Loci

MA 341 – Topics in Geometry
Lecture 25

It’s loci with a soft “c” otherwise you get

Locus

Set of points satisfying a given property is called the locus of that property.

Examples:

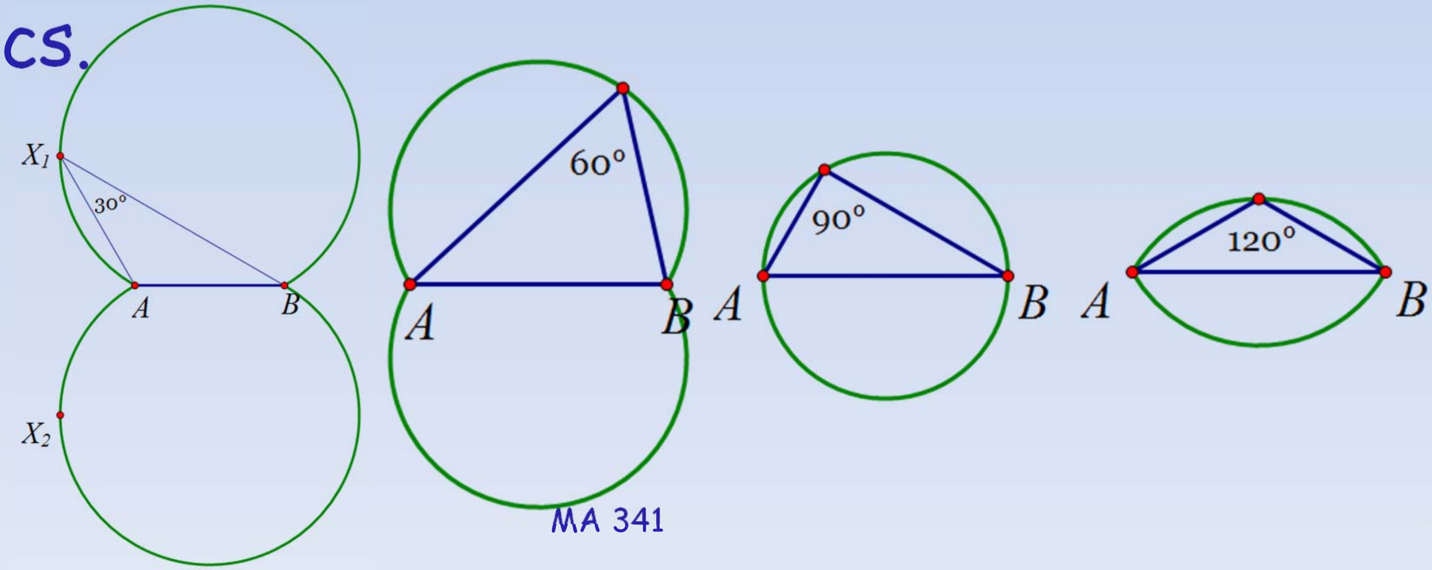
1. Circle of radius r centered at O: locus of points at a distance r from O.

2. Locus of points equidistant from 2 points is perpendicular bisector of segment joining the points.

Locus

3. The locus of points that are equidistant from the sides of a given angle is the angle bisector.

4. Given points A and B and real number r, $0 < r < \pi$, locus of all points X so that $\angle AXB = r$ is the union of two circular arcs.

