Chapter 4

Introduction to Hyperbolic Geometry

The major difference that we have stressed throughout the semester is that there is one *small* difference in the parallel postulate between Euclidean and hyperbolic geometry. We have been working with eight axioms. Let's recall the first seven and then add our new parallel postulate.

Axiom 1: *We can draw a unique line segment between any two points.*

Axiom 2: *Any line segment may be continued indefinitely.*

Axiom 3: *A circle of any radius and any center can be drawn.*

Axiom 4: *Any two right angles are congruent.*

Axiom 6: *Given any two points P and Q, there exists an isometry f such that $f(P) = Q$.*

Axiom 7: *Given a point P and any two points Q and R which are equidistant from P, there exists an isometry which fixes P and sends Q to R.*

Axiom 8: *Given any line ℓ, there exists a map which fixes every point in ℓ and fixes no other points.*

Our new postulate is one of the negations of Playfair's Postulate.

Axiom 5H: *Given any line ℓ and any point P not on ℓ, there exist two distinct lines ℓ_1 and ℓ_2 through P which do not intersect ℓ.*

Note that in negating Playfair's Postulate, we have to choose whether we want to have no parallel lines (leading us to elliptic geometry) or more than one parallel line through the given point. We shall show that the existence of two distinct parallel lines leads to the existence of an infinite number of distinct parallel lines.

What could such an animal look like? How could we have multiple parallels? Recall that the concept of no parallels sounded questionable until we looked at the sphere.

4.1 Neutral Geometry

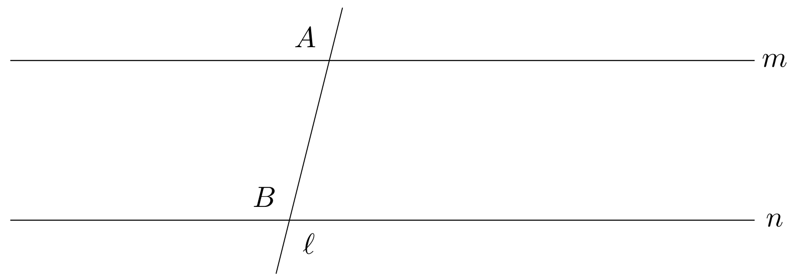
We have not spent too much time considering the ramifications of the axioms unrelated to the Parallel Axiom. What can we derive from these alone. Remember, the purpose of a lot of mathematics in the time between Euclid and Bolyai-Lobachevsky-Gauss was to prove that the Parallel Postulate did depend on the others.

4.1.1 Alternate Interior Angles

Definition 4.1 Let $\mathcal{L}$ be a set of lines in the plane. A line ℓ is **transversal** of $\mathcal{L}$ if

1. $\ell \notin \mathcal{L}$, and
2. $\ell \cap m \neq \emptyset$ for all $m \in \mathcal{L}$.

Let ℓ be transversal to m and n at points A and B, respectively. We say that each of the angles of intersection of ℓ and m and of ℓ and n has a *transversal side* in ℓ and a *non-transversal side* not contained in ℓ.



Definition 4.2 An angle of intersection of m and k and one of n and k are **alternate interior angles** if their transversal sides are opposite directed and intersecting, and if their non-transversal sides lie on opposite sides of ℓ. Two of these angles are **corresponding angles** if their transversal sides have like directions and their non-transversal sides lie on the same side of ℓ.

Definition 4.3 If k and ℓ are lines so that $k \cap \ell = \emptyset$, we shall call these lines **parallel**.

Theorem 4.1 (Alternate Interior Angle Theorem) *If two lines cut by a transversal have a pair of congruent alternate interior angles, then the two lines are non-intersecting.*

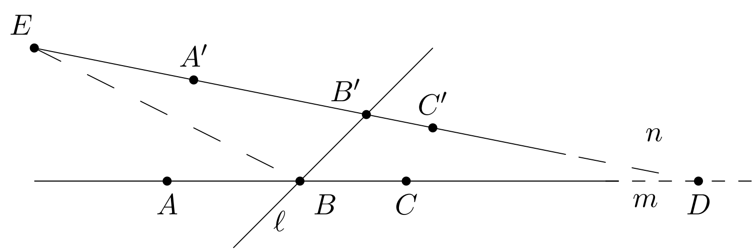


Figure 4.1:

Proof: Let m and n be two lines cut by the transversal ℓ. Let the points of intersection be B and B', respectively. Choose a point A on m on one side of ℓ, and choose $A' \in n$ on the same side of ℓ as A. Likewise, choose $C \in m$ on the opposite side of ℓ from A. Choose $C' \in n$ on the same side of ℓ as C. Then it is on the opposite side of ℓ from A'.

