Chapter 5

Compactness

Compactness is the generalization to topological spaces of the property of closed and bounded subsets of the real line: the Heine-Borel Property. While compact may infer "small" size, this is not true in general. We will show that $[0, 1]$ is compact while $(0, 1)$ is not compact.

Compactness was introduced into topology with the intention of generalizing the properties of the closed and bounded subsets of $\mathbb{R}^n$.

5.1 Compact Spaces and Subspaces

Definition 5.1 *Let A be a subset of the topological space X. An **open cover** for A is a collection $\mathscr{O}$ of open sets whose union contains A. A **subcover** derived from the open cover $\mathscr{O}$ is a subcollection $\mathscr{O}'$ of $\mathscr{O}$ whose union contains A.*

Example 5.1.1 Let $A = [0, 5]$ and consider the open cover

$$\mathscr{O} = \{(n-1, n+1) \mid n = -\infty, \dots, \infty\}.$$

Consider the subcover $\mathscr{P} = \{(-1, 1), (0, 2), (1, 3), (2, 4), (3, 5), (4, 6)\}$ is a subcover of A, and happens to be the smallest subcover of $\mathscr{O}$ that covers A.

Definition 5.2 *A topological space X is **compact** provided that every open cover of X has a finite subcover.*

This says that however we write X as a union of open sets, there is always a finite subcollection $\{O_i\}_{i=1}^n$ of these sets whose union is X. A subspace A of X is ***compact*** if A is a compact space in its subspace topology. Since relatively open sets in the subspace topology are the intersections of open sets in X with the subspace A, the definition of compactness for subspaces can be restated as follows.

Alternate Definition: *A subspace A of X is **compact** if and only if every open cover of A by open sets in X has a finite subcover.*

Example 5.1.2 1. Any space consisting of a finite number of points is compact.

2. The real line $\mathbb{R}$ with the finite complement topology is compact.

3. An infinite set X with the discrete topology is not compact.

4. The open interval $(0,1)$ is not compact. $\mathscr{O} = \{(1/n, 1) \mid n = 2, \ldots, \infty\}$ is an open cover of $(0,1)$. However, no finite subcollection of these sets will cover $(0,1)$.

5. $\mathbb{R}^n$ is not compact for any positive integer n, since $\mathscr{O} = \{B(\mathbf{0}, n) \mid n = 1, \ldots, \infty\}$ is an open cover with no finite subcover.

A sequence of sets $\{S_n\}_{n-1}^{\infty}$ is **nested** if $S_{n+1} \subset S_n$ for each positive integer n.

Theorem 5.1 (Cantor's Nested Intervals Theorem) *If $\{[a_n, b_n]\}_{n=1}^{\infty}$ is a nested sequence of closed and bounded intervals, then $\cap_{n=1}^{\infty}[a_n, b_n] \neq \emptyset$. If, in addition, the diameters of the intervals converge to zero, then the intersection consists of precisely one point.*

PROOF: Since $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ for each $n \in \mathbb{Z}^+$, the sequences $\{a_n\}$ and $\{b_n\}$ of left and right endpoints have the following properties:

(i) $a_1 \leq a_2 \leq \cdots \leq a_n \leq \ldots$ and $\{a_n\}$ is an increasing sequence;

(ii) $b_1 \geq b_2 \geq \cdots \geq b_n \geq \ldots$ and $\{b_n\}$ is a decreasing sequence;

(iii) each left endpoint is less than or equal to each right endpoint.

Let c denote the least upper bound of the left endpoints and d the greatest lower bound of the right endpoints. The existence of c and d are guaranteed by the Least Upper Bound Property. Now, by property (iii), $c \leq b_n$ for all n, so $c \leq d$. Since $a_n \leq c \leq d \leq b_n$, then $[c, d] \subset [a_n, b_n]$ for all n. Thus, $\cap_{n=1}^{\infty}[a_n, b_n]$ contains the closed interval $[c, d]$ and is thus non-empty.

If the diameters of $[a_n, b_n]$ go to zero, then we must have that $c = d$ and c is the one point of the intersection. ∎

Theorem 5.2 *The interval* $[0,1]$ *is compact.*

PROOF: Let $\mathscr{O}$ be an open cover. Assume that $[0,1]$ is not compact. Then either $[0, \frac{1}{2}]$ or $[\frac{1}{2}, 1]$ is not covered by a finite number of members of $\mathscr{O}$. Let $[a_1, b_1]$ be the half that is not covered by a finite number of members of $\mathscr{O}$.

Apply the same reasoning to the interval $[a_1, b_1]$. One of the halves, which we will call $[a_2, b_2]$, is not finitely coverable by $\mathscr{O}$ and has length $\frac{1}{4}$. We can continue this reasoning inductively to create a nested sequence of closed intervals $\{[a_n, b_n]\}_{n=1}^{\infty}$, none of which is finitely coverable by $\mathscr{O}$. Also, by construction, we have that

$$b_n - a_n = \frac{1}{2^n},$$

so the diameters of these intervals goes to zero.