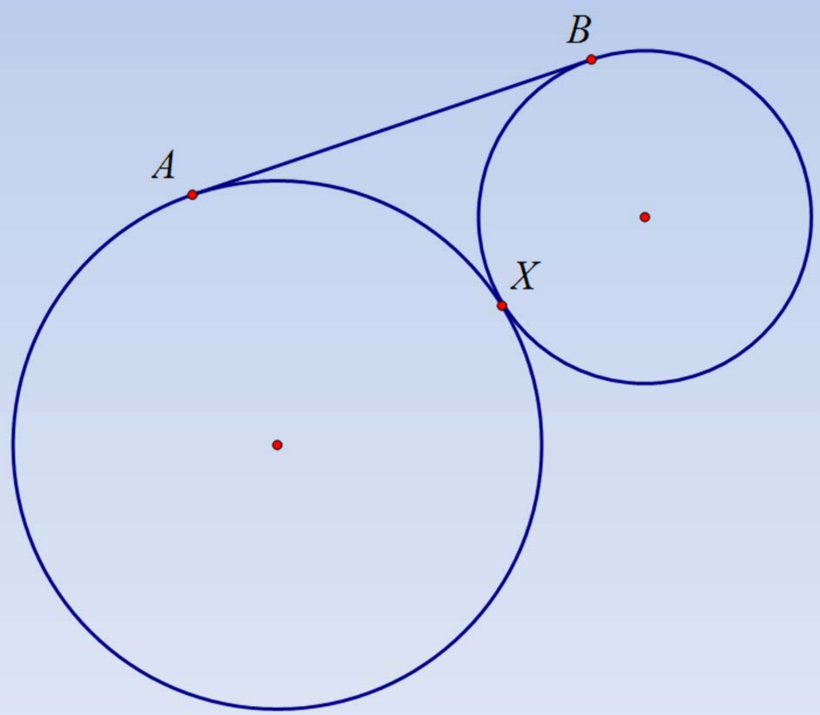


Locus

5. The locus of points that are equidistant from a line segment is:

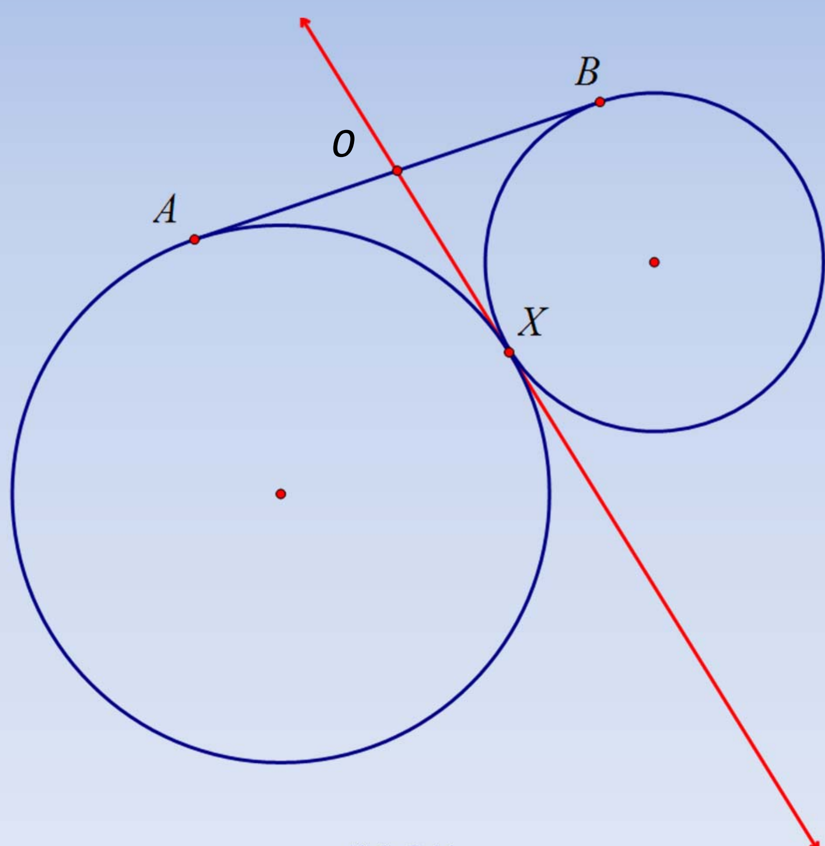
Locus

6. Given points A and B, consider 2 circles on one side of AB, tangent to AB at A and at B and externally tangent to each other at X. Find the locus of X.



Locus

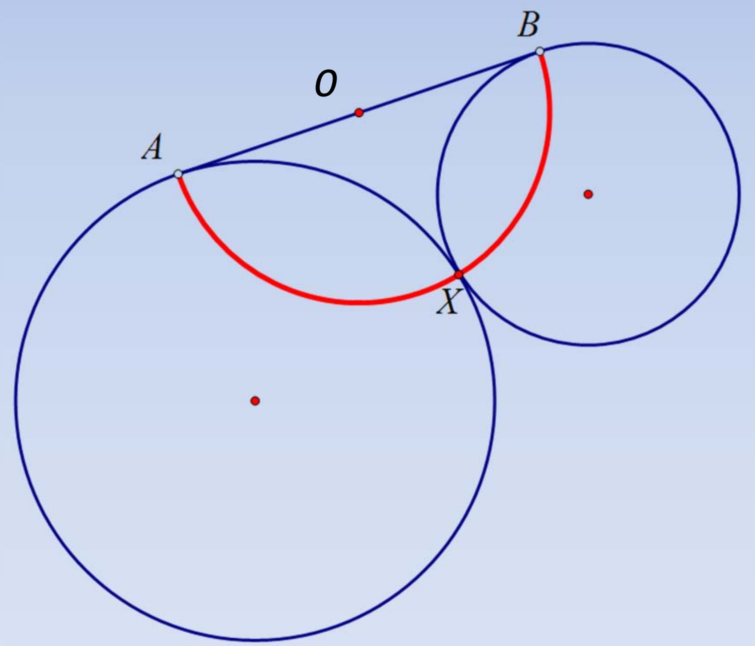
Let l be the radical axis through X.
It intersects AB at O.
Claim: OA = OB = OX.