We are given that $\angle A'B'B \cong \angle CBB'$. Assume that the lines m and n are not parallel; *i.e.*, they have a nonempty intersection. Let us denote this point of intersection by D. D

is on one side of ℓ, so by changing the labelling, if necessary, we may assume that D lies on the same side of ℓ as C and C'. There is a unique point E on the ray $B'A'$ so that $B'E \cong BD$. Since, $BB' \cong BB'$, we may apply the SAS Axiom to prove that

$$\triangle EBB' \cong \triangle DBB'.$$

From the definition of congruent triangles, it follows that $\angle DB'B \cong \angle EBB'$. Now, the supplement of $\angle DBB'$ is congruent to the supplement of $\angle EB'B$. The supplement of $\angle EB'B$ is $\angle DB'B$ and $\angle DB'B \cong \angle EBB'$. Therefore, $\angle EBB'$ is congruent to the supplement of $\angle DBB'$. Since the angles share a side, they are themselves supplementary. Thus, $E \in n$ and we have shown that $\{D, E\} \subset n$ or that $m \cap n$ is more that one point. Thus, m and n must be parallel. ∎

Corollary 3 *If m and n are distinct lines both perpendicular to the line ℓ, then m and n are parallel.*

PROOF: ℓ is the transversal to m and n. The alternate interior angles are right angles. All right angles are congruent, so the *Alternate Interior Angle Theorem* applies. m and n are parallel. ∎

Corollary 4 *If P is a point not on ℓ, then the perpendicular dropped from P to ℓ is unique.*

PROOF: Assume that m is a perpendicular to ℓ through P, intersecting ℓ at Q. If n is another perpendicular to ℓ through P intersecting ℓ at R, then m and n are two distinct lines perpendicular to ℓ. By the above corollary, they are parallel, but each contains P. Thus, the second line cannot be distinct, and the perpendicular is unique. ∎

The point at which this perpendicular intersects the line ℓ, is called the *foot* of the perpendicular.

Corollary 5 *If ℓ is any line and P is any point not on ℓ, there exists at least one line m through P which does not intersect ℓ.*

PROOF: By Corollary 2 there is a unique line, m, through P perpendicular to ℓ. Now there is a unique line, n, through P perpendicular to m. By Corollary 1 ℓ and n are parallel. ∎

Note that while we have proved that there is a line through P which does not intersect ℓ, we have not (and cannot) proved that it is *unique*.

4.2 Weak Exterior Angle Theorem

Let $\triangle ABC$ be any triangle in the plane. This triangle gives us not just three segments, but in fact three lines.

Definition 4.4 An angle supplementary to an angle of a triangle is called an **exterior angle** of the triangle. The two angles of the triangle not adjacent to this exterior angle are called the **remote interior angles**.

Theorem 4.2 (Exterior Angle Theorem) *An exterior angle of a triangle is greater than either remote interior angle. See Figure 4.2*

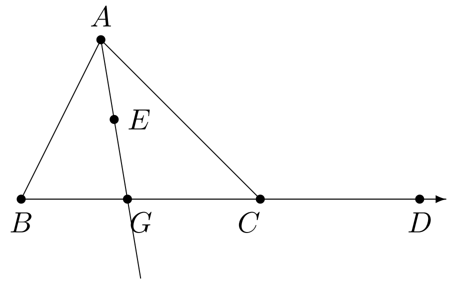


Figure 4.2:

Proof: We shall show that $\angle ACD > \angle A$. In a like manner, you can show that $\angle ACD > \angle B$. Then by using the same techniques, you can prove the same for the other two exterior angles.

Now, either:

$$\angle A < \angle ACD \quad \angle A \cong \angle ACD \quad \text{or } \angle A > \angle ACD.$$

If $\angle A = \angle BAC \cong \angle ACD$, then by the *Alternate Interior Angle Theorem*, lines AB and CD are parallel. This is impossible, since they both contain B.

Assume, then, that $\angle A > \angle ACD$. Then there exists a ray AE between rays AB and AC so that

$$\angle CAE \cong \angle ACD.$$

By what is known as the *Crossbar Theorem*, ray AE intersects BC in a point G. Again by the *Alternate Interior Angle Theorem* lines AE and CD are parallel. This is a contradiction.

