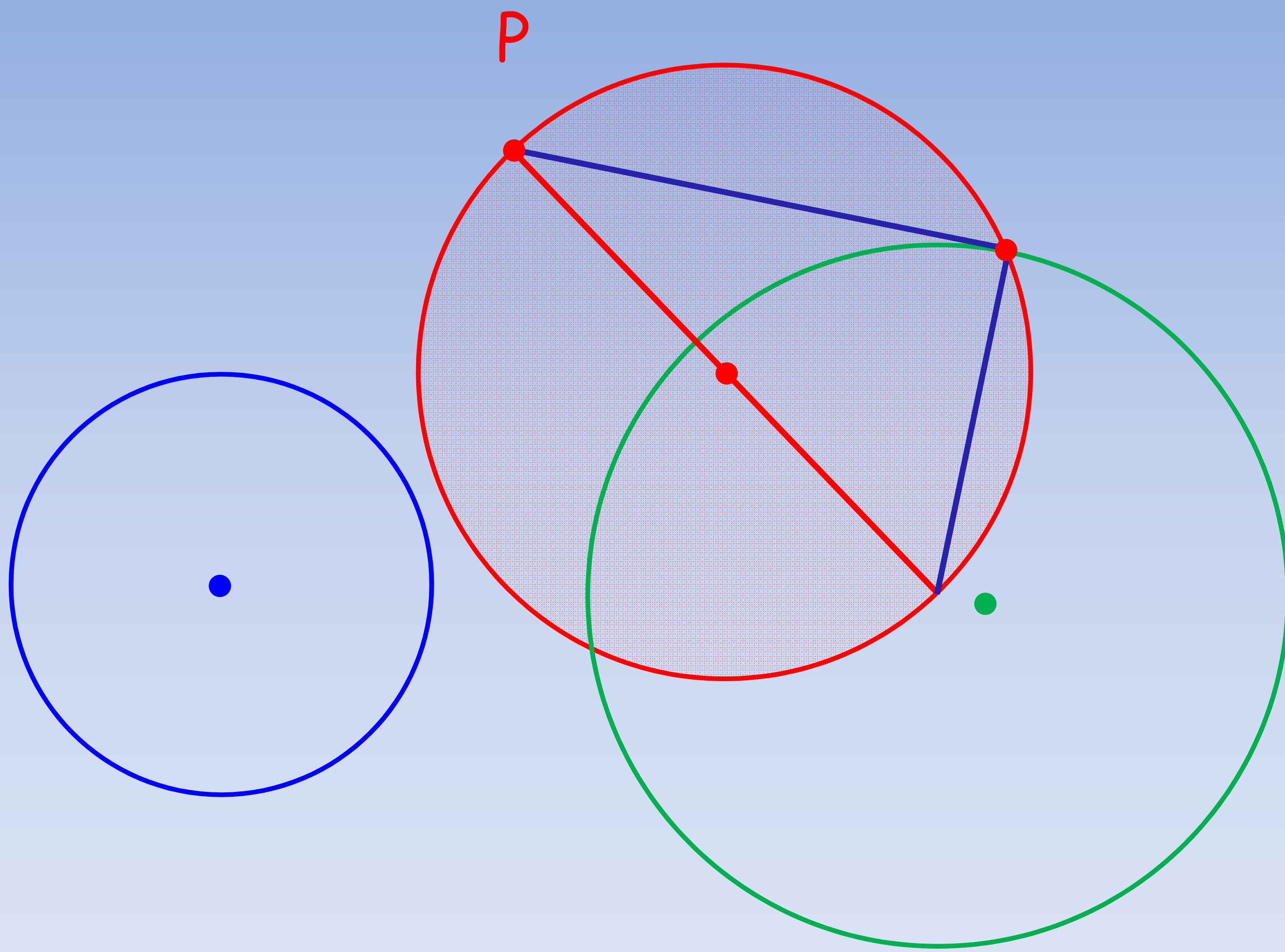
What is the Radical Axis?



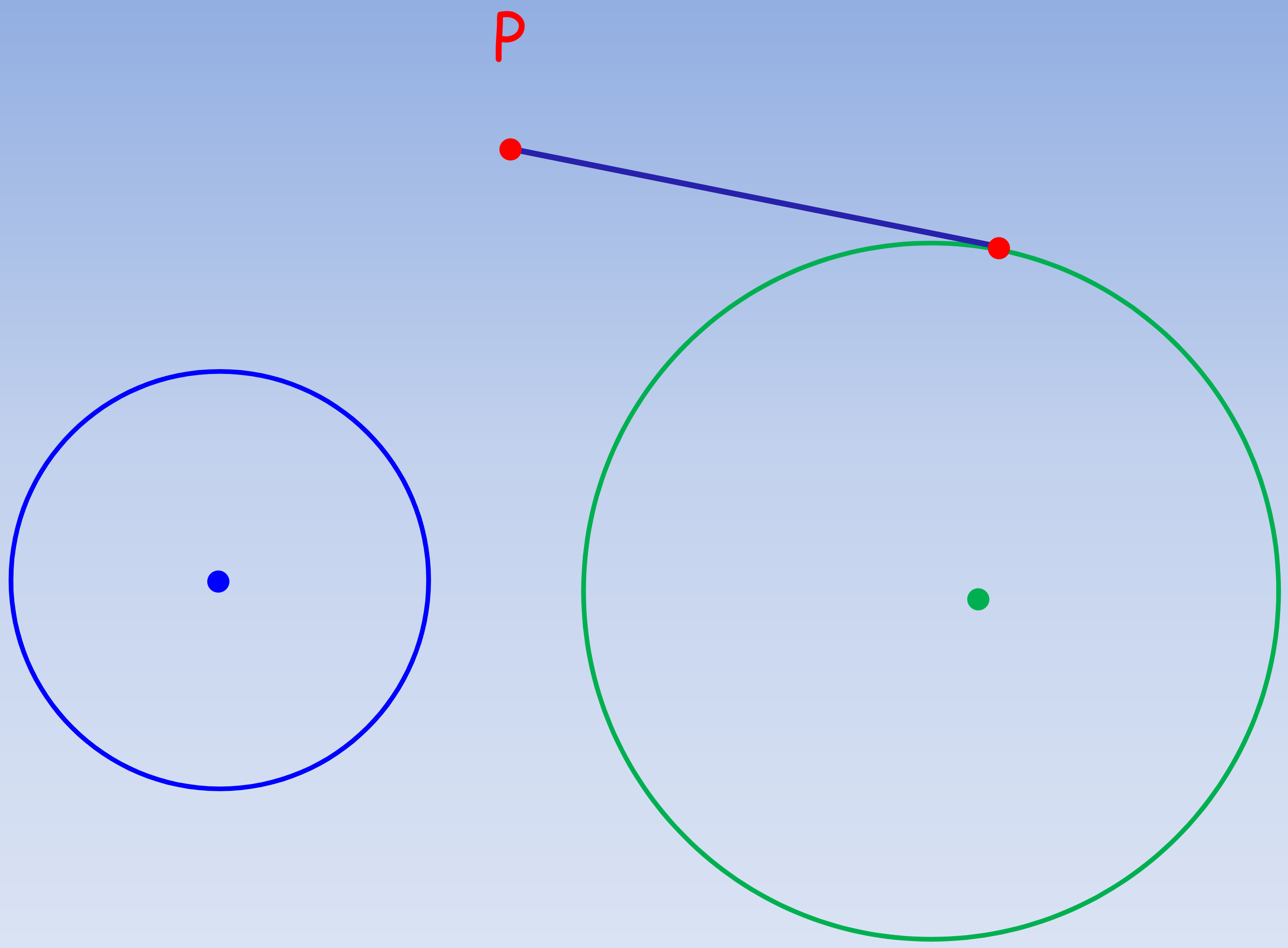
What is the Radical Axis?