Locus

Locus of points X with desired property lies on semicircle centered at O with radius AB/2.

As circles change, we get entire semicircle



Classical Loci

Ellipse

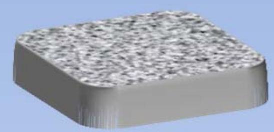
Hyperbola

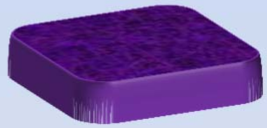
Parabola

How many have seen the description of these as

a) conic sections?
b) quadratic equations?
c) loci of points?

Conic Sections



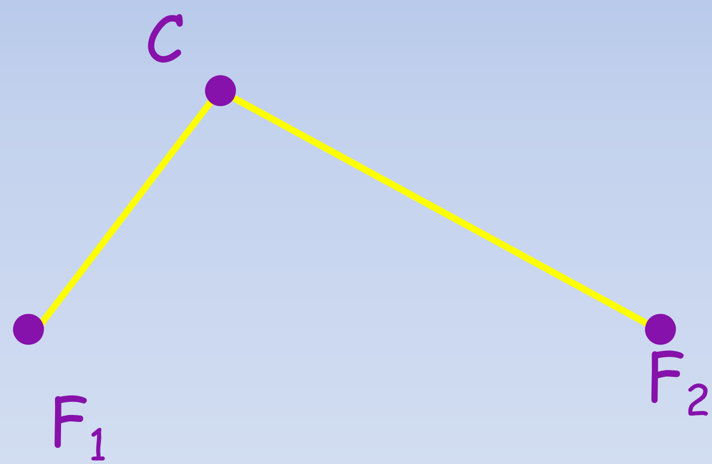


Ellipse

Given two points F_1 and F_2 and $s > 0$.

Find the locus of points, C, so that

$$CF_1 + CF_2 = s$$



Ellipse

Called an ellipse with foci F_1 and F_2.
CF_1 and CF_2 = focal radii

Three cases:
i) $s < F_1F_2$
ii) $s = F_1F_2$
iii) $s > F_1F_2$

i) If $s < F_1F_2$, then by triangle inequality
$s = CF_1 + CF_2 \geq F_1F_2$. Thus, it is empty.
ii) If $s = F_1F_2$, then the only points will be those on F_1F_2. The ellipse is the segment

Ellipse

Let C be point on ray F_1F_2 so that

$$CF_2 = \tfrac{1}{2}(s - F_1F_2)$$

$$CF_1 + CF_2 = (CF_2 + F_1F_2) + CF_2 = 2\,CF_2 + F_1F_2$$
$$= s$$

Similar point C' on other side of F_1.