Thus, $\angle A < \angle ACD$. ∎

Proposition 4.1 (SAA Congruence) *In triangles $\triangle ABC$ and $\triangle DEF$ given that $AC \cong DF$, $\angle A \cong \angle D$, and $\angle B \cong \angle E$, then $\triangle ABC \cong \triangle DEF$.*

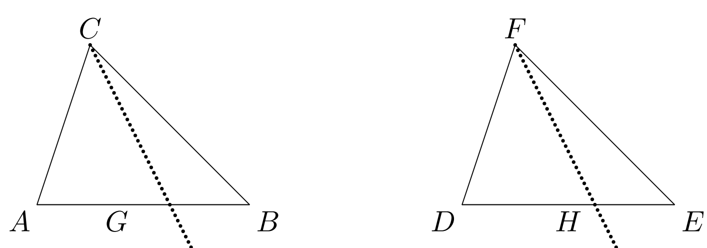


Figure 4.3:

Proof: If $AB \cong DE$, we are done by *Angle-Side-Angle*. Thus, let us assume that $AB \ncong DE$. Then, by we must have that either $AB < DE$ or $AB > DE$.

If $AB < DE$, then there is a point $H \in DE$ so that $AB \cong DH$. Then by the *SAS Theorem* $\triangle ABC \cong \triangle DHF$. Thus, $\angle B \cong \angle DHF$. But $\angle DHF$ is exterior to $\triangle FHE$, so by the *Exterior Angle Theorem* $\angle DHF > \angle E \cong \angle B$. Thus, $\angle DHF > \angle B$, and we have a contradiction. Therefore, AB is not less than DE. By a similar argument, we can show that assuming that $AB > DE$ leads to a similar contradiction.

Thus, our hypothesis that $AB \ncong DE$ cannot be valid. Thus, $AB \cong DE$ and $\triangle ABC \cong \triangle DEF$ by ASA. ∎

Proposition 4.2 *Two right triangles are congruent if the hypotenuse and a leg of one are congruent respectively to the hypotenuse and a leg of the other.*

Proposition 4.3 *Every segment has a unique midpoint.*

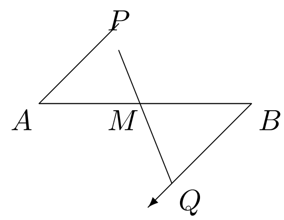


Figure 4.4:

Proof: Let AB be any segment in the plane, and let C be any point not on line AB. There exists a unique ray BX on the opposite side of line AB from P such that $\angle PAB \cong \angle XBA$. There is a unique point Q on the ray BX so that $AP \cong BQ$. Q is on the opposite side of line AB from P. Since P and Q are on opposite sides of line AB, $PQ \cap AB \neq \emptyset$. Let M denote this point of intersection. Either M lies between A and B, A lies between M and B, B lies between A and M, $M = A$, or $M = B$.

We want to show that M lies between A and B, so assume not. Since $\angle PAB \cong \angle QBA$, by construction, we have from the *Alternate Interior Angle Theorem* that lines AP and BQ are parallel. If $M = A$ then A, P ,and M are collinear on the line AP and lines $AP = AB$ which intersects line BQ. We can dispose of the case $M = B$ similarly.

Thus, assume that A lies between M and B. This will mean that the line PA will intersect side MB of $\triangle MBQ$ at a point between M and B. Thus, by *Pasch's Theorem* it must intersect either MQ or BQ. It cannot intersect side BQ as lines AP and BQ are parallel. If line AP intersects MQ then it must contain MQ for P, Q, and M are collinear. Thus, $M = A$ which we have already shown is impossible. Thus, we have shown that A cannot lie between M and B.

In the same manner, we can show that B cannot lie between A and M. Thus, we have that M must lie between A and B. This means that $\angle AMP \cong \angle BMQ$ since they are vertical angles. By *Angle-Angle-Side* we have that $\triangle AMP \cong \triangle BMQ$. Thus, $AM \cong MB$ and M is the midpoint of AB. ∎

Proposition 4.4 *i) Every angle has a unique bisector.*

ii) Every segment has a unique perpendicular bisector.

Proposition 4.5 *In a triangle $\triangle ABC$ the greater angle lies opposite the greater side and the greater side lies opposite the greater angle;* i.e., *$AB > BC$ if and only if $\angle C > \angle A$.*

Proposition 4.6 *Given $\triangle ABC$ and $\triangle A'B'C'$, if $AB \cong A'B'$ and $BC \cong B'C'$, then $\angle B < \angle B'$ if and only if $AC < A'C'$.*

4.2.1 Measure of Angles and Segments

At some point we have to introduce a measure for angles and for segments.

The proofs of these results require the axioms of continuity. We need the measurement of angles and segments by real numbers, and for such measurement Archimedes's axiom is required.

Theorem 4.3 *A There is a unique way of assigning a degree measure to each angle such that the following properties hold:*

i) $\angle A$ *is a real number such that* $0 < \angle A < 180°$.

ii) $\angle A = 90°$ *if and only if* $\angle A$ *is a right angle.*

iii) $\angle A = \angle B$ *if and only if* $\angle A \cong \angle B$.

iv) If the ray AC *is interior to* $\angle DAB$, *then* $\angle DAB = \angle DAC + \angle CAB$.

v) For every real number x *between 0 and 180, there exists an angle* $\angle A$ *such that* $\angle A = x°$.

vi) If $\angle B$ *is supplementary to* $\angle A$, *then* $\angle A + \angle B = 180\circ$.