By the Cantor Nested Intervals Theorem, we know that there is precisely one point in the intersection of all of these intervals; $p \in [a_n, b_n]$, for all n. Since $p \in [0,1]$ there is an open interval $O \in \mathscr{O}$ with $p \in O$. Thus, there is a positive number, $\epsilon > 0$ so that $(p-\epsilon, p+\epsilon) \subset O$. Let N be a positive integer so that $1/2^N < \epsilon$. Then since $p \in [a_N, b_n]$ it follows that

$$[a_n, b_n] \subset (p-\epsilon, p+\epsilon) \subset O.$$

This contradicts the fact that $[a_N, b_N]$ is not finitely coverable by $\mathscr{O}$ since we just covered it with one set from $\mathscr{O}$. This contradiction shows that $[0,1]$ is finitely coverable by $\mathscr{O}$ and is compact. ∎

Compactness is defined in terms of open sets. The duality between open and closed sets and if $C_\alpha = X \setminus O_\alpha$,

$$X \setminus \left(\bigcap_{\alpha \in I} C_\alpha \right) = \bigcup_{\alpha \in I} O_\alpha$$

leads us to believe that there is a characterization of compactness with closed sets.

Definition 5.3 *A family $\mathscr{A}$ of subsets of a space X has the **finite intersection property** provided that every finite subcollection of $\mathscr{A}$ has non-empty intersection.*

Theorem 5.3 *A space X is compact if and only if every family of closed sets in X with the finite intersection property has non-empty intersection.*

This says that if $\mathscr{F}$ is a family of closed sets with the finite intersection property, then we must have that $\bigcap_{\mathscr{F}} C_\alpha \neq \emptyset$.

PROOF: Assume that X is compact and let $\mathscr{F} = \{C_\alpha \mid \alpha \in I\}$ be a family of closed sets with the finite intersection property. We want to show that the intersection of all members of $\mathscr{F}$ is non-empty. Assume that the intersection is empty. Let $\mathscr{O} = \{O_\alpha = X \setminus C_\alpha \mid \alpha \in I\}$. $\mathscr{O}$ is a collection of open sets in X. Then,

$$\bigcup_{\alpha \in I} O_\alpha = \bigcup_{\alpha \in I} X \setminus C_\alpha = X \setminus \bigcap_{\alpha \in I} C_\alpha = X \setminus \emptyset = X.$$

Thus, $\mathscr{O}$ is an open cover for X. Since X is compact, it must have a finite subcover; *i.e.*,

$$X = \bigcup_{i=1}^{n} O_{\alpha_i} = \bigcup_{i=1}^{n} (X \setminus C_{\alpha_i}) = X \setminus \bigcap_{i=1}^{n} C_{\alpha_i}.$$

This means that $\cap_{i=1}^{n} C_{\alpha_i}$ must be empty, contradicting the fact that $\mathscr{F}$ has the finite intersection property. Thus, if $\mathscr{F}$ has the finite intersection property, then the intersection of all members of $\mathscr{F}$ must be non-empty.

The opposite implication is left as an exercise. ∎

Is compactness hereditary? No, because $(0,1)$ is not a compact subset of $[0,1]$. It is *closed hereditary.*

Theorem 5.4 *Each closed subset of a compact space is compact.*

PROOF: Let A be a closed subset of the compact space X and let $\mathscr{O}$ be an open cover of A by open sets in X. Since A is closed, then $X \setminus A$ is open and

$$\mathscr{O}^* = \mathscr{O} \cup \{X \setminus A\}$$

is an open cover of X. Since X is compact, it has a finite subcover, containing only finitely many members $O_1, \dots, O_n$ of $\mathscr{O}$ and may contain $X \setminus A$. Since

$$X = (X \setminus A) \cup \bigcup_{i=1}^{n} O_i,$$

it follows that

$$A \subset \bigcup_{i=1}^{n} O_i$$

and A has a finite subcover. ∎

Is the opposite implication true? Is every compact subset of a space closed? Not necessarily. The following though is true.

Theorem 5.5 *Each compact subset of a Hausdorff space is closed.*

PROOF: Let A be a compact subset of the Hausdorff space X. To show that A is closed, we will show that its complement is open. Let $x \in X \setminus A$. Then for each $y \in A$ there are disjoint sets U_y and V_y with $x \in V_y$ and $y \in U_y$. The collection of open sets $\{U_y \mid y \in A\}$ forms an open cover of A. Since A is compact, this open cover has a finite subcover, $\{U_{y_i} \mid i = 1, \dots, n\}$. Let

$$U = \bigcup_{i=1}^{n} U_{y_i} \qquad V = \bigcap_{i=1}^{n} V_{y_i}.$$

Since each U_{y_i} and V_{y_i} are disjoint, we have U and V are disjoint. Also, $A \subset U$ and $x \in V$. Thus, for each point $x \in X \setminus A$ we have found an open set, V, containing x which is disjoint from A. Thus, $X \setminus A$ is open, and A is closed. ∎

Corollary 6 *Let X be a compact Hausdorff space. A subset A of X is compact if and only if it is closed.*

The following results are left to the reader to prove.