Ellipse

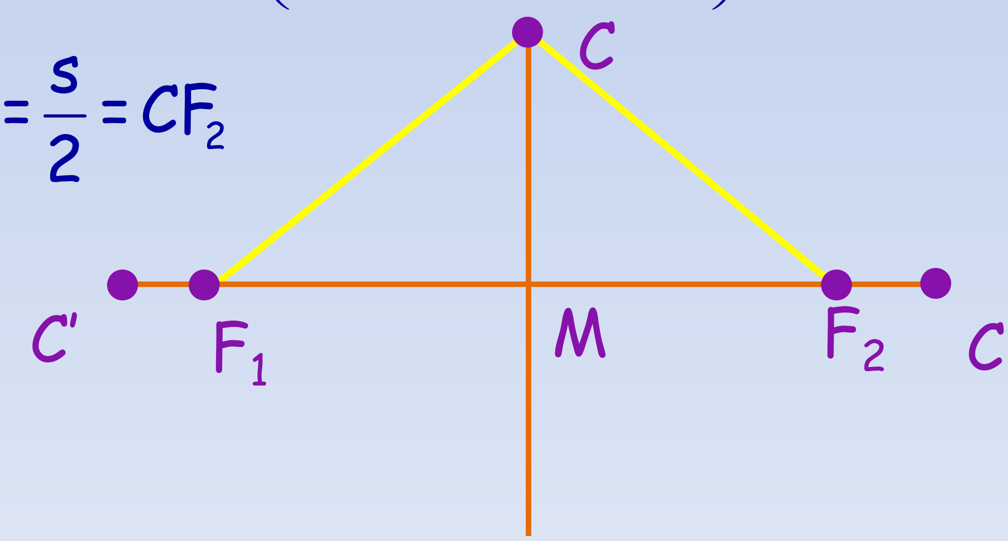
M = midpoint of F_1F_2

$\frac{1}{2}s > \frac{1}{2}F_1F_2$ so $\exists$ C on perp bisector so that

$$CM = \sqrt{\left(\frac{s}{2}\right)^2 - \left(\frac{F_1F_2}{2}\right)^2}$$

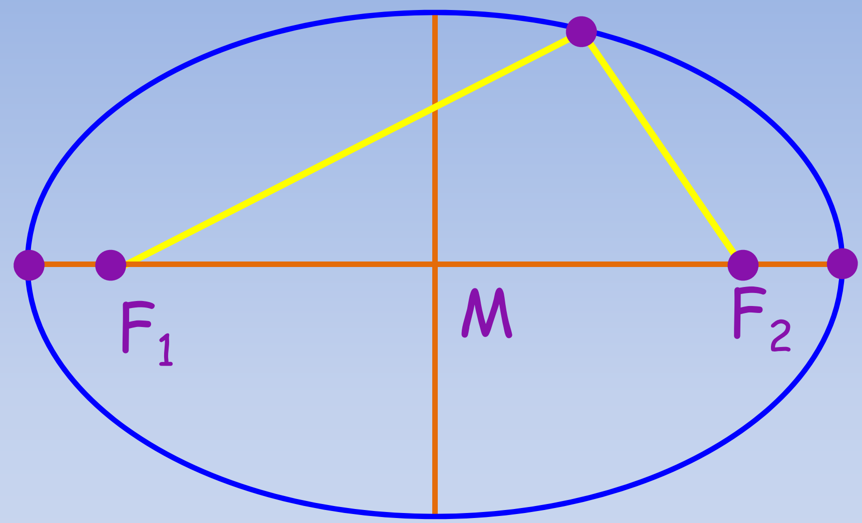
$$(CF_1)^2 = (F_1M)^2 + \left(\sqrt{\left(\frac{s}{2}\right)^2 - \left(\frac{F_1F_2}{2}\right)^2}\right)^2 = \left(\frac{s}{2}\right)^2$$

$$CF_1 = \frac{s}{2} = CF_2$$



Ellipse

Use reflective symmetry to complete the curve.



What is the interior of the ellipse?

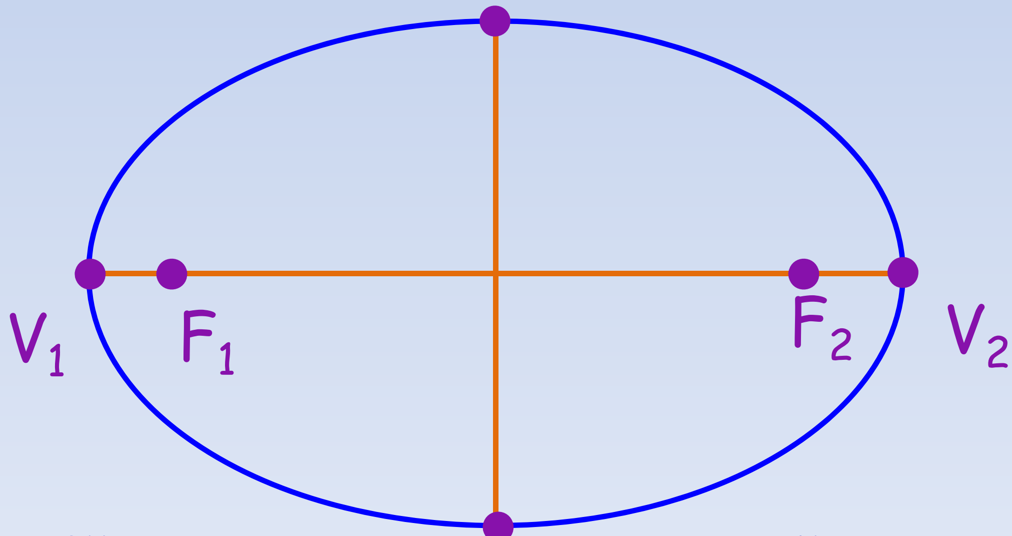
$$CF_1 + CF_2 < s$$

Ellipse

Major axis is the line containing F_1 and F_2
Minor axis is the perpendicular bisector of F_1F_2
Eccentricity: e or ε is the ratio of F_1F_2 to V_1V_2.