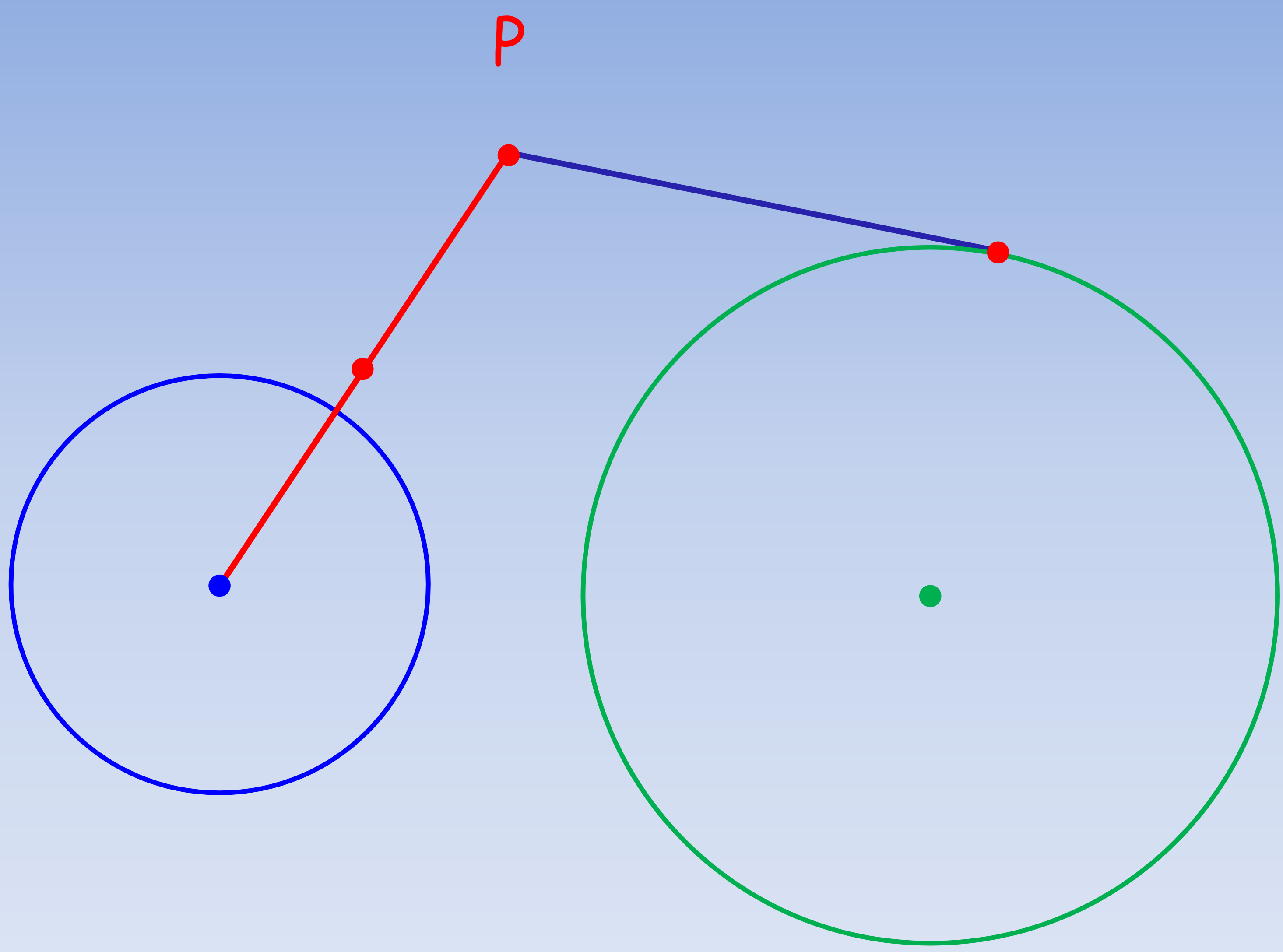
What is the Radical Axis?



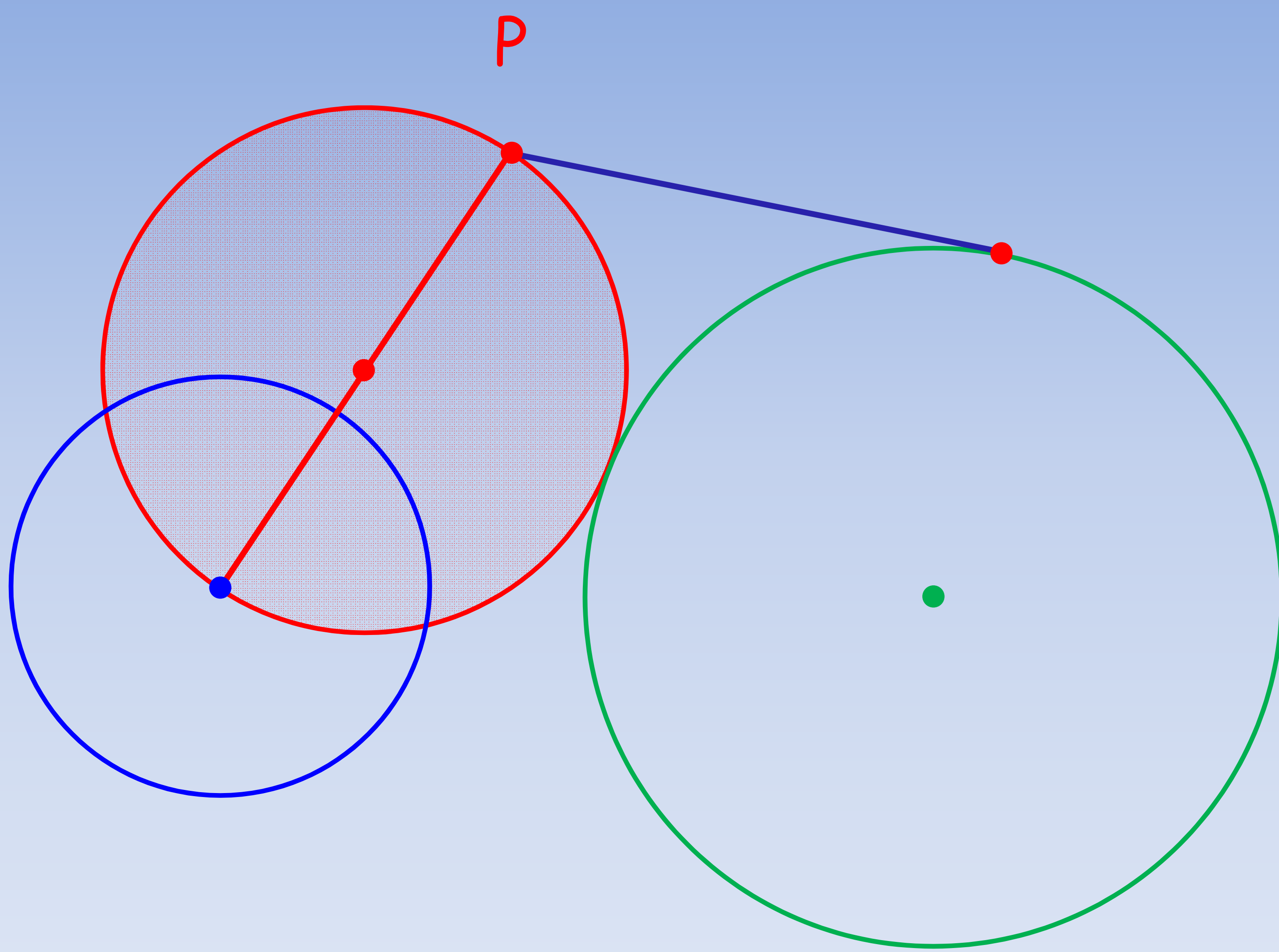
What is the Radical Axis?