Theorem 5.6 *If A and B are disjoint compact subsets of a Hausdorff space X, then there exist disjoint open sets U and V in X such that $A \subset U$ and $B \subset V$.*

Corollary 7 *If A and B are disjoint closed subsets of a compact Hausdorff space X, then there exist disjoint open sets U and V in X such that $A \subset U$ and $B \subset V$.*

5.2 Compactness and Continuity

Theorem 5.7 *Let X be a compact space and $f\colon X \to Y$ a continuous function from X onto Y. Then Y is compact.*

PROOF: We will outline this proof. Start with an open cover for Y. Use the continuity of f to pull it back to an open cover of X. Use compactness to extract a finite subcover for X, and then use the fact that f is onto to reconstruct a finite subcover for Y. ∎

Corollary 8 *Let X be a compact space and $f\colon X \to Y$ a continuous function. The image $f(X)$ of X in Y is a compact subspace of Y.*

Corollary 9 *Compactness is a topological invariant.*

Theorem 5.8 *Let X be a compact space, Y a Hausdorff space, and $f\colon X \to Y$ a continuous one-to-one function. Then f is a homeomorphism.*

5.3 Locally Compact and One-Point Compactifications

Is it always possible to consider a topological space as a subspace of a compact topological space? We can consider the real line as an open interval (they are homeomorphic). Can we always do something of this sort?

Definition 5.4 *A space X is* ***locally compact at a point*** *$x \in X$ provided that there is an open set U containing x for which $\overline{U}$ is compact. A space is* ***locally compact*** *if it is locally compact at each point.*

Note that every compact space is locally compact, since the whole space X satisfies the necessary condition. Also, note that locally compact is a topological property. However, locally compact does not imply compact, because the real line is locally compact, but not compact.

Definition 5.5 *Let X be a topological space and let ∞ denote an ideal point, called the* ***point at infinity****, not included in X. Let $X_\infty = X \cup \infty$ and define a topology $\mathscr{T}_\infty$ on X_∞ by specifying the following open sets:*

(a) the open sets of X, considered as subsets of X_∞;

(b) the subsets of X_∞ whose complements are closed, compact subsets of X; and

(c) the set X_∞.

The space $(X_\infty, \mathscr{T}_\infty$ is called the ***one point compactification*** *of X.*

Theorem 5.9 *Let X be a topological space and X_∞ its one-point compactification. Then*

a) X_∞ is compact.

b) $(X, \mathscr{T})$ is a subspace of $(X_\infty, \mathscr{T}_\infty)$.

c) X_∞ is Hausdorff if and only if X is Hausdorff and locally compact.

d) X is a dense subset of X_∞ if and only if X is not compact.

PROOF:

a) Any open cover $\mathscr{O}$ of X_∞ must have a member U containing ∞. Since the complement $X_\infty \setminus U$ is compact, it has a finite subcover $\{O_i\}_{i=1}^n$ derived from $\mathscr{O}$. Thus, $U, O_1, \ldots, O_n$ is a finite subcover of X_∞.

b) The fact that $(X, \mathscr{T})$ is the subspace topology in $(X_\infty, \mathscr{T}_\infty)$ basically follows from the definition of the extended topology. It also requires that we look at what open sets containing the point at infinity look like. One such set is $U = X_\infty$ itself and $U \cap X = X$ is open in X. The second type is a subset of X_∞ so that $X \setminus U$ is closed and compact in X. In this case $U \cap X$ is open since its complement is closed.

c) Suppose that X_∞ is Hausdorff. Then X is Hausdorff since the property is hereditary. Now, let $p \in X$. Since X_∞ is Hausdorff, there are open, disjoint sets U and V in X_∞ so that $\infty \in U$ and $p \in V$. Thus, $V \subset X_\infty \setminus U$ and this latter set is closed and compact in X. Hence $\overline{V} \subset X_\infty \setminus U$, so $\overline{V}$ is compact, since it is a closed subset of a compact set. Thus, X is locally compact at p.

   Now, suppose that X is Hausdorff and locally compact. To show that X_∞ is Hausdorff, we only need to be able to separate ∞ from any point in $p \in X$. Since X is locally compact, there is an open set O so that $p \in O$ and $\overline{O}$ is compact. Then O and $X_\infty \setminus \overline{O}$ are two disjoint open sets in X_∞ containing p and ∞ respectively.

d) If X is compact, then $\{\infty\}$ is an open set in X_∞, since $\{\infty\} = X_\infty \setminus X$. Thus, ∞ is not a limit point of X, and $\overline{X} \neq X_\infty$. Hence, X is not dense.

   If X is not dense in X_∞, then $\overline{X} = X$, since $\infty \notin \overline{X}$. Hence, $\{\infty\}$ is open in X_∞. Thus, X is compact. ∎

Example 5.3.1 What is the one-point compactification of the open interval $(0, 1)$? You can define a function $f\colon (0,1)_\infty \to S^1$ by

$$f(t) = \begin{cases} (\cos(2\pi t), \sin(2\pi t)) & \text{if } 0 < t < 1 \\ (1, 0) & \text{if } t = \infty \end{cases}$$

This f is a one-to-one continuous function from $(0,1)_\infty$ onto the unit circle. By Theorem 5.8, this is a homeomorphism.