Coordinatization of the Plane

MA 341 – Topics in Geometry
Lecture 29

Background

Used by Egyptians & Romans for surveying

Used by Greeks for mapmaking

Nicole Oresme graphed one variable against another in 14th century

Credit goes to Rene Descartes and Pierre Fermat for formalizing the process of analytic geometry

The Line

Ruler Postulate guarantees that each line looks like the real line

There is a one-to-one correspondence between each line and the real line

We can pick a point that corresponds to 0, then each point will be given some value – say A gets coordinate a, B gets coordinate b, and $d(A,B) = |b - a|$

Line

Given two points A and B and k>0 there is at least one point C so that

$$\frac{AC}{CB} = k$$

Assume a < b. Find C so that

$$\frac{AC}{CB} = \frac{|c-a|}{|c-b|} = k$$

Line

$$\frac{AC}{CB}=\frac{|c-a|}{|c-b|}=k \iff \begin{cases} \dfrac{c-a}{b-c}=k & \text{if } a<c<b \\ \dfrac{c-a}{c-b}=k & \text{if } c<a \text{ or } c>b \end{cases}$$

$$\iff \begin{cases} c-a=kb-kc & \text{if } a<c<b \\ c-a=kc-kb & \text{if } c<a \text{ or } c>b \end{cases}$$

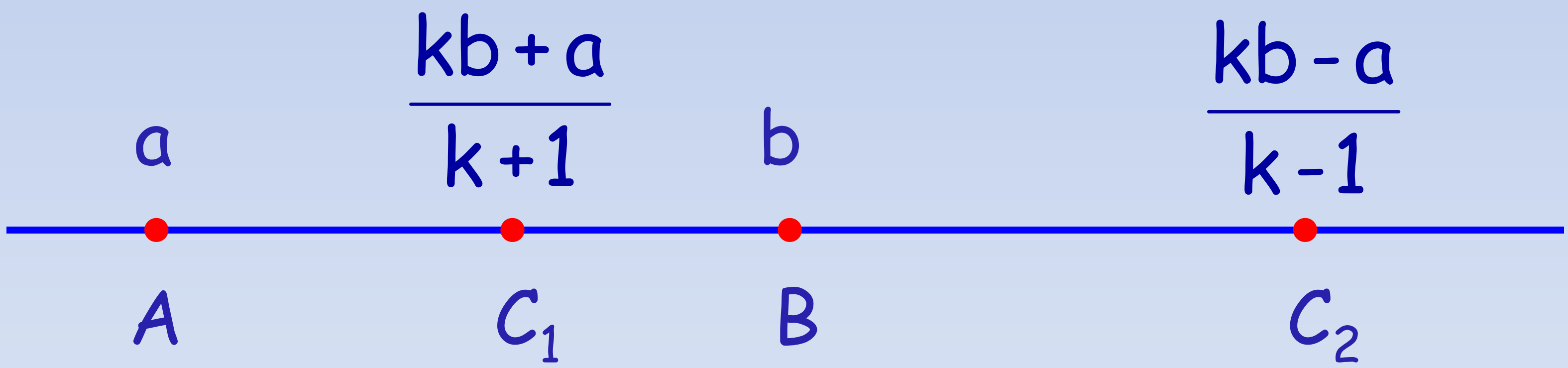
$$\iff \begin{cases} c=\dfrac{kb+a}{k+1} & \text{if } a<c<b \\ c=\dfrac{kb-a}{k-1} & \text{if } c<a \text{ or } c>b, k\neq 1 \end{cases}$$

Line

Unless k=1

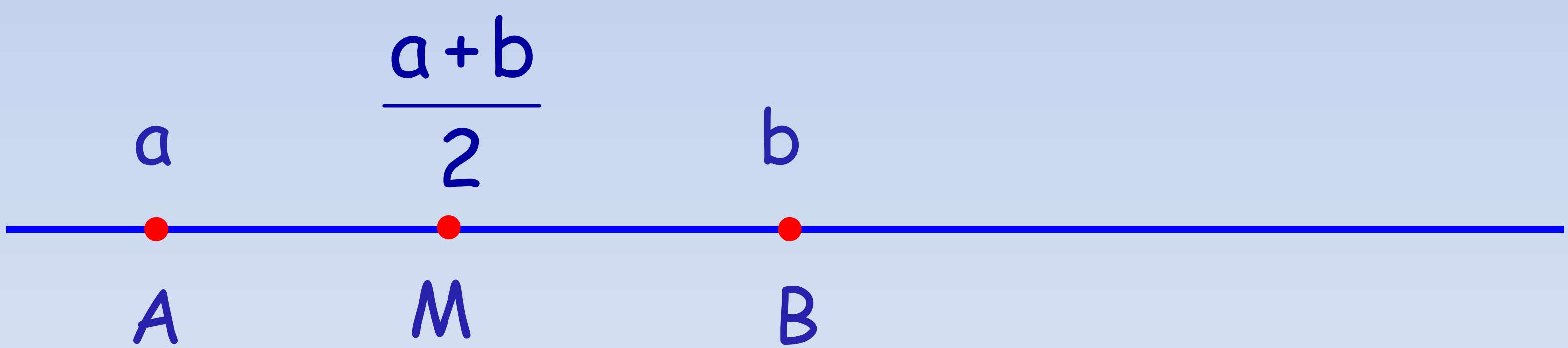
There are 2 points c to satisfy the ratio:

- one between A and B
- other not between A and B



Line

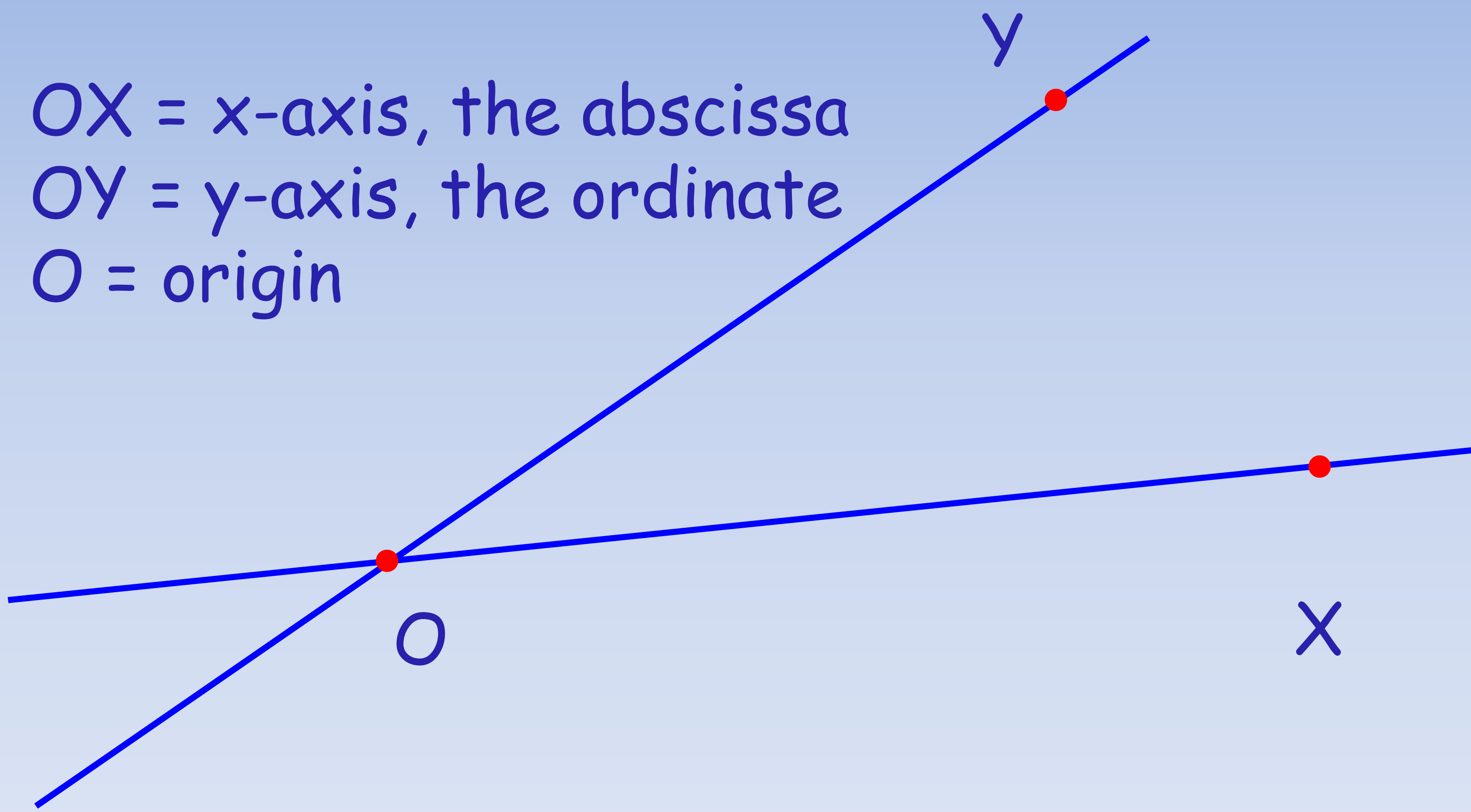
Note k=1 is a special point - the midpoint which can be shown GEOMETRICALLY to be unique



Plane

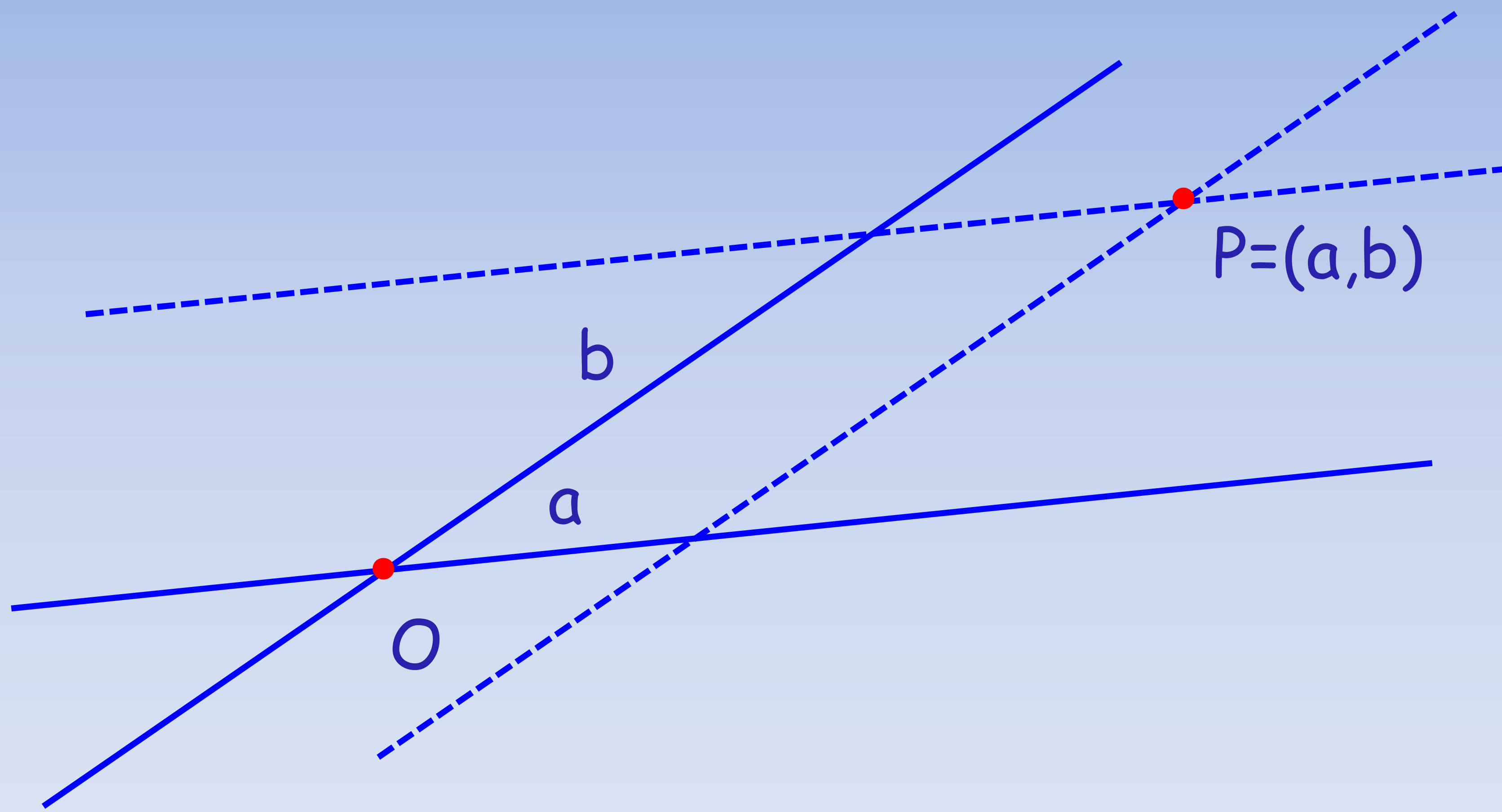
Let O, X, Y be three non-collinear points in plane.