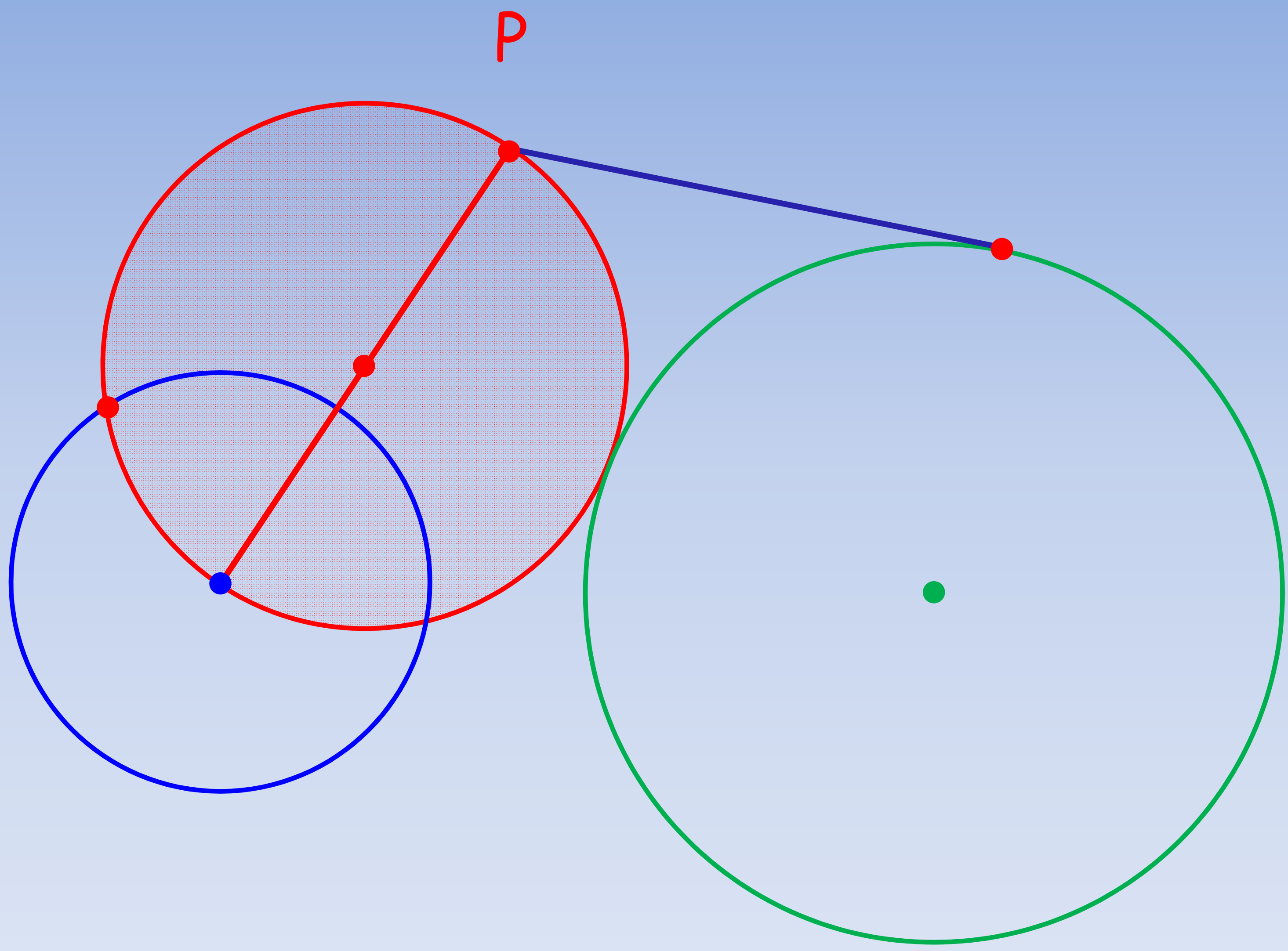
What is the Radical Axis?



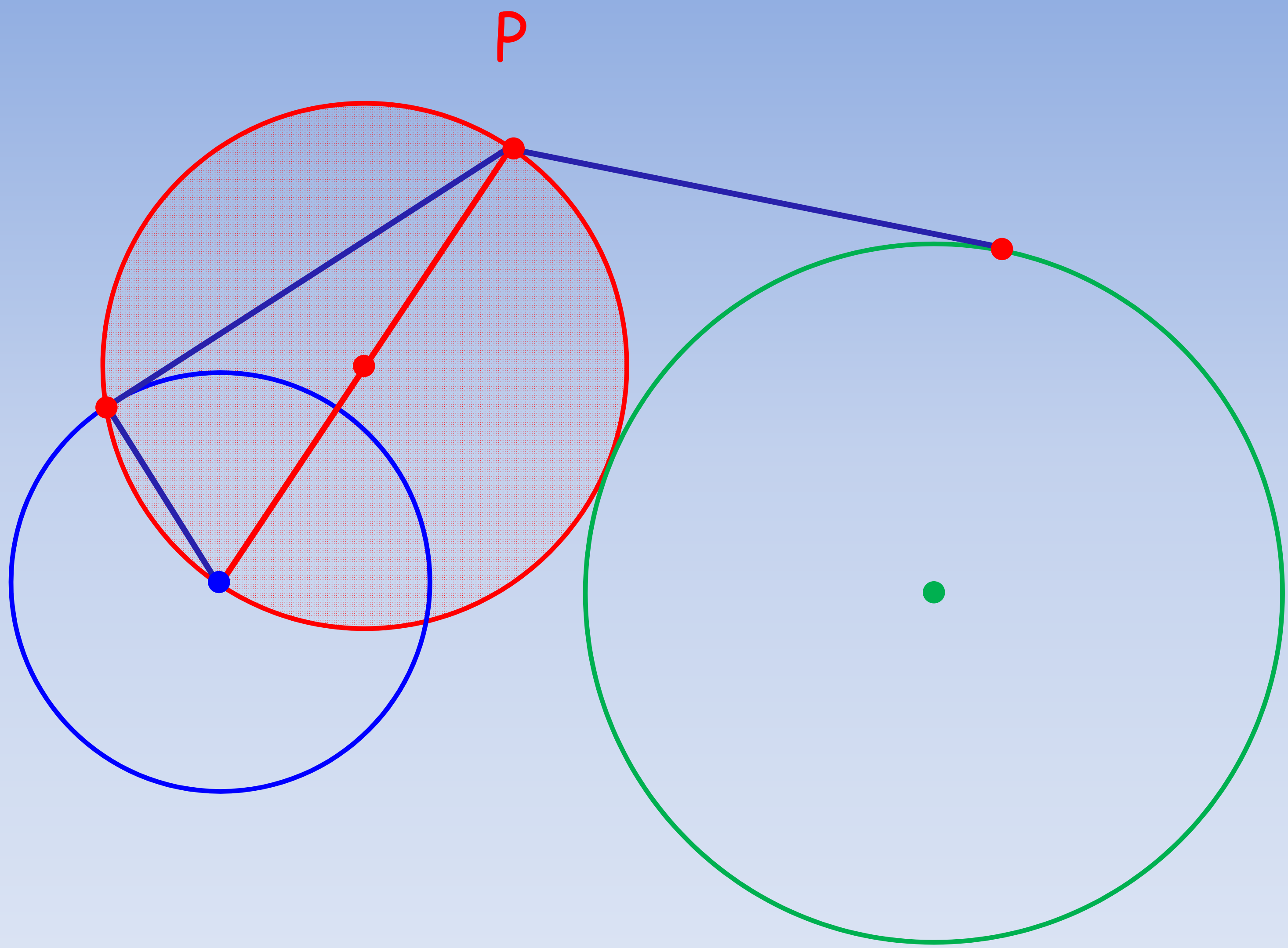
What is the Radical Axis?