$$\varepsilon = \frac{F_1F_2}{V_1V_2}$$

$$0 < \varepsilon < 1$$

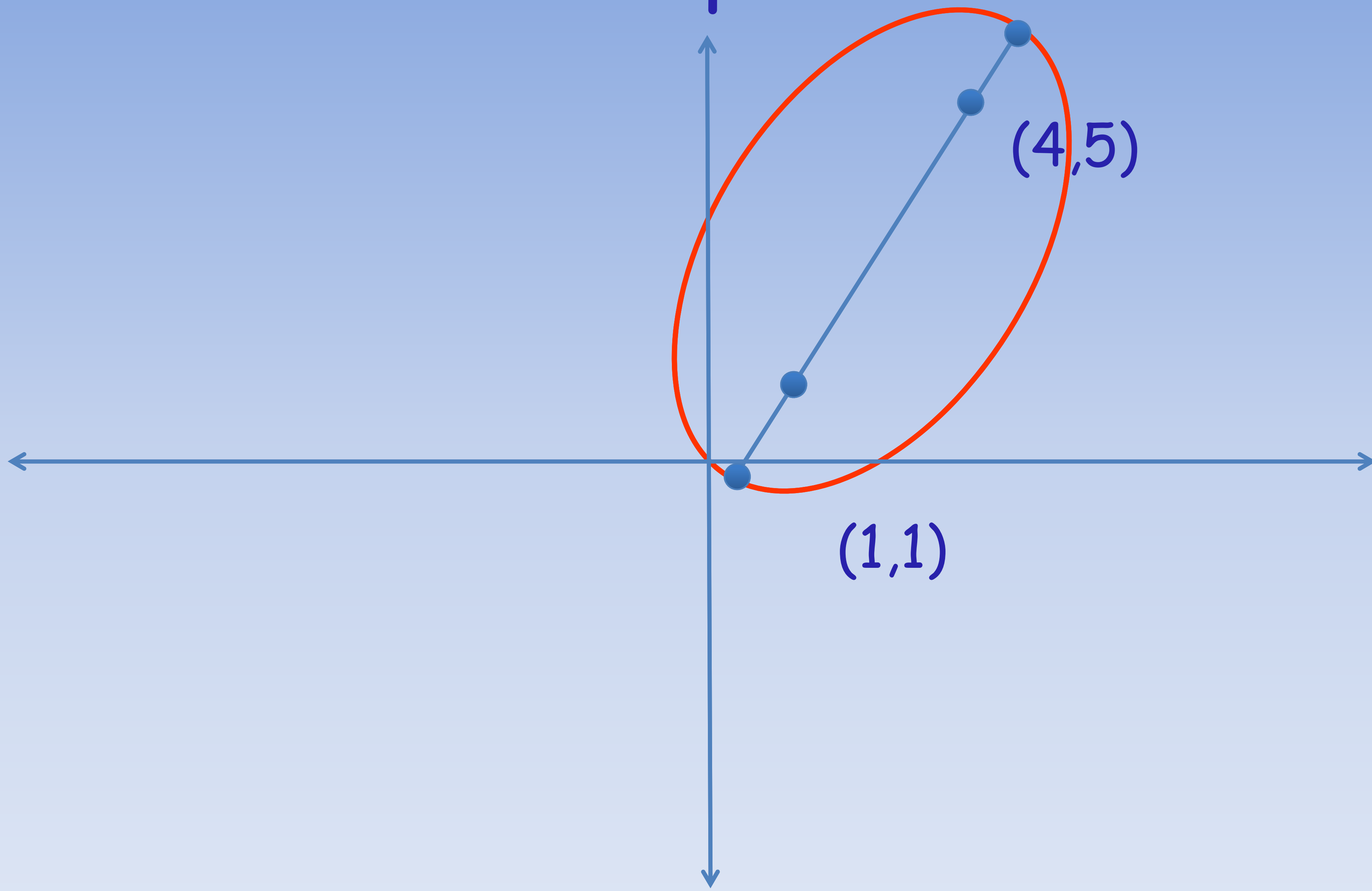


Ellipse