OX = x-axis, the abscissa
OY = y-axis, the ordinate
O = origin



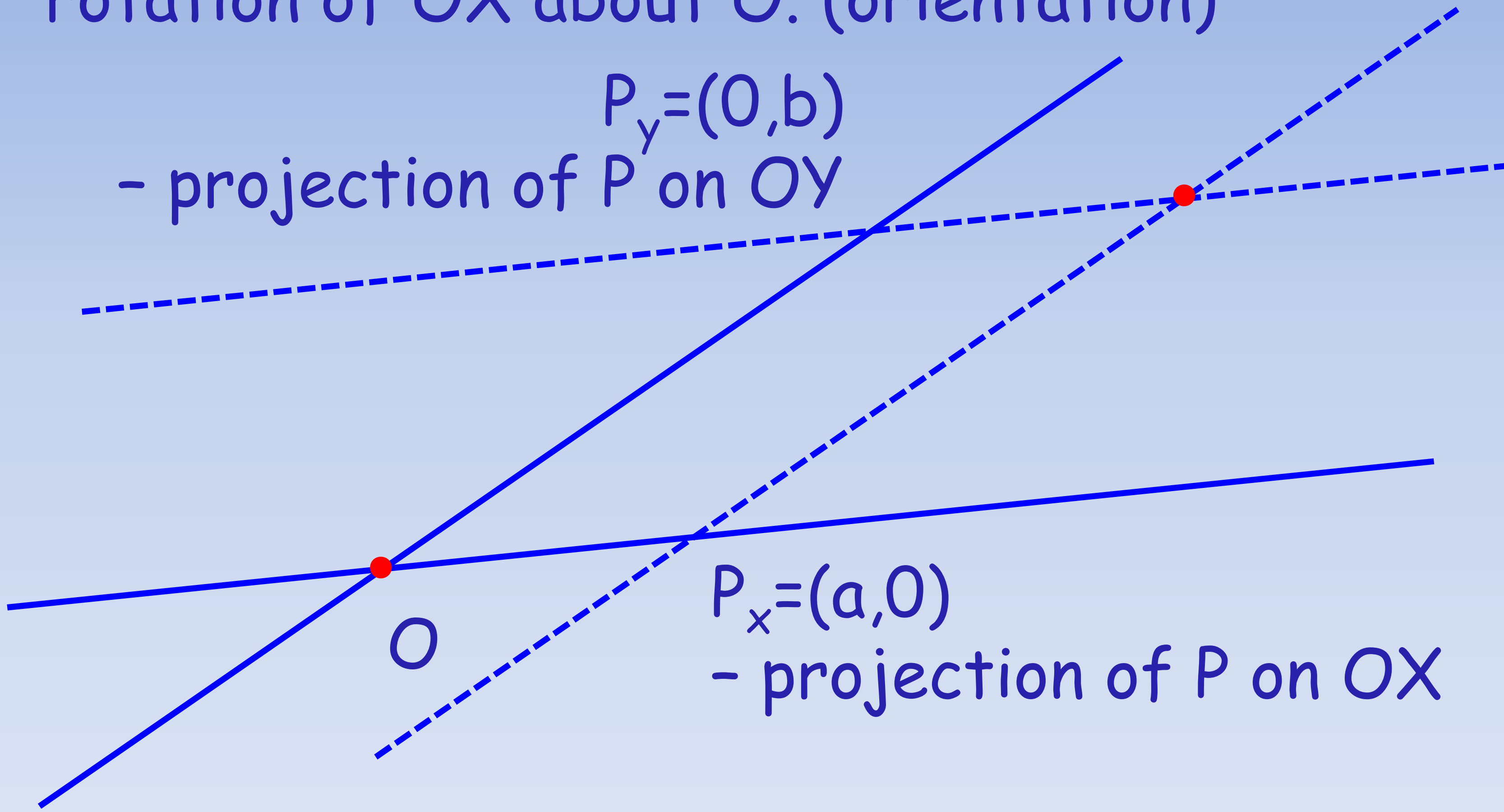
Plane

Lines run parallel to the axes



Plane

Regular assumptions:
OY comes from OX by a counterclockwise rotation of OX about O. (orientation)



Plane

There is a one-to-one correspondence between the points in the plane and the set of ordered pairs of real numbers