B Given a segment OI, *called the* unit segment. *Then there is a unique way of assigning a length* $|AB|$ *to each segment* AB *such that the following properties hold:*

i) $|AB|$ *is a positive real number and* $|OI| = 1$.

ii) $|AB| = |CD|$ *if and only if* $AB \cong CD$.

iii) B *lies between* A *and* C *if and only if* $|AC| = |AB| + |BC|$.

iv) $|AB| < |CD|$ *if and only if* $AB < CD$.

v) For every positive real number x, *there exists a segment* AB *such that* $|AB| = x$.

Definition 4.5 An angle $\angle A$ is **acute** if $\angle A < 90°$, and is **obtuse** if $\angle A > 90°$.

Corollary 1 *The sum of the degree measures of any* two *angles of a triangle is less than* $180°$.

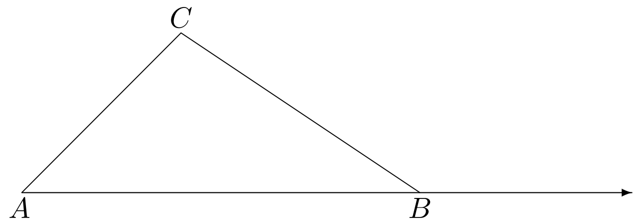
This follows from the Exterior Angle Theorem.

PROOF: We want to show that $\angle A + \angle B < 180°$. From the Exterior Angle Theorem,

$$\angle A < \angle CBD$$
$$\angle A + \angle B < \angle CBD + \angle B = 180°,$$

since they are supplementary angles. ∎

Corollary 2 (Triangle Inequality) *If* A, B, *and* C *are three noncollinear points, then* $|AC| < |AB| + |BC|$.



4.3 Saccheri-Legendre Theorem

Theorem 4.4 (Saccheri-Legendre Theorem) *The sum of the degree measures of the three angles in any triangle is* less than or equal to 180°*;*

$$\angle A + \angle B + \angle C \leq 180^\circ.$$

Proof: Let us assume not; *i.e.*, assume that we have a triangle $\triangle ABC$ in which $\angle A + \angle B + \angle C > 180^\circ$. So there is an $x \in R^+$ so that

$$\angle A + \angle B + \angle C = 180^\circ + x.$$

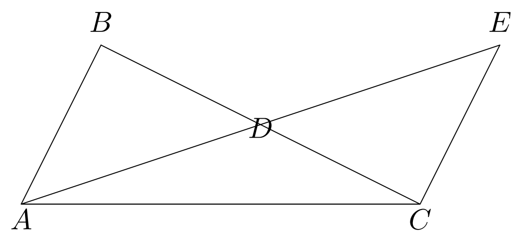


Figure 4.5: Saccheri-Legendre Theorem

Compare Figure 4.5. Let D be the midpoint of BC and let E be the unique point on the ray AD so that $DE \cong AD$. Then by SAS $\triangle BAD \cong \triangle CED$. This makes

$$\angle B = \angle DCE \qquad \angle E = \angle BAD.$$

Thus,

$$\begin{aligned}\angle A + \angle B + \angle C &= (\angle BAD + \angle EAC) + \angle B + \angle ACB \\ &= \angle E + \angle EAC + (\angle DCE + \angle ACD) \\ &= \angle E + \angle A + \angle C\end{aligned}$$

So, $\triangle ABC$ and $\triangle ACE$ have the same angle sum, even though they need not be congruent. Note that $\angle BAE + \angle CAE = \angle BAC$, hence

$$\angle CEA + \angle CAE = \angle BAC.$$

It is impossible for both of the angles $\angle CEA$ and $\angle C^\circ AE$ to have angle measure greater than $1/2\angle BAC^\circ$, so at least one of the angles has angle measure less than or equal to $1/2\angle BAC^\circ$.

Therefore, there is a triangle $\triangle ACE$ so that the angle sum is $180° + x$ but in which one angle has measure less than or equal to $1/2\angle A°$. Repeat this construction to get another triangle with angle sum $180° + x$ but in which one angle has measure less than or equal to $1/4\angle A°$. Now there is an $n \in Z^+$ so that

$$\frac{1}{2^n}\angle A \leq x,$$

by the Archimedean property of the real numbers. Thus, after a finite number of iterations of the above construction we obtain a triangle with angle sum $180° + x$ in which one angle has measure less than or equal to

$$\frac{1}{2^n}\angle A \leq x.$$

Then the other two angles must sum to a number greater than $180°$ contradicting Corollary 1. ∎

Corollary 1 *In $\triangle ABC$ the sum of the degree measures of two angles is less than or equal to the degree measure of their remote exterior angle.*

4.3.1 The Defect of a Triangle

Since the angle sum of any triangle in neutral geometry is not more than $180°$, we can compute the difference between the number 180 and the angle sum of a given triangle.