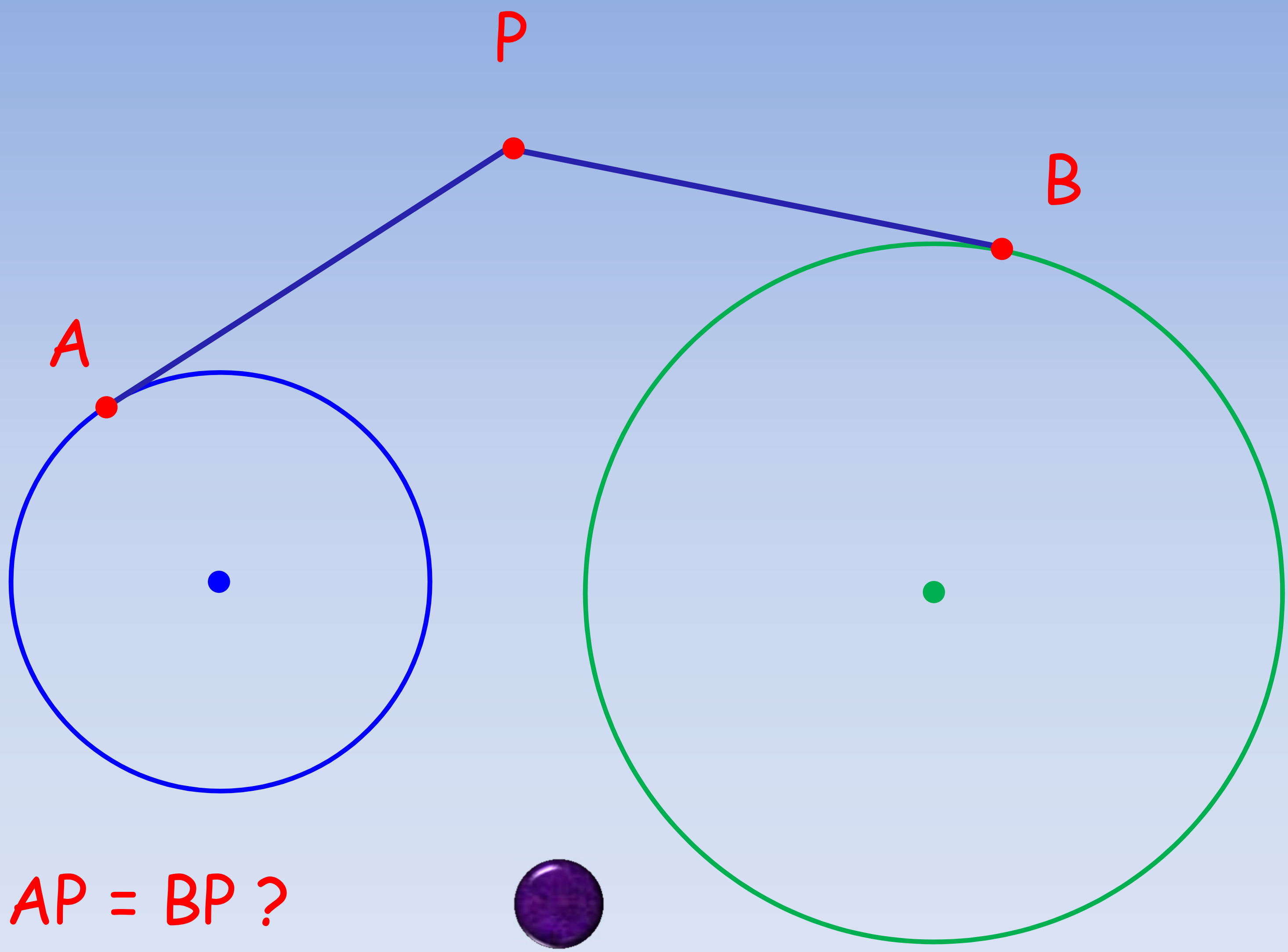
What is the Radical Axis?



What is the Radical Axis?



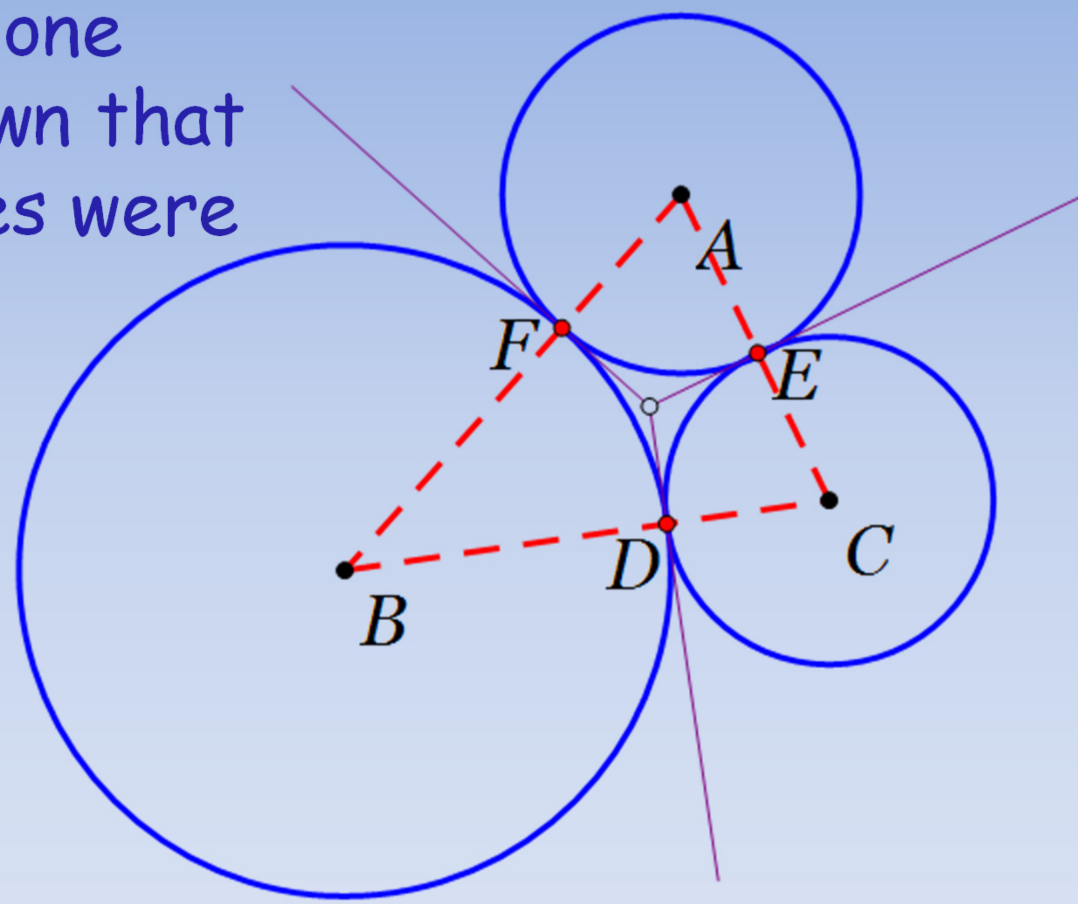
What is the Radical Axis?