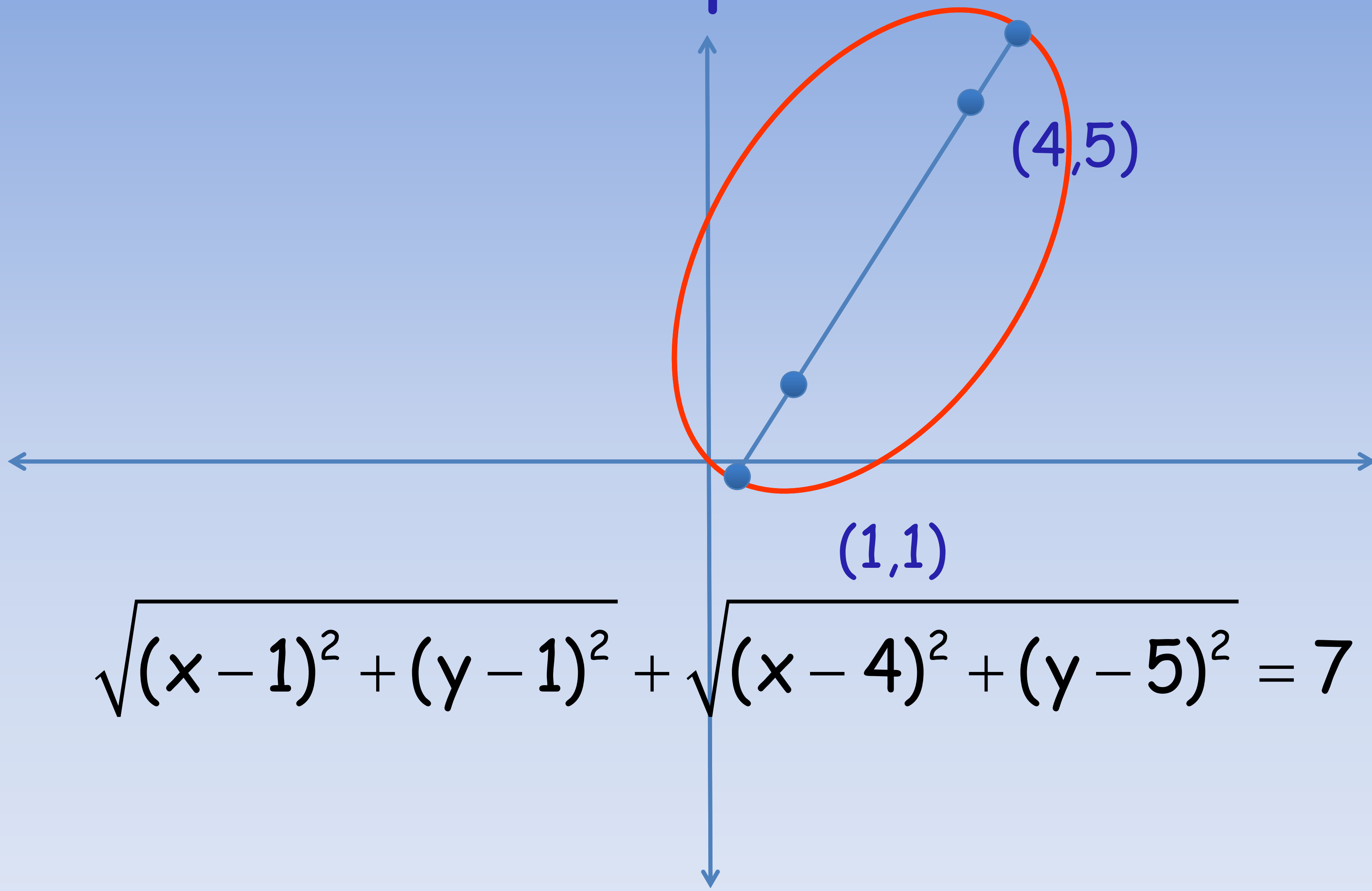
What do we get if $\varepsilon = 0$?

What do we get if $\varepsilon = 1$?