Definition 4.6 The **defect** of a triangle $\triangle ABC$ is the number

$$\textit{defect}\,(\triangle ABC) = 180° - (\angle A + \angle B + \angle C).$$

In euclidean geometry we are accustomed to having triangles whose defect is zero. Is this always the case? The Saccheri-Legrendre Theorem indicates that it may not be so. However, what we wish to see is that the *defectiveness* of triangles is preserved. That is, if we have one defective triangle, then all of the sub and super-triangles are defective. By defective, we mean that the triangles have positive defect.

Theorem 4.5 (Additivity of Defect) *Let $\triangle ABC$ be any triangle and let D be a point between A and B. Then* $\text{defect}\,(\triangle ABC) = \text{defect}\,(\triangle ACD) + \text{defect}\,(\triangle BCD)$.

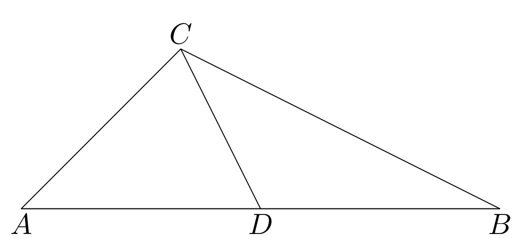


Figure 4.6:

PROOF: Since the ray CD lies in $\angle ACB$, we know that

$$\angle ACB = \angle ACD + \angle BCD,$$

and since $\angle ADC$ and $\angle BDC$ are supplementary angles $\angle ADC + \angle BDC = 180°$. Therefore,

$$\begin{aligned}
\mathit{defect}(\triangle ABC) &= 180° - (\angle A + \angle B + \angle C) \\
&= 180° - (\angle A + \angle B + \angle ACD + \angle BCD) \\
&= 180° + 180° - (\angle A + \angle B \\
&\qquad + \angle ACD + \angle BCD + \angle ADC + \angle BDC) \\
&= \mathit{defect}(\triangle ACD) + \mathit{defect}(\triangle BCD)\,.
\end{aligned}$$

$\blacksquare$

Corollary 1 defect $(\triangle ABC) = 0$ *if and only if* defect $(\triangle ACD) =$ defect $(\triangle BCD) = 0$.

A **rectangle** is a quadrilateral all of whose angles are right angles. We cannot prove the existence or non-existence of rectangles in Neutral Geometry. Nonetheless, the following result is extremely useful.

Theorem 4.6 *If there exists a triangle of defect 0, then a rectangle exists. If a rectangle exists, then every triangle has defect 0.*

Let me first outline the proof in five steps.

1. Construct a *right* triangle having defect 0.

2. From a right triangle of defect 0, construct a rectangle.

3. From one rectangle, construct arbitrarily large rectangles.

4. Prove that all *right* triangles have defect 0.

5. If every *right* triangle has defect 0, then *every* triangle has defect 0.

Having outlined the proof, each of the steps is relatively straightforward.

1. Construct a *right* triangle having defect 0.
   Let us assume that we have a triangle $\triangle ABC$ so that $\mathit{defect}(\triangle ABC) = 0$. We may assume that $\triangle ABC$ is not a right triangle, or we are done. Now, at least two angles are acute since the angle sum of any two angles is always less than $180°$. Let us assume that $\angle A$ and $\angle B$ are acute. Also, let D be the foot of C on line AB. We need to know that D lies between A and B.

   Assume not; *i.e.*, assume that $\between DAB$. See Figure 4.7. This means that $\angle CAB$ is exterior to $\triangle CAD$ and, therefore, $\angle A > \angle CDA = 90°$. This makes $\angle A$ obtuse, a contradiction. Similarly, if $\between ABD$ we can show that $\angle B$ is obtuse. Thus, we must have that D lies between A and B.

   This makes $\triangle ADC$ and $\triangle BDC$ right triangles. By Corollary 1 above, since $\triangle ABC$ has defect 0, each of them has defect 0, and we have two right triangles with defect 0.

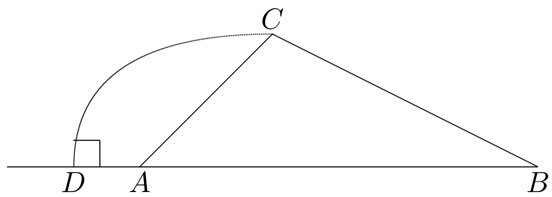


Figure 4.7:

2. From a right triangle of defect 0, construct a rectangle.
   We now have a right triangle of defect 0. Take $\triangle CBD$ from Step 1, which has a right angle at D. There is a unique ray CX on the opposite side of BC from D so that

$$\angle DBC \cong \angle BCX.$$

   Then there is a unique point E on ray CX such that $CE \cong BD$.