Is AP = BP ?

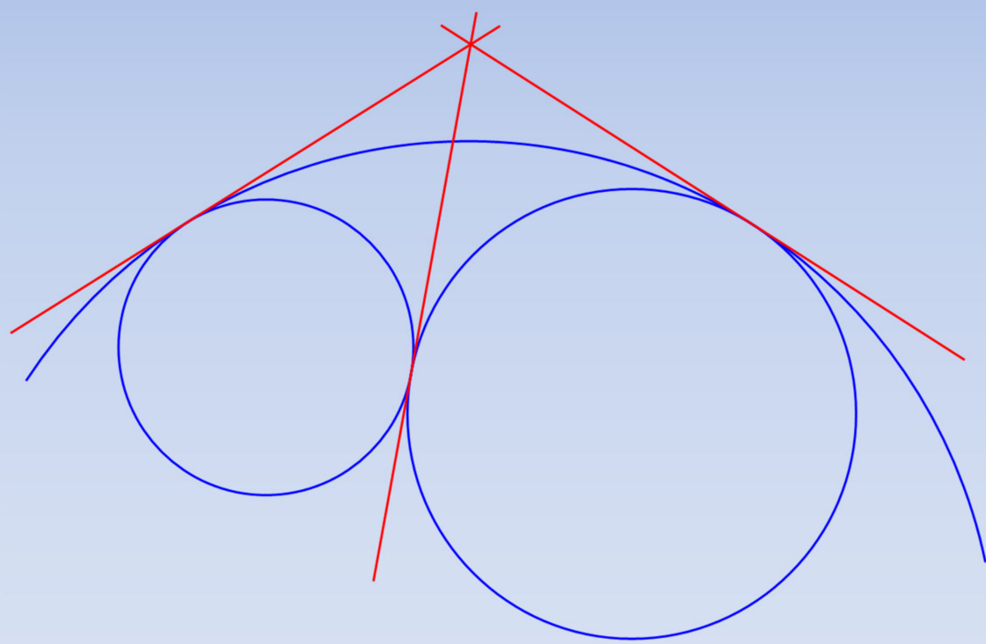
Previously

We had looked at three circles that were externally tangent to one another. We had shown that the three tangent lines were concurrent.
(See Incircle)

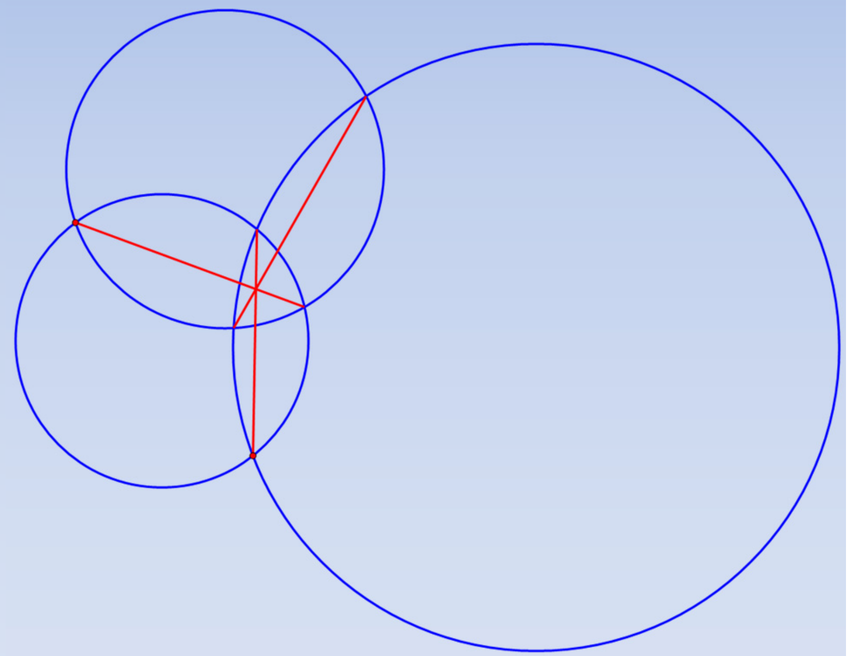


Is there something more?

Consider the following figures:



External tangency is not necessary!



Tangency is not necessary!

Circle-Line Concurrency

Theorem: Given 3 circles with noncollinear centers and with every two have a point in common. For each pair of circles draw either common secant or common tangent, then these three constructed lines are concurrent.

Power of a point

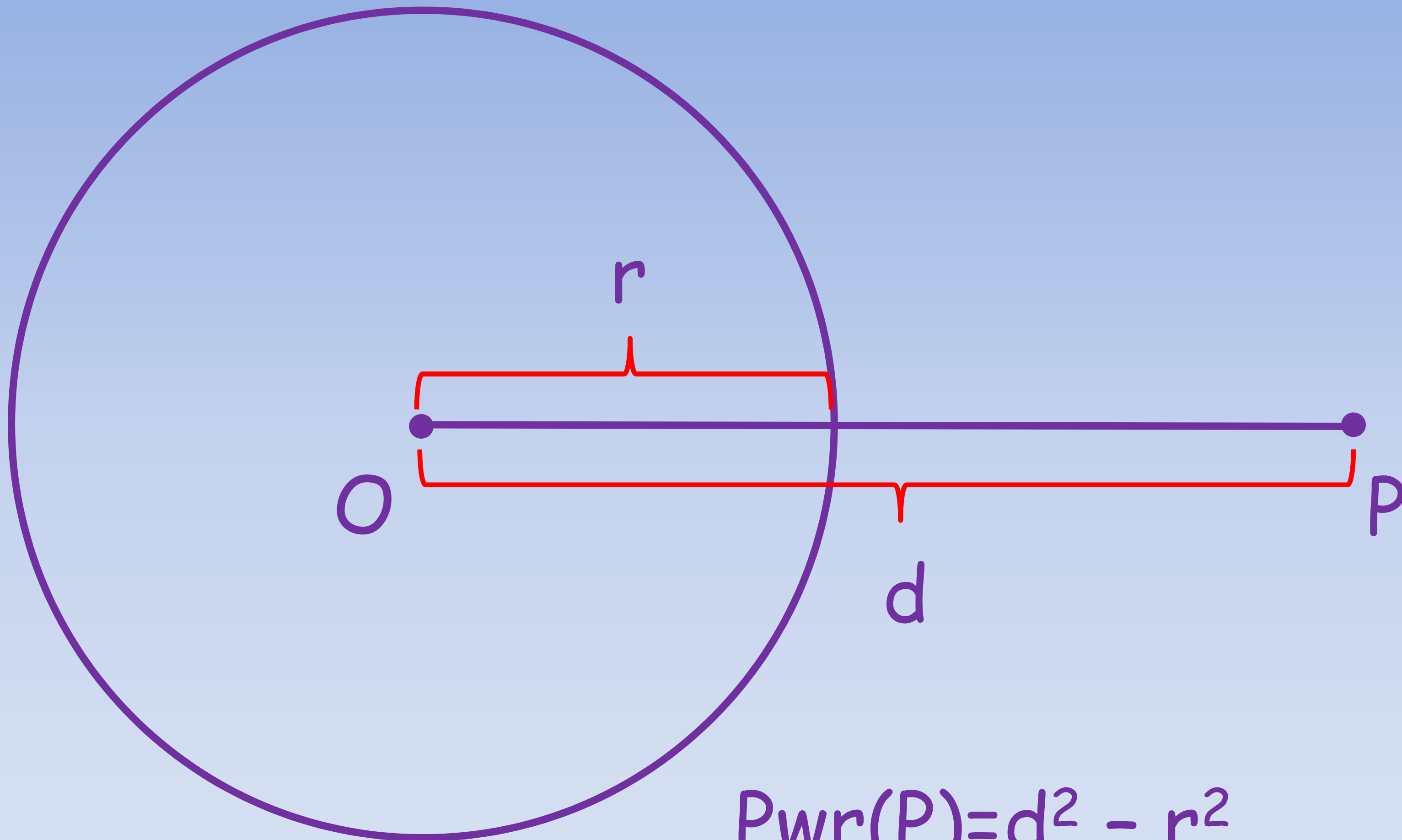
Let P be a point and c_1 be a circle of radius r centered at O.