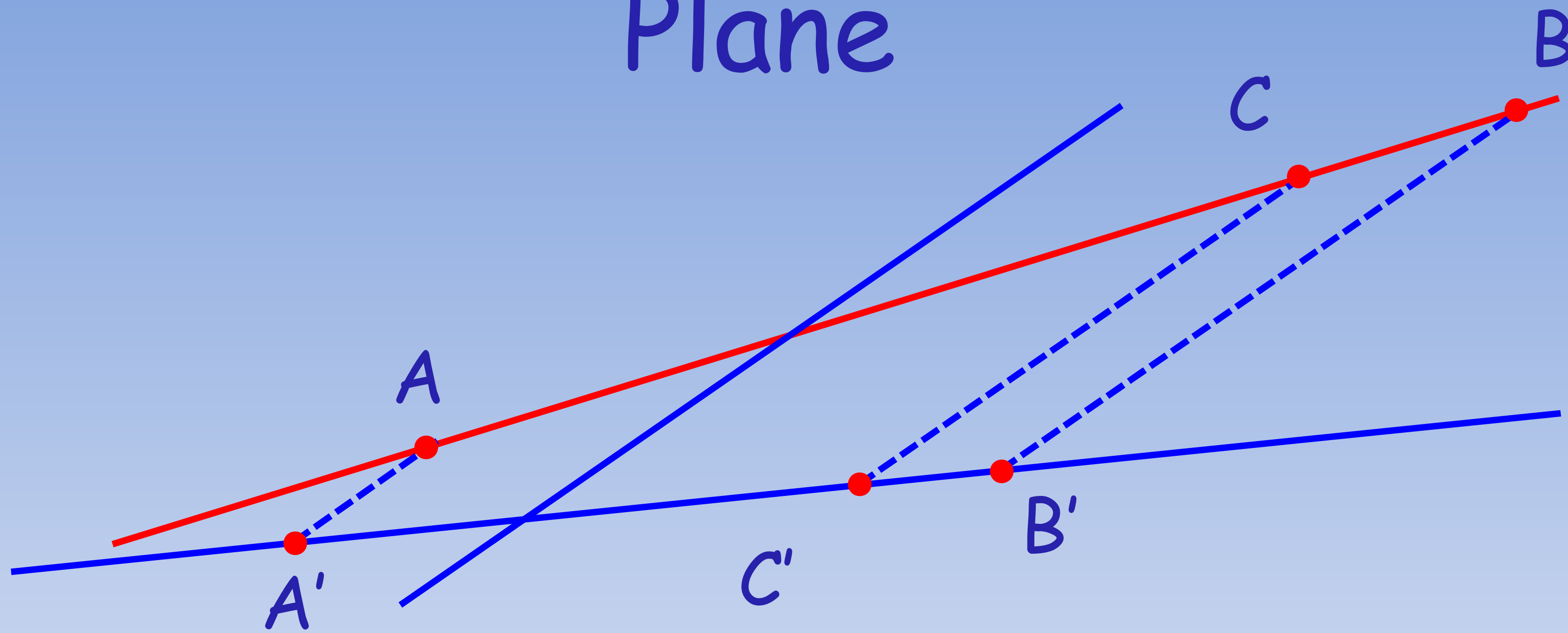
$$\{ (x,y) \mid x,y \in R\}$$

Plane

Given two points A and B and k>0 there is at least one point C that divides the line segment AB in the ratio k:1.

Assume $x_A \leq x_B$. Find coordinates of point (x_C, y_C) that internally divides AB in a ratio k:1.

Plane

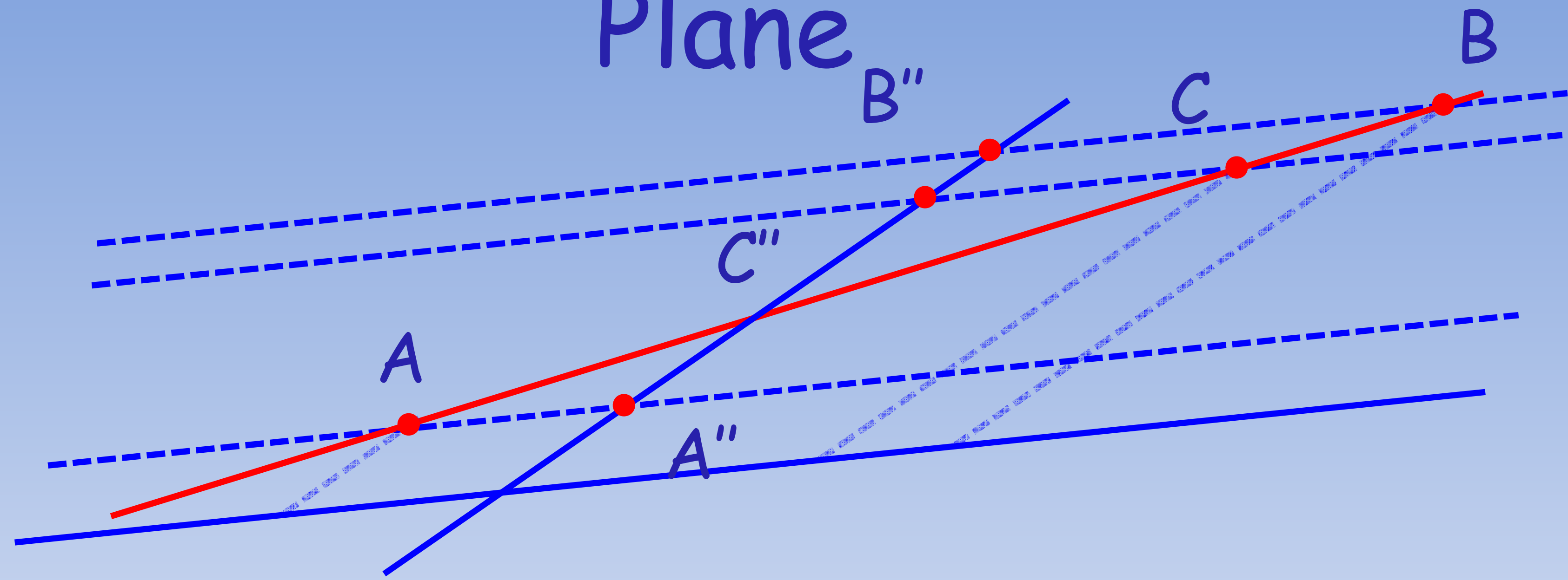


AA' || BB' || CC'