Ellipse



Ellipse



$$\sqrt{(x-1)^2+(y-1)^2}+\sqrt{(x-4)^2+(y-5)^2}=7$$

Ellipse

$$\sqrt{(x-1)^2+(y-1)^2}+\sqrt{(x-4)^2+(y-5)^2}=7$$

$$\left(\sqrt{(x-1)^2+(y-1)^2}+\sqrt{(x-4)^2+(y-5)^2}\right)^2=49$$

[illegible]

$$2\sqrt{(x-1)^2+(y-1)^2}\sqrt{(x-4)^2+(y-5)^2}=49-\left((x-1)^2+(y-1)^2+(x-4)^2+(y-5)^2\right)$$

$$2\sqrt{(x-1)^2+(y-1)^2}\sqrt{(x-4)^2+(y-5)^2}=49-\left(2x^2+2y^2-10x-12y+43\right)$$

$$=6-\left(2x^2+2y^2-10x-12y\right)$$

$$\sqrt{(x-1)^2+(y-1)^2}\sqrt{(x-4)^2+(y-5)^2}=3-\left(x^2+y^2-5x-6y\right)$$

$$\left(\sqrt{(x-1)^2+(y-1)^2}\sqrt{(x-4)^2+(y-5)^2}\right)^2=\left(3-\left(x^2+y^2-5x-6y\right)\right)^2$$

Ellipse

$$\left((x-1)^2+(y-1)^2\right)\left((x-4)^2+(y-5)^2\right)=\left(3-\left(x^2+y^2-5x-6y\right)\right)^2$$

$$\left(x^2+y^2-2x-2y+2\right)\left(x^2+y^2-8x-10y+41\right)=$$

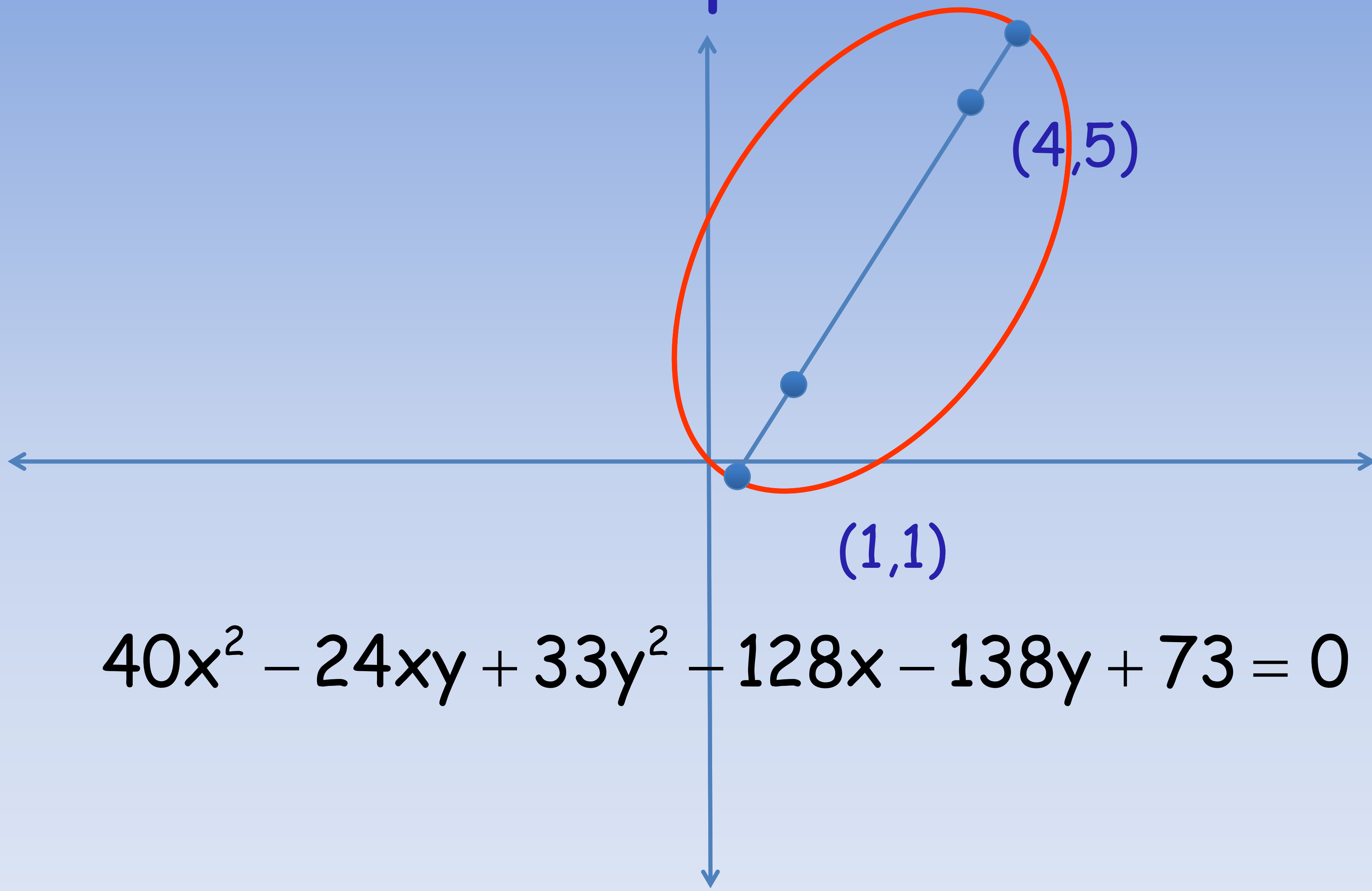
$$x^4+2x^2y^2+y^4-10x^3-12x^2y-10xy^2-12y^3+19x^2+60xy+30y^2+30x+36y+9$$