The power of P with respect to c_1 is defined to be:

$$\mathrm{Pwr}(P) = d^2 - r^2,$$

where d = OP.

Power of a point



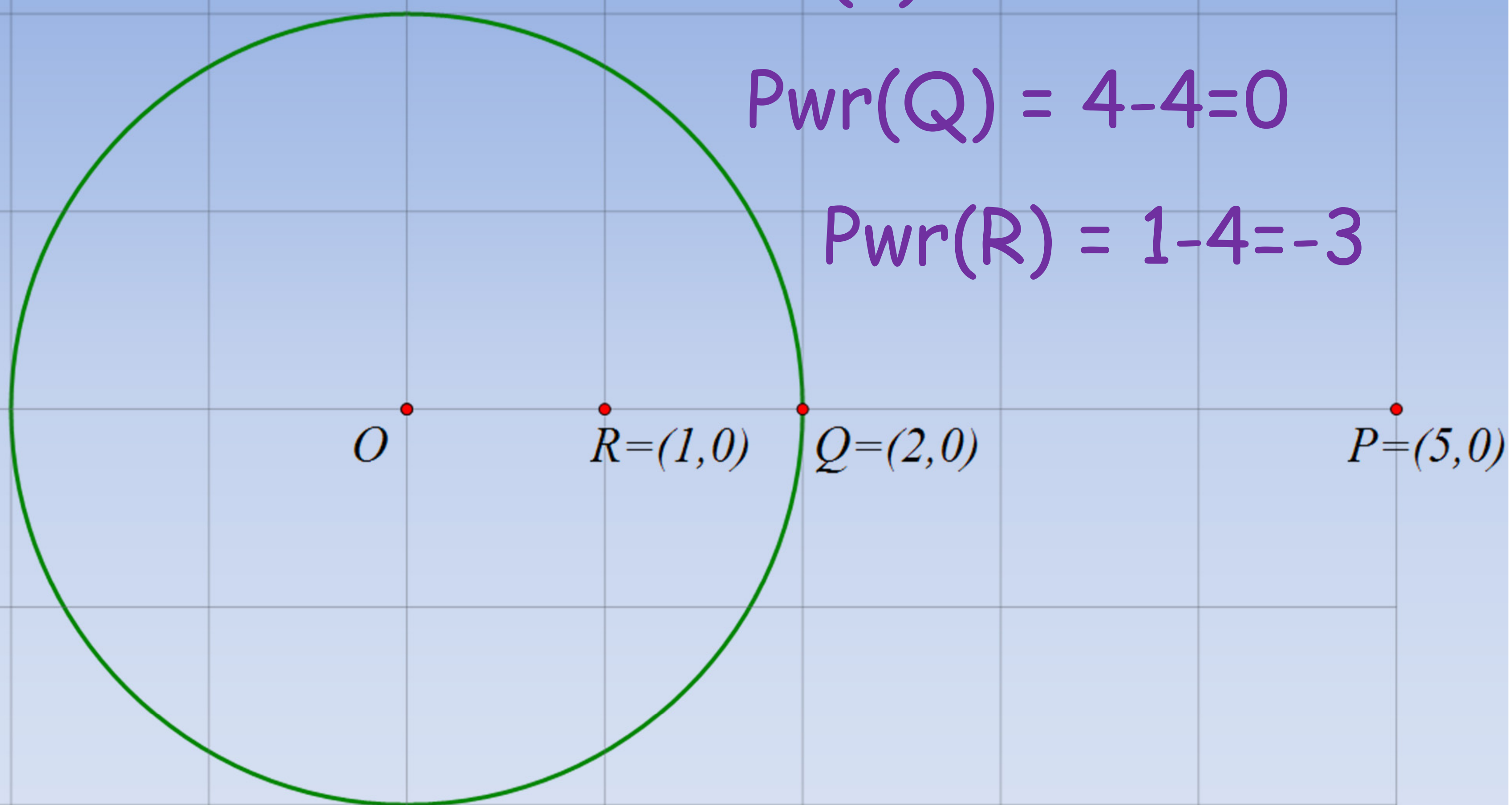
$Pwr(P) = d^2 - r^2$

Power of a point

Pwr(P) = 25-4=21

Pwr(Q) = 4-4=0

Pwr(R) = 1-4=-3



Power of a point

Lemma: Let P be a point and c_1 be a circle of radius r centered at O.

1. Pwr(P) > 0 iff P lies outside c_1;
2. Pwr(P) < 0 iff P lies inside c_1;
3. Pwr(P) = 0 iff P lies on c_1.

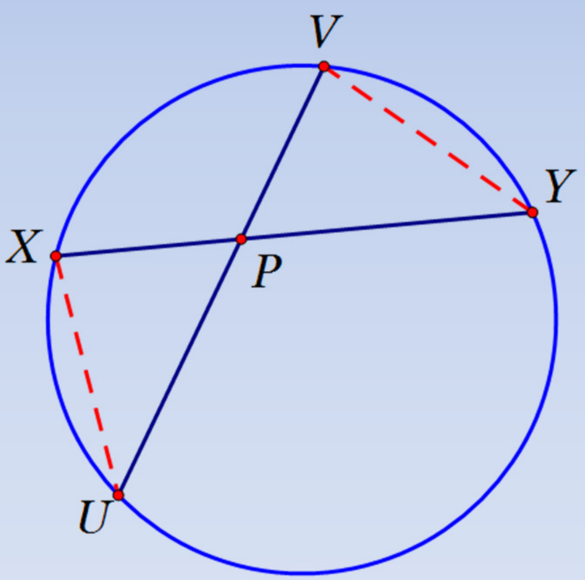
Proof:

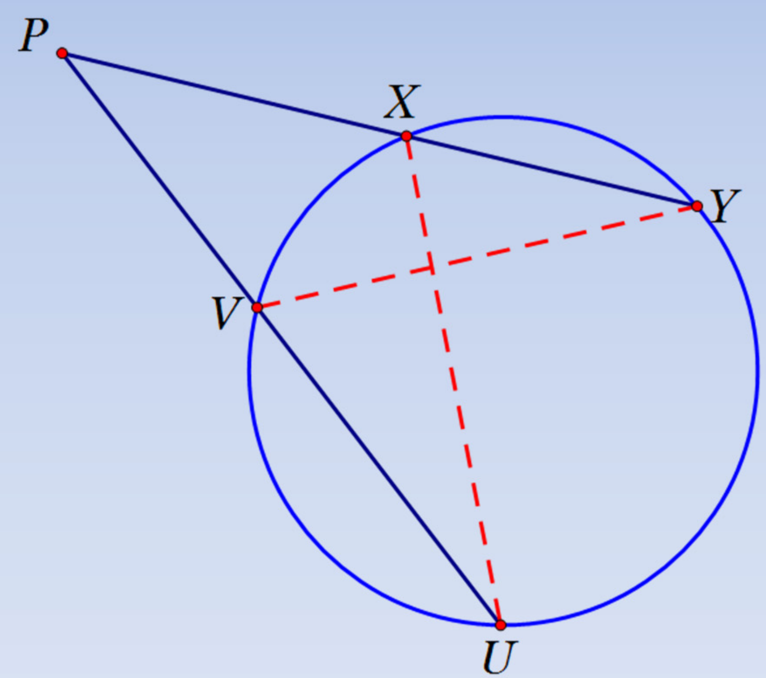
Background on Power

Theorem [1.35]: Given a circle and a point P not on the circle, choose an arbitrary line through P meeting the circle at X and Y. The quantity $PX \cdot PY$ depends only on P and is independent of the choice of line through P.

Background on Power

Let a second line through P intersect circle at U and V.
Need to show PU PV = PX PY.





Background on Power

Draw UX and VY.

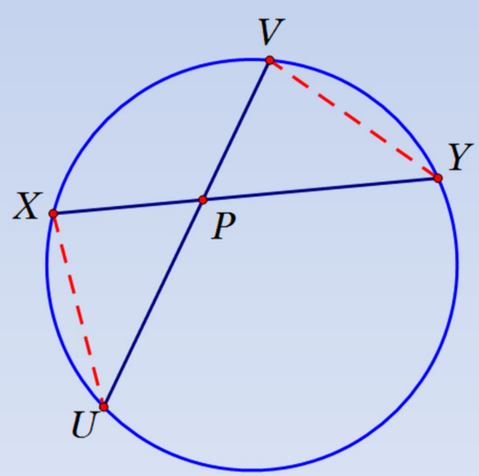
∠U = ∠Y (Why?)

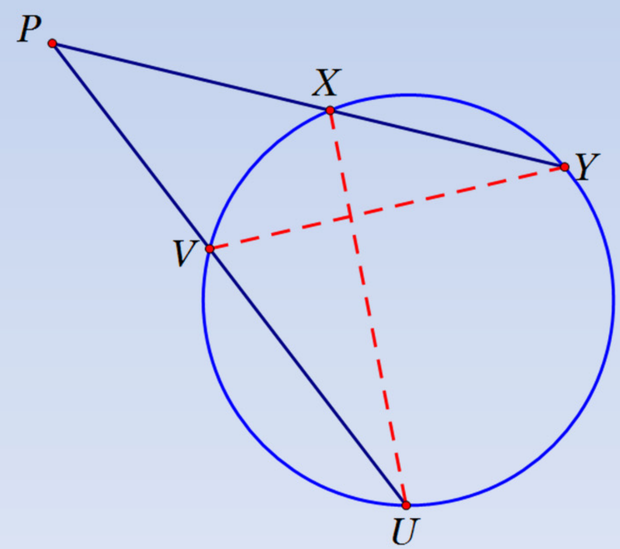
∠XPU = ∠VPY

ΔXPU ~ ΔVPY

PX/PV = PU/PY

PX PY = PU PV





The Power Lemma

Lemma: Fix a circle and a point P. Let p be the power of P with respect to the circle.

a) If P lies outside the circle and a line through P cuts the circle at X and Y, then $p = PX \cdot PY$.

b) If P is inside the circle on chord XY, then $p = -PX \cdot PY$.

c) If P lies on a tangent to the circle at point T, then $p = (PT)^2$.

Proof:

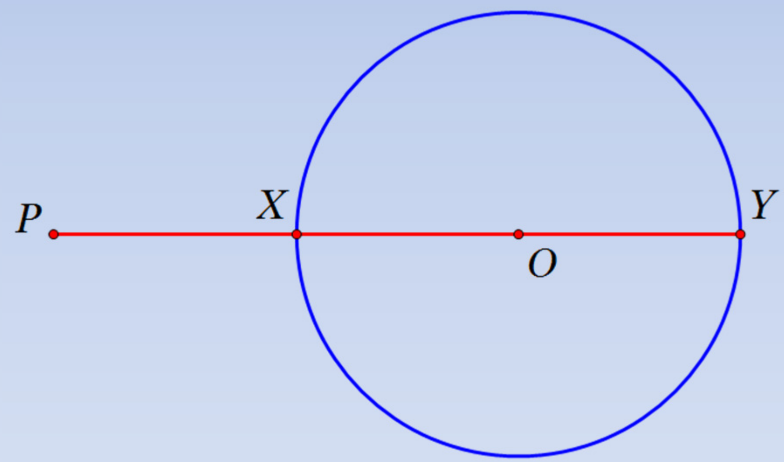
1. $PX \cdot PY$ does not depend on the choice of line. Let the line go through O, the center of the circle.

XY = diameter = $2r$

$PO = d$

$PX = PO - XO = d - r$

$PY = PO + OY = d + r$



$PX \cdot PY = (d - r)(d+r) = d^2 - r^2$

Proof:

2. PX·PY does not depend on the choice of line. Let the line go through O, the center of the circle.

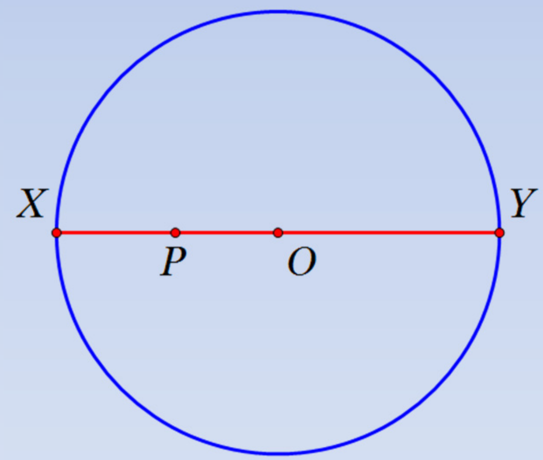
XY = diameter = 2r

PO = d

PX = PO - XO = r - d

PY = PO + OY = r + d

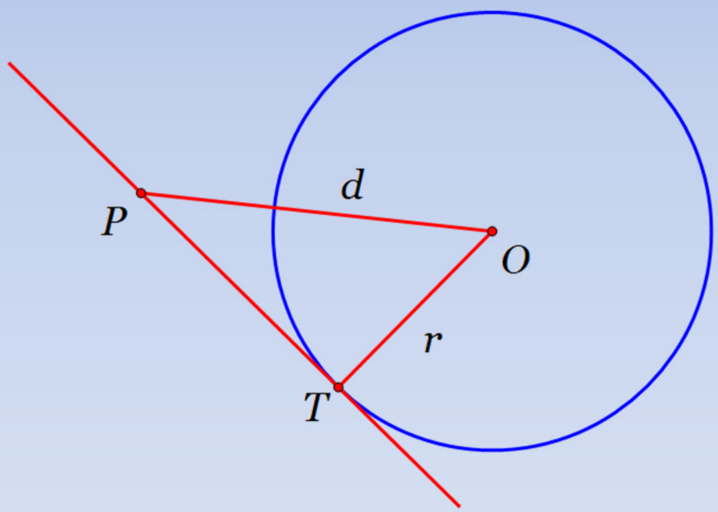
$PX \cdot PY = (r - d)(r + d) = r^2 - d^2 = -p$



Proof:

3. ΔPTO is a right triangle.