Then $\frac{AC}{CB} = \frac{A'C'}{C'B'}$

thus $x_C = \frac{kx_B + x_A}{k+1}$

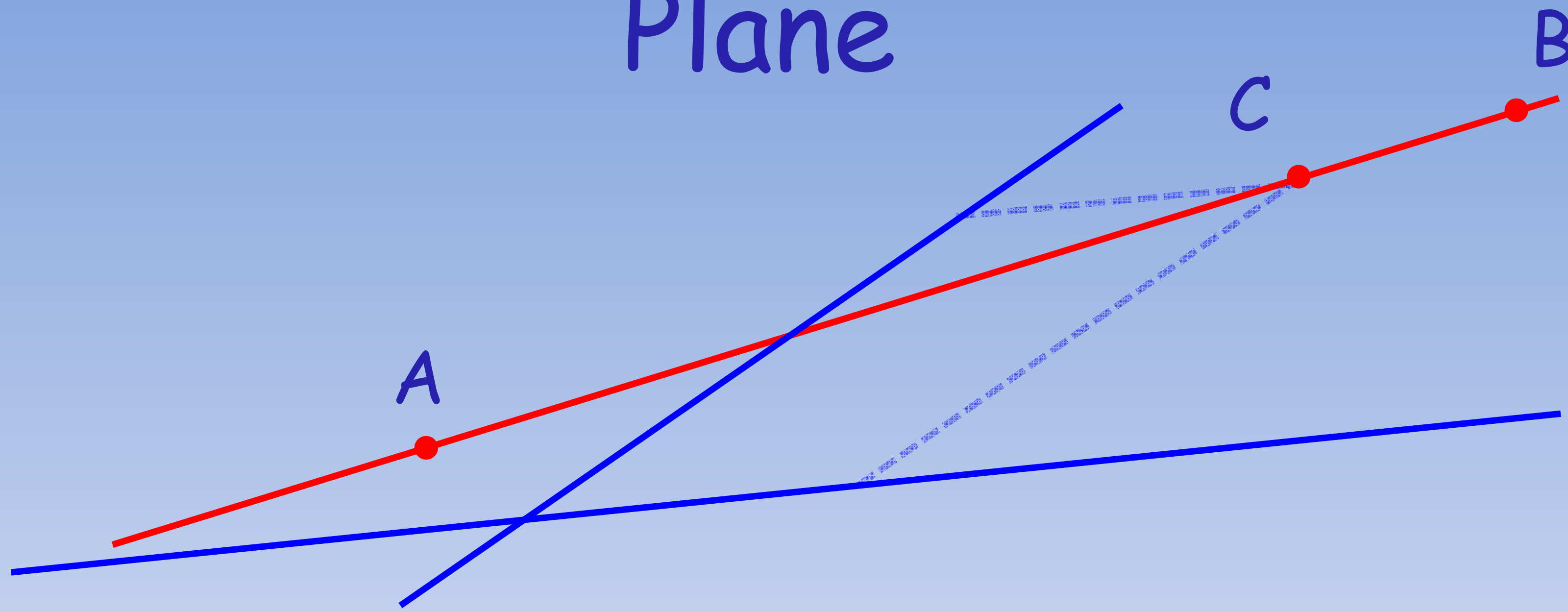
Plane



AA'' || BB'' || CC'' || OY

Then $\frac{AC}{CB} = \frac{A''C''}{C''B''}$ thus $y_C = \frac{ky_B + y_A}{k+1}$

Plane

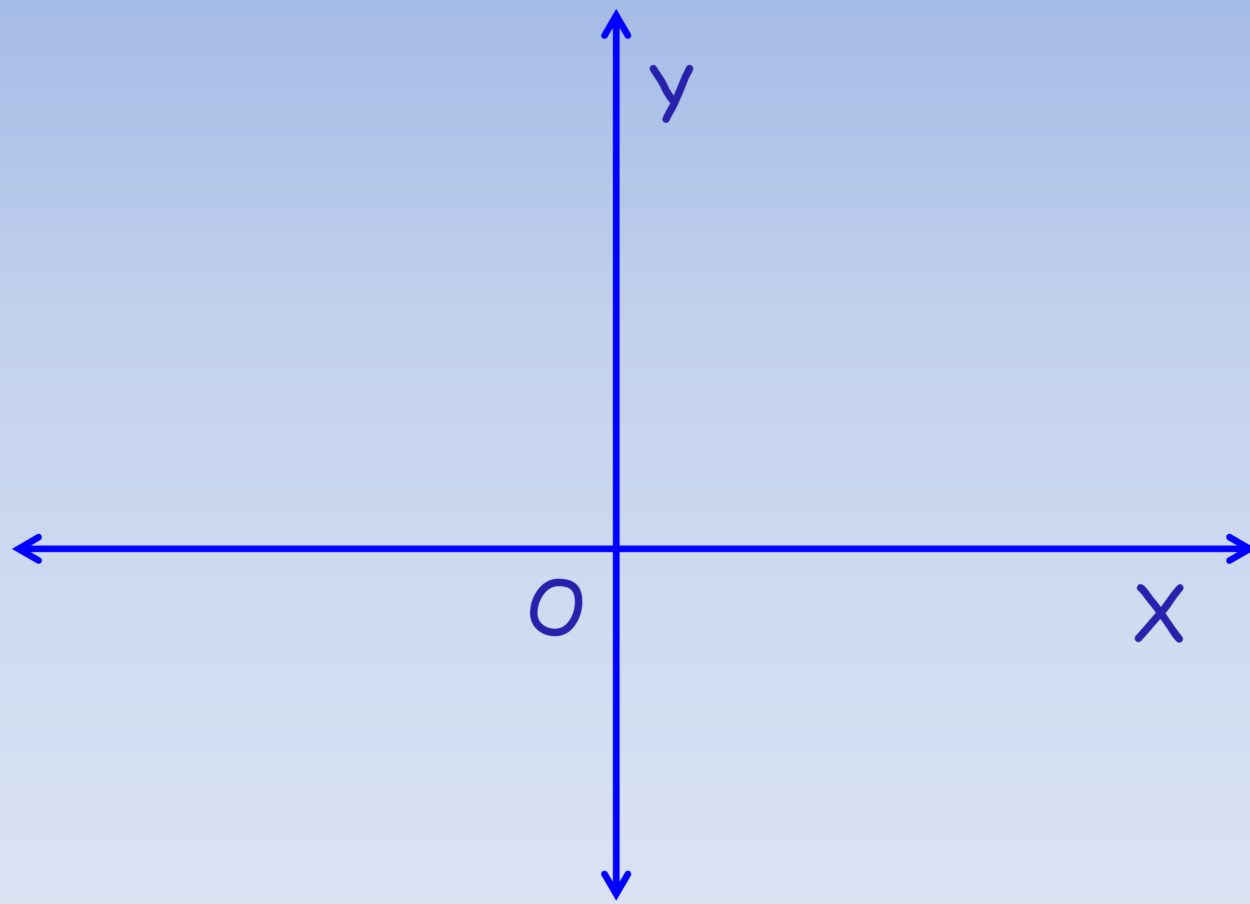


Thus, $$C = \left(\frac{kx_B + x_A}{k+1}, \frac{ky_B + y_A}{k+1} \right)$$

Midpoint: $$M = \left(\frac{x_B + x_A}{2}, \frac{y_B + y_A}{2} \right)$$

Plane

Normally take OX and OY perpendicular!!