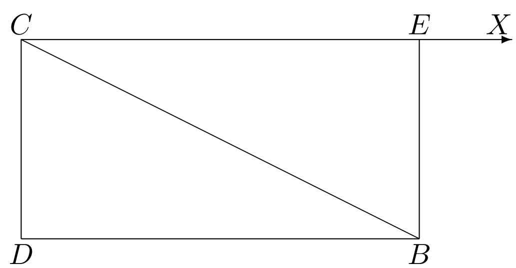


Figure 4.8:

   Thus, $\triangle CDB \cong \triangle BEC$ by *SAS*. Then $\angle BEC = 90°$ and $\triangle BEC$ must also have defect 0. Now, clearly, since *defect*$(\triangle CDB) = 0$

$$\angle DBC + \angle BCD = 90°$$

   and, hence,

$$\angle ECB + \angle BCD = \angle ECD = 90°.$$

   Likewise, $\angle EBD = 90°$ and $\square CDBE$ is a rectangle.

3. From one rectangle, construct arbitrarily large rectangles.
   Given any right triangle $\triangle XYZ$, we can construct a rectangle $\square PQRS$ so that $PS > XZ$ and $RS > YZ$. By applying Archimedes Axiom, we can find a number n so that we lay off segment BD in the above rectangle on the ray ZX to reach the point P so that $n \cdot BD \cong PZ$ and X lies between P and Z. We make n copies of our rectangle sitting on $PZ = PS$. This gives us a rectangle with vertices P, $Z = S$, Y, and some other point. Now, using the same technique, we can find a number m and a point R on the ray ZY so that $m \cdot BE \cong RZ$ and Y lying between R and Z. Now, constructing m copies of the long rectangle, gives us the requisite rectangle containing $\triangle XYZ$.

4. Prove that all *right* triangles have defect 0.
   Let $\triangle XYZ$ be an arbitrary right triangle. By Step 3 we can embed it in a rectangle $\square PQRS$.

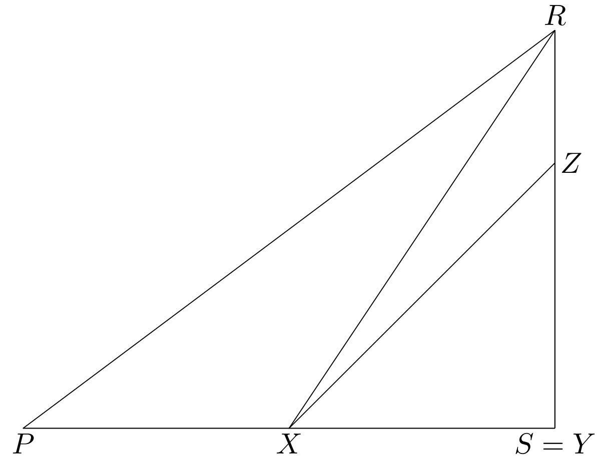


Figure 4.9:

Since $\triangle PQR \cong \triangle PSR$, we have that $\angle RPS + \angle PRS = 90°$ and then, $\triangle PRS$ has defect 0. Using Corollary 1 to Theorem 4.5 we find $defect(\triangle RXY) = 0$ and thus, $defect(\triangle XYZ) = 0$. Therefore, each triangle has defect 0.

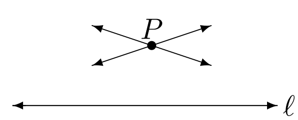
5. If every *right* triangle has defect 0, then *every* triangle has defect 0.
   As in the first step, use the foot of a vertex to decompose the triangle into two right triangles, each of which has defect 0, from Step 4. Thus, the original triangle has defect 0.

Corollary 1 *If there is a triangle with positive defect, then all triangles have positive defect.*

4.4 Hyperbolic Axiom Results

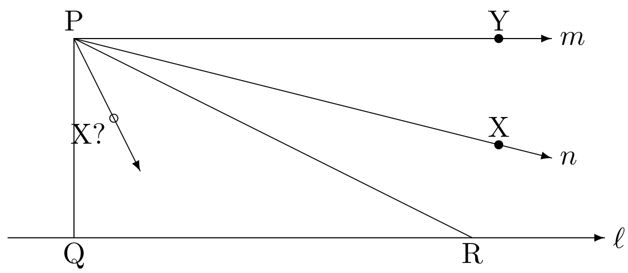
Hyperbolic geometry is often called *Bolyai-Lobachevskiian geometry* after two of its *discovers* János Bolyai and Nikolai Ivanovich Lobachevsky. Bolyai first announced his discoveries in a 26 page appendix to a book by his father, the *Tentamen*, in *1831*. Another of the great mathematicians who seems to have preceded Bolyai in his work is Carl Fredrich Gauss. He seems to have done some work in the area dating from 1792, but never published it. The first to publish a complete account of non-euclidean geometry was Lobachevsky in *1829*. It was first published in Russian and was not widely read. In *1840* he published a treatise in German.