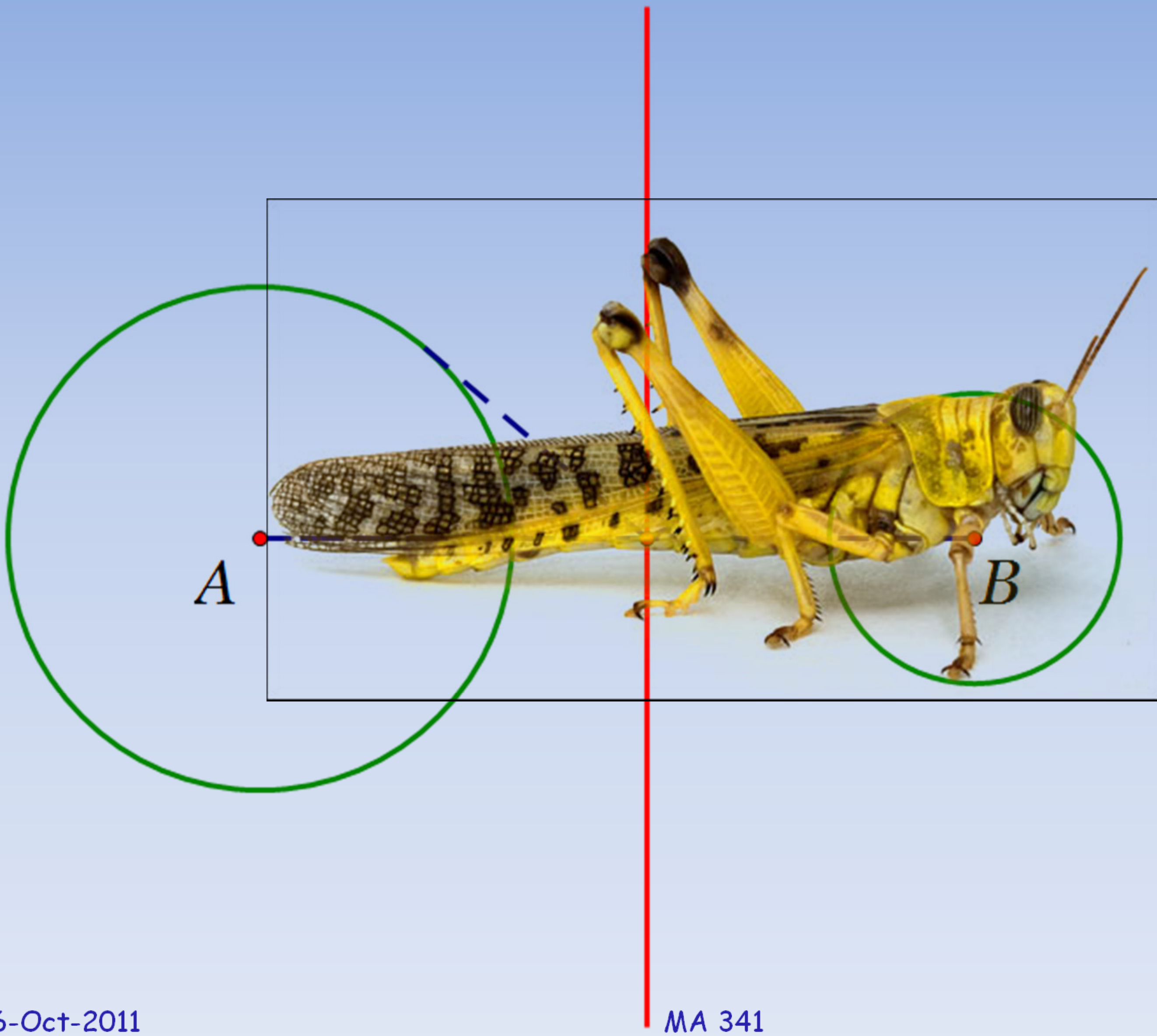
$$(PT)^2 = d^2 - r^2 = p$$



Locus of Points

Lemma: Fix 2 circles centered at A and B, $A \neq B$. There exist points whose powers with respect to the two circles are equal. The locus of all points is a line perpendicular to AB.

Locust of Points



Proof

Suppose that A and B lie on the x-axis.

(Is this a reasonable assumption? Why?)

Let $A=(a,0)$ and $B=(b,0)$, $a \neq b$.

Let $P=(x,y)$, then

$(PA)^2=(x-a)^2 + y^2$ and $(PB)^2=(x-b)^2 + y^2$

Let r = radius of circle at A, s=radius at B

Powers of P are equal IFF

$(x-a)^2 + y^2 - r^2 = (x-b)^2 + y^2 - s^2$

Proof

$$(x-a)^2 + \cancel{y}^2 - r^2 = (x-b)^2 + \cancel{y}^2 - s^2$$

$$(x-a)^2 - r^2 = (x-b)^2 - s^2$$

$$x^2 - 2ax + a^2 - r^2 = x^2 - 2bx + b^2 - s^2$$

$$2(a-b)x = r^2 - s^2 + b^2 - a^2$$

$$x = \frac{r^2 - s^2 + b^2 - a^2}{2(a-b)}$$

Radical Axis

Given two circles with different centers their radical axis is the line consisting of all points that have equal powers with respect to the two circles.

Radical Axis

Corollary:

a) If two circles intersect at two points A and B, then their radical axis is their common secant AB.

b) If two circles are tangent at T, their radical axis is their common tangent line.

Proof of (a)

A point common to two circles has power 0 with respect to BOTH circles.

Pwr(A)=0=Pwr(B), which is radical axis.

Radical axis is line containing A and B.

Proof of (b)

T lies on both circles, so $Pwr_1(T)=Pwr_2(T)=0$ and T lies on the radical axis.

If P lies on radical axis of one circle and lies on one circle, then Pwr(P)=0 so it also lies on other circle since it is on radical axis.

Thus, P lies on both circles, but T is the only point that lies on both circles.

Radical Axis

Corollary:

Given three circles with noncollinear centers, the three radical axes of the circles taken in pairs are distinct concurrent lines.

Proof

Radical axis is perpendicular to the line between the centers of the circles.

Centers non-collinear implies radical axes distinct and nonparallel.

Each pair intersects!!

Let P be a point and let p_1, p_2, and p_3 be the powers of P with respect to the 3 circles.

Proof

On one radical axis we have $p_1 = p_2$

On another we have, and $p_2 = p_3$

At P the radical axes meet and we have

$$p_1 = p_2 = p_3$$

Thus, $p_1 = p_3$ and P lies on the third radical axis.