Distance Formula

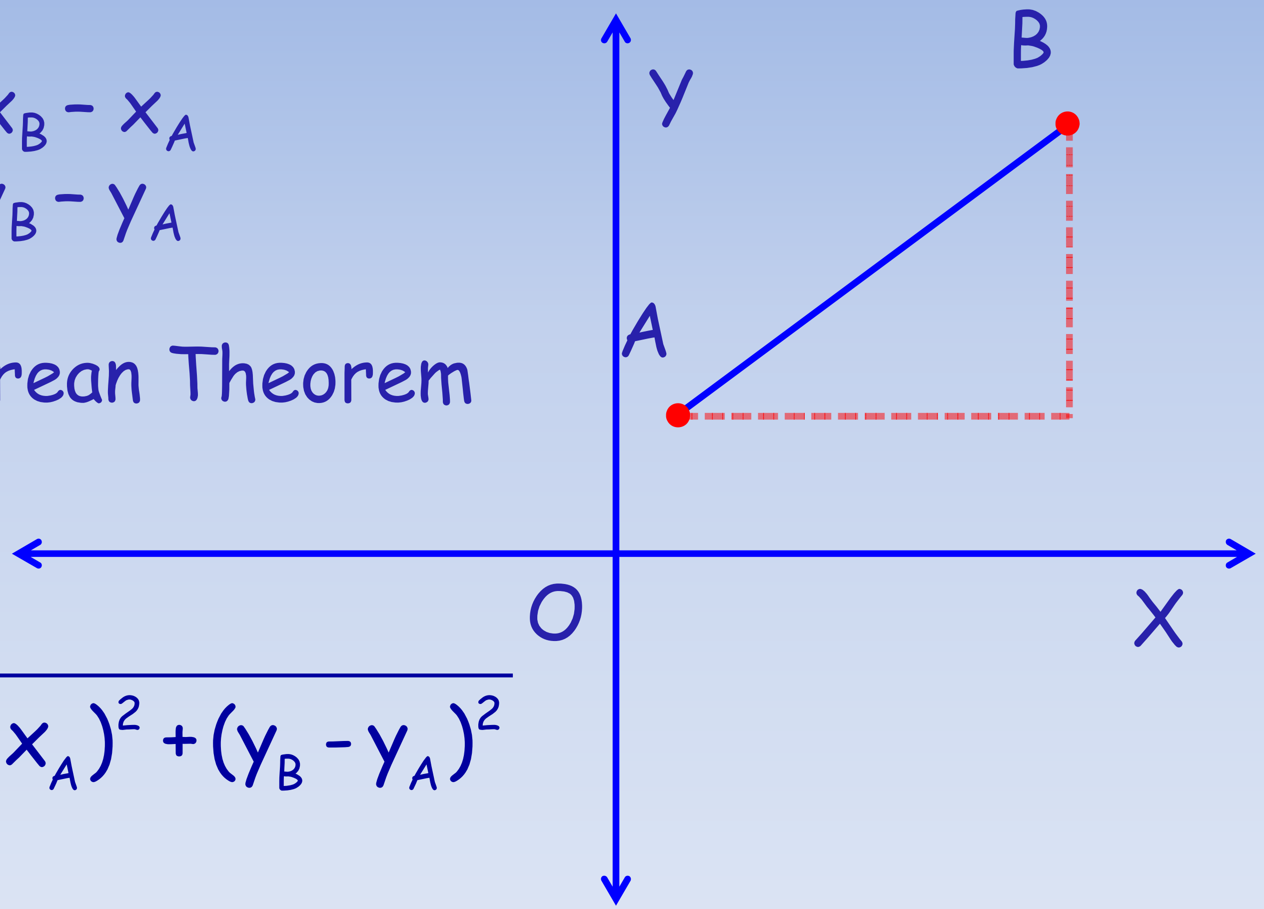
$A = (x_A, y_A)$ and $B = (x_B, y_B)$

AB = hypotenuse of right triangle

x-leg = $x_B - x_A$

y-leg = $y_B - y_A$

Pythagorean Theorem



$$AB = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$$

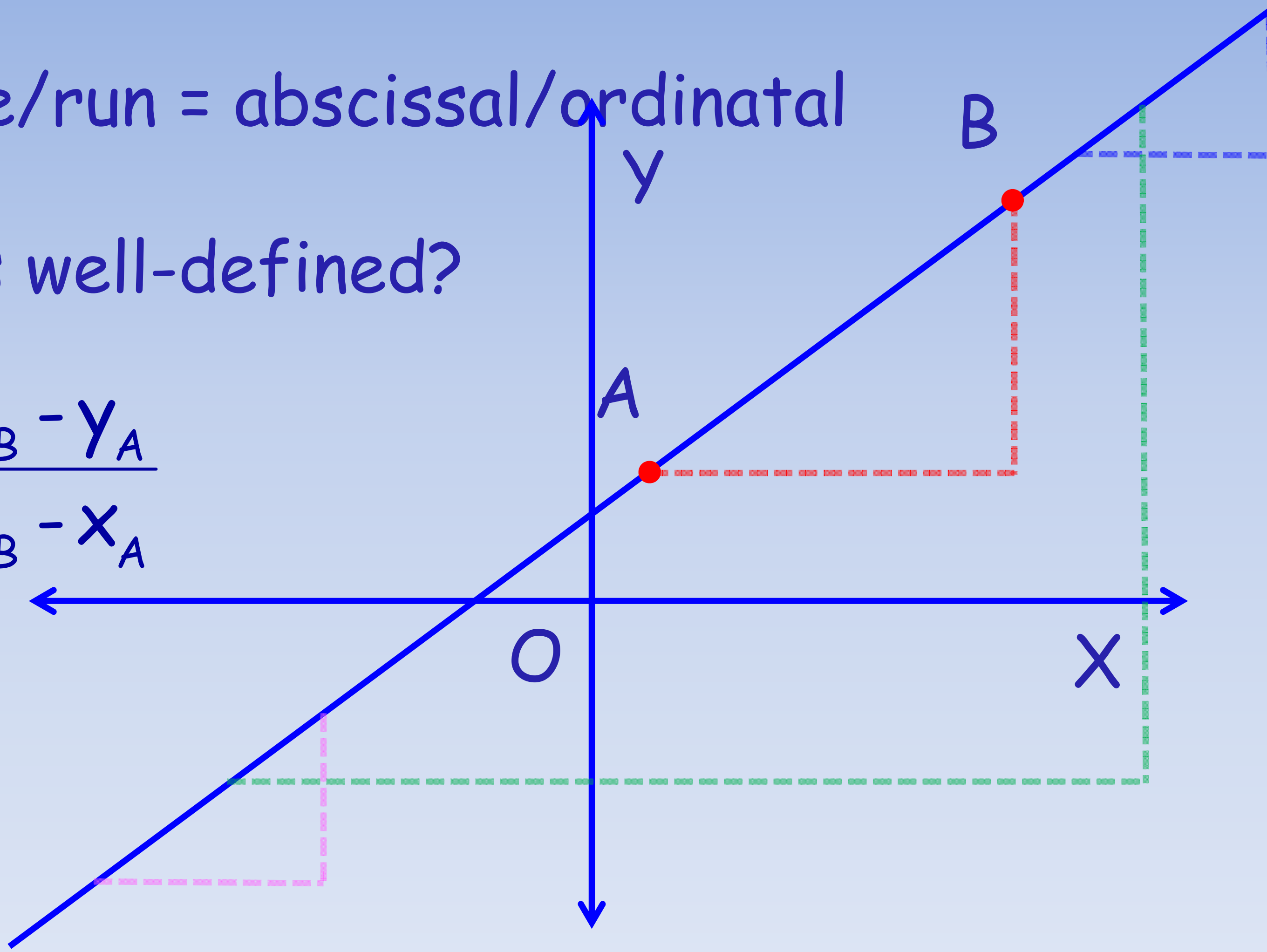
Lines

$A = (x_A, y_A)$ and $B = (x_B, y_B)$

Slope = rise/run = abscissal/ordinatal

Why is this well-defined?

$$m_{AB} = \frac{y_B - y_A}{x_B - x_A}$$



Line Theorem

Let l be a line with slope m and let A be a point on l. A point P lies on l if and only if its coordinates satisfy the equation

$$y - y_A = m(x - x_A)$$

Proof: Let P = (x_P, y_P) lie on l, P $\neq$ A. Then l = AP, so

$$m = \frac{y_P - y_A}{x_P - x_A}$$

Therefore, (x_P, y_P) satisfies the equation

$$y - y_A = m(x - x_A)$$

Line Theorem

Proof: Assume that $P = (x_P, y_P)$ satisfies the equation

$$y - y_A = m(x - x_A)$$

Therefore

$$y_P - y_A = m(x_P - x_A)$$

Let P' lie on l and let P' have the same x-coordinate as P: $x_P = x_{P'}$

Then

$$\begin{aligned} y_{P'} - y_A &= m(x_{P'} - x_A) \\ &= m(x_P - x_A) \\ &= y_P - y_A \end{aligned}$$

$y_{P'} = y_p$ and $P = P'$

Equations of Lines

$y - y_0 = m(x - x_0)$ - point-slope form

$y = mx + b$ - slope-intercept form

$$\frac{x}{a} + \frac{y}{b} = 1$$ - xy intercept form

$ax + by + c = 0$ - standard form

Corollary: Lines are parallel if and only if they have the same slope or no slope.

Inclination of Lines

For any non-horizontal line define its inclination to be the smallest positive measure, θ, of angle measured counterclockwise from the positive x-axis.

If horizontal define its inclination to be 0.