We shall call our added axiom the Hyperbolic Axiom.



We shall denote the set of all points in the plane by H^2, and call this the *hyperbolic plane*.

Lemma 4.1 *There exists a triangle whose angle sum is less than* 180°.



Proof: Let ℓ be a line and P a point not on ℓ such that two parallels to ℓ pass through P. We can construct one of these parallels as previously done using perpendiculars. Let Q be the foot of the perpendicular to ℓ through P. Let m be the perpendicular to the line PQ through P. Then m and ℓ are parallel. Let n be another line through P which does not intersect ℓ. This line exists by the *Hyperbolic Axiom*. Let PX be a ray of n lying between PQ and a ray PY of m.
claim: There is a point $R \in \ell$ on the same side of the line PQ as X and Y so that $\angle QRP < \angle XPY$.

proof of claim. The idea is to construct a sequence of angles

$$\angle QR_1P,\ \angle QR_2P,\ \ldots,\ \angle QR_nP,\ \ldots$$

so that $\angle QR_{j+1}P < \frac{1}{2}\angle QR_jP$. We will then apply Archimedes Axiom for real numbers to complete the proof.

There is a point $R_1 \in \ell$ so that $QR_1 \cong PQ$. Then $\triangle QR_1P$ is isosceles and $\angle QR_1P^\circ \leq 45^\circ$. Also, there is a point $R_2 \in \ell$ so that $\between FR_1R_2$ and $R_1R_2 \cong PR_1$. Then $\triangle PR_1R_2$ is isosceles and $\angle R_1PR_2 \cong \angle QR_2P$. Since $\angle QR_1P$ is exterior to $\triangle PR_1R_2$ it follows that

$$\angle R_1PR_2 + \angle QR_2P \leq \angle QR_1P,$$

so then $\angle QR_2P \leq 22\frac{1}{2}^\circ$. Continuing with this construction, we find a point $R_n \in \ell$ so that $\between QR_{n-1}R_n$ and

$$\angle QR_nP \leq \left(\frac{45}{2^n}\right)^\circ.$$

Applying the Archimedean axiom we see that for any positive real number, for example $\angle XPY$, there is a point $R \in \ell$ so that R is on the same side of the line PQ as X and Y and $\angle QRP < \angle XPY$. Thus, we have proved our claim.

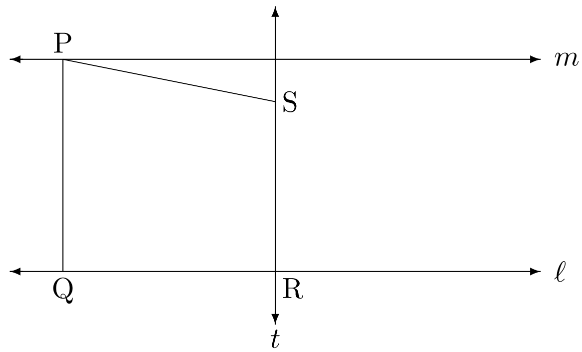
Now, the ray PR lies in the interior of $\angle QPX$, for if not then the ray PX is in the interior of $\angle QRP$. By the *Crossbar Theorem* it follows that the ray $PX \cap \ell \neq \emptyset$ which implies that n and ℓ are not parallel—a contradiction. Thus, $\angle RPQ < \angle XPQ$. Then,

$$\angle RPQ + \angle QRP < \angle XPQ + \angle QRP < \angle XPQ + \angle XPY = 90^\circ.$$

Therefore, $\angle P + \angle Q + \angle R < 180^\circ$ and *defect*$(\triangle PQR) > 0$. ∎

The *Hyperbolic Axiom* only hypothesizes the existence of one line and one point not on that line for which there are two parallel lines. With the above theorem we can now prove a much stronger theorem.

Theorem 4.7 (Universal Hyperbolic Theorem) *In H^2 for every line ℓ and for every point P not on ℓ there pass through P at least two distinct lines, neither of which intersect ℓ.*



PROOF: Drop a perpendicular PQ to ℓ and construct a line m through P perpendicular to PQ. Let R be any other point on ℓ, and construct a perpendicular t to ℓ through R. Now, let S be the foot of the perpendicular to t through P. Now, the line PS does not intersect ℓ since both are perpendicular to t. At the same time $PS \neq m$. Assume that $S \in m$, then $\square PQRS$ is a rectangle. By Theorem 4.6, if one rectangle exists all triangles have defect 0. We have a contradiction to Lemma 4.1. Thus, $PS \neq m$, and we are done. ∎

4.5 Angle Sums (again)

We have just proven the following theorem.