$$x^4+2x^2y^2+y^4-10x^3-12x^2y-10xy^2-12y^3+59x^2+36xy+63y^2-98x-102y+82=$$

$$x^4+2x^2y^2+y^4-10x^3-12x^2y-10xy^2-12y^3+19x^2+60xy+30y^2+30x+36y+9$$

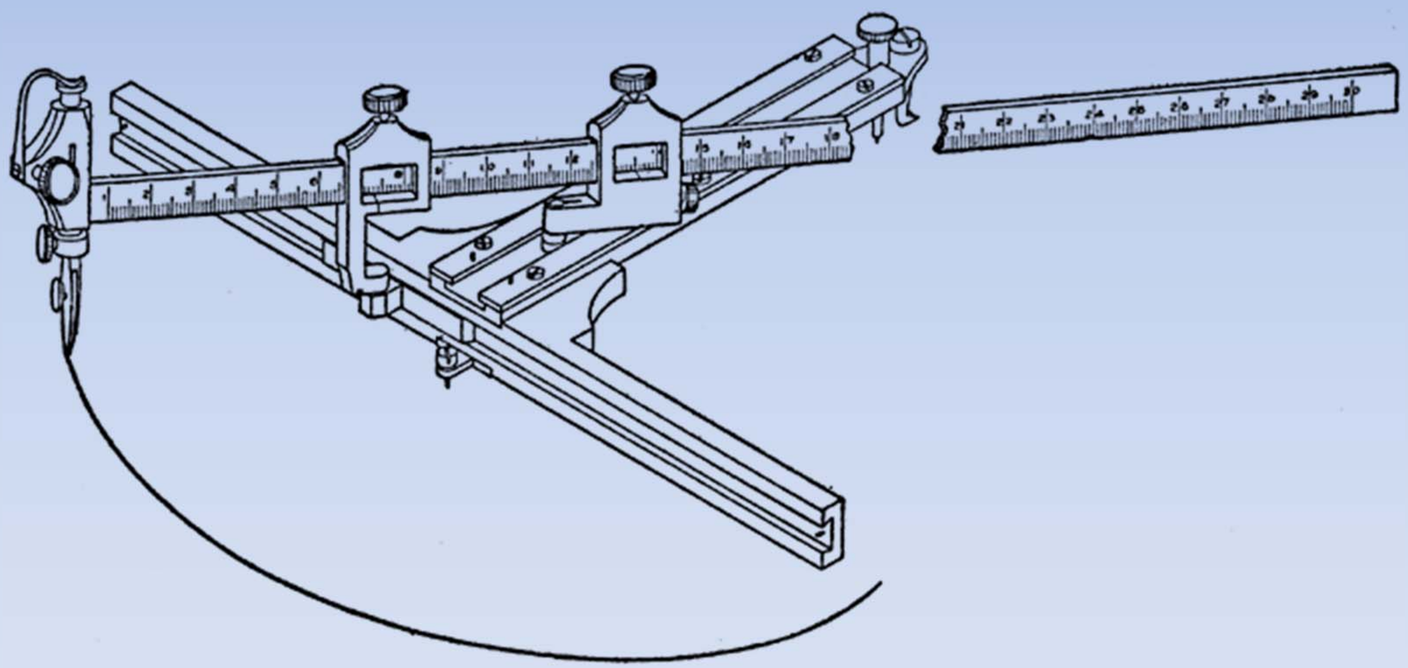
$$40x^2-24xy+33y^2-128x-138y+73=0$$

Ellipse



$$40x^2 - 24xy + 33y^2 - 128x - 138y + 73 = 0$$

Ellipsograph



Ellipse

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

$$x = x'\cos(\theta) - y'\sin(\theta)$$

$$y = x'\sin(\theta) + y'\cos(\theta)$$