Note: $0^\circ \leq \theta \leq 180^\circ$

$m = \tan\theta$

Intersection of Lines

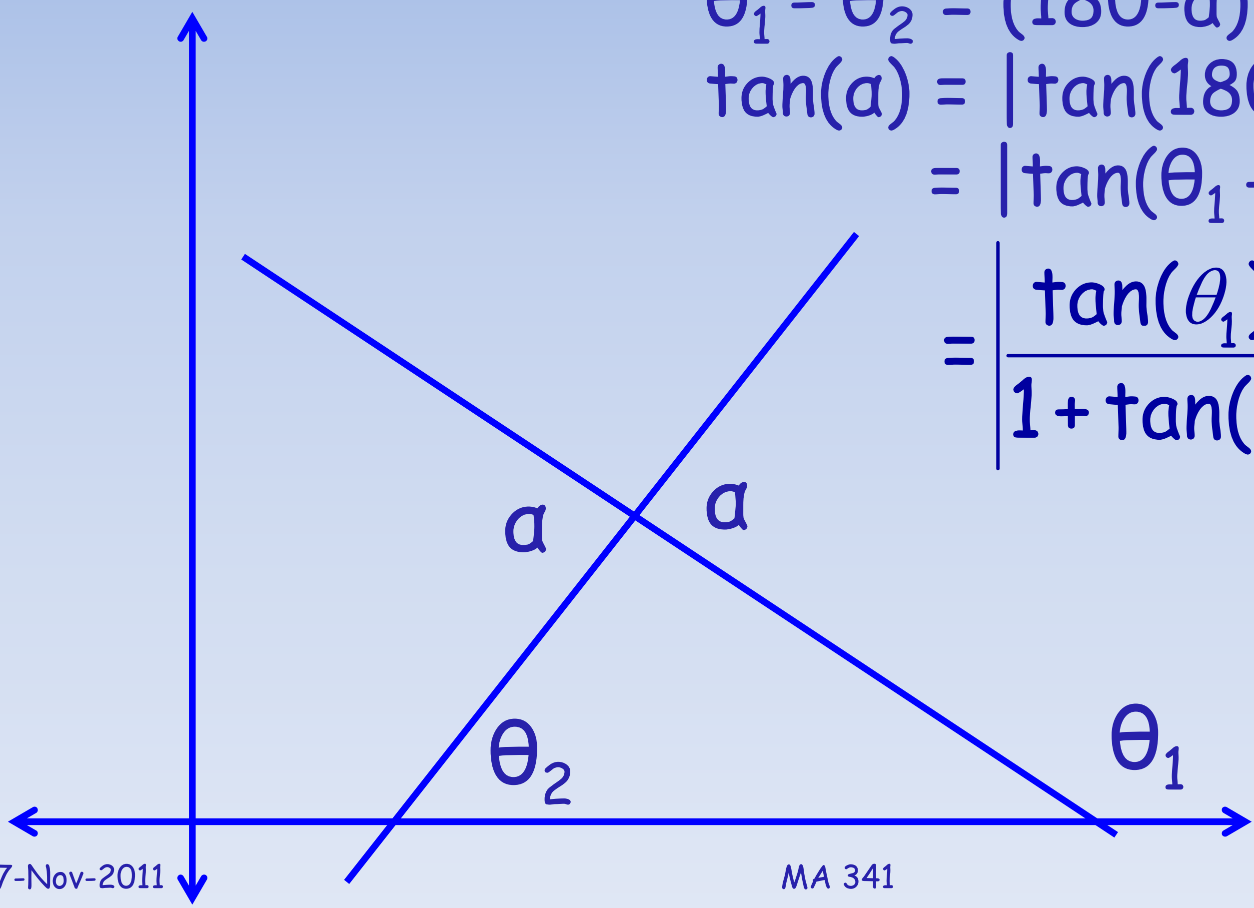
Let α be angle between intersecting non-vertical lines l_1 and l_2 having slopes m_1 and m_2. Then $\alpha = 0$ iff $m_1m_2=-1$. Otherwise

$$\tan(\alpha) = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$$

Intersection of Lines

$m_1 = \tan(\theta_1)$ and $m_2 = \tan(\theta_2)$

$$\theta_1 = \theta_2 + (180-\alpha)$$
$$\theta_1 - \theta_2 = (180-\alpha)$$
$$\tan(\alpha) = |\tan(180-\alpha)|$$
$$= |\tan(\theta_1 - \theta_2)|$$
$$= \left|\frac{\tan(\theta_1) - \tan(\theta_2)}{1 + \tan(\theta_1)\tan(\theta_2)}\right|$$



Distance from a Point to a Line

If a vertical or horizontal line, easy.

$P = (x_p, y_p)$ and line l: $ax + by + c = 0$

Assume $ab \neq 0$. Then $m = -a/b$, so line through P perpendicular to l is $y - y_p = b/a(x - x_p)$

Find point of intersection:

$$x_q = \frac{b^2 x_p - ac - aby_p}{a^2 + b^2} \qquad y_q = \frac{a^2 y_p - bc - abx_p}{a^2 + b^2}$$

$$x_p - x_q = \frac{a(ax_p + by_p + c)}{a^2 + b^2} \qquad y_p - y_q = \frac{b(ax_p + by_p + c)}{a^2 + b^2}$$

Distance from a Point to a Line

$$d(P,l) = d(P,Q) = \sqrt{(x_p - x_q)^2 + (y_p - y_q)^2}$$

$$= \sqrt{\left(\frac{a(ax_p + by_p + c)}{a^2 + b^2}\right)^2 + \left(\frac{b(ax_p + by_p + c)}{a^2 + b^2}\right)^2}$$

$$= \sqrt{\frac{(a^2 + b^2)(ax_p + by_p + c)^2}{(a^2 + b^2)^2}}$$

$$= \frac{|ax_p + by_p + c|}{\sqrt{a^2 + b^2}}$$

Applications

Medians are concurrent and centroid divides each in a 2:1 ratio.

a) Brute force method – find midpoint of each side, equation of each median, points of intersection of each pair of lines – algebraically tedious

$$G = \left(\frac{x_a + x_b + x_c}{3}, \frac{y_a + y_b + y_c}{3} \right)$$

Applications

b) Create coordinate system so that median of AC is origin, A=(a,0) and B=(0,1) and C=(-a,0)
Gives simple equations for each median

c) Don't find equations of lines at all, but use earlier result on ratios. D=midpoint of BC so

$$D = \left(\frac{x_b + x_c}{2}, \frac{y_b + y_c}{2} \right)$$

Let P=point on AD so that AP/PD=2/1.

$$P = \left(\frac{x_a + 2x_d}{3}, \frac{y_a + 2y_d}{3} \right) = \left(\frac{x_a + x_b + x_c}{3}, \frac{y_a + y_b + y_c}{3} \right)$$

Applications

Given two intersecting lines and a positive number k, find the locus of all points the sum of whose distances to the lines is k.

Choose OX and OY so that they are the angle bisectors of the angles formed by the two lines.

CLAIM: OX and OY are perpendicular.

Line l_1 has equation $y = mx$, $m>0$, line l_2 has equation $y=-mx$.

Applications

(x,y) on the locus if and only if

$$d((x,y),l_1) + d((x,y),l_2) = k$$

$$\frac{|mx-y|}{\sqrt{m^2+1}} + \frac{|-mx-y|}{\sqrt{(-m)^2+1}} = k$$

Let $K = k\sqrt{m^2+1}$

We have equation: $|mx - y| + |mx + y| = K$
Note: If (x,y) satisfies this then so do (-x,y), (-x,-y) and (x,-y)
We only need work in first quadrant

Applications

In first quadrant $mx + y \geq 0$, so

$$|mx - y| + mx + y = K$$

Reduces to 2 cases: $mx - y \geq 0$ or $mx - y \leq 0$

If $0 \leq y \leq mx$, then

$$|mx - y| + mx + y = mx - y + mx + y = K$$

$$x = K/(2m)$$

and the graph is a vertical segment

If $0 \leq mx < y$, then

$$|mx - y| + mx + y = -mx = y + mx + y = K$$

$y = K/2$ and the graph is a horizontal segment

Applications