Theorem 4.8 *In H^2 rectangles do not exist and all triangles have angle sum less than $180°$.*

This tells us that in hyperbolic geometry the defect of any triangle is a *positive* real number. We shall see that it is a very important quantity in hyperbolic geometry.

Corollary 1 *In H^2 all convex quadrilaterals have angle sum less than $360°$.*

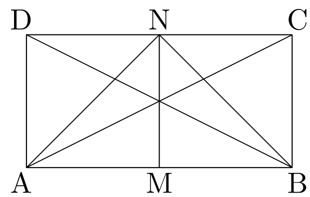
4.6 Saccheri Quadrilaterals

Girolamo Saccheri was a Jesuit priest living from 1667 to 1733. Before he died he published a book entitled *Euclides ab omni nævo vindicatus* (*Euclid Freed of Every Flaw*). It sat unnoticed for over a century and a half until rediscovered by the Italian mathematician Beltrami.

He wished to prove Euclid's Fifth Postulate from the other axioms. To do so he decided to use a *reductio ad absurdum* argument. He assumed the negation of the Parallel Postulate and tried to arrive at a contradiction. He studied a family of quadrilaterals that have come to be called *Saccheri quadrilaterals*. Let S be a convex quadrilateral in which two adjacent angles are right angles. The segment joining these two vertices is called the **base**. The side opposite the base is the **summit** and the other two sides are called the **sides**. If the sides are congruent to one another then this is called a **Saccheri quadrilateral**. The angles containing the summit are called the **summit angles**.

Theorem 4.9 *In a Saccheri quadrilateral* *i) the summit angles are congruent, and*

ii) the line joining the midpoints of the base and the summit—called the **altitude**—*is perpendicular to both.*



Proof: Let M be the midpoint of AB and let N be the midpoint of CD.

1. We are given that
$$\angle DAB = \angle ABC = 90°.$$
Now, $AD \cong BC$ and $AB \cong AB$, so that by SAS $\triangle DAB \cong \triangle CBA$, which implies that $BD \cong AC$. Also, since $CD \cong CD$ then we may apply the SSS criterion to see that $\triangle CDB \cong \triangle DCA$. Then, it is clear that $\angle D \cong \angle C$.

2. We need to show that the line MN is perpendicular to both lines AB and CD. Now $DN \cong CN$, $AD \cong BC$, and $\angle D \cong \angle C$. Thus by SAS $\triangle ADN \cong \triangle BCN$. This means then that $AN \cong BN$. Also, $AM \cong BM$ and $MN \cong MN$. By SSS $\triangle ANM \cong \triangle BNM$ and it follows that $\angle AMN \cong \angle BMN$. They are supplementary angles, hence they must be right angles. Therefore MN is perpendicular to AB.

   Using the analogous proof and triangles $\triangle DMN$ and $\triangle CMN$, we can show that MN is perpendicular to CD.

Thus, we are done. ∎

Theorem 4.10 *In a Saccheri quadrilateral the summit angles are acute.*

Proof: Recall from Corollary 1 to Theorem 4.8 that the angle sum for any convex quadrilateral is less that 360°. Thus, since the Saccheri quadrilateral is convex,

$$\begin{aligned} \angle A + \angle B + \angle C + \angle D &< 360° \\ 2\angle C &< 180° \\ \angle C &< 90° \end{aligned}$$

Thus, $\angle C$ and $\angle D$ are acute. ∎

A convex quadrilateral three of whose angles are right angles is called a **Lambert quadrilateral**.

Theorem 4.11 *The fourth angle of a Lambert quadrilateral is acute.*

Theorem 4.12 *The side adjacent to the acute angle of a Lambert quadrilateral is greater than its opposite side.*

Theorem 4.13 *In a Saccheri quadrilateral the summit is greater than the base and the sides are greater than the altitude.*

PROOF: Using Theorem 4.9 if M is the midpoint of AB and N is the midpoint of CD, then $\square AMND$ is a Lambert quadrilateral. Thus, $AB > MN$ and, since $BC \cong AB$, both sides are greater than the altitude.

Also, applying Theorem 4.9 $DN > AM$. Since $CD \cong 2DN$ and $AB \cong 2AM$ it follows that $CD > AB$, so that the summit is greater than the base. ∎

4.7 Similar Triangles

In euclidean geometry we are used to having two triangles similar if their angles are congruent. It is obvious that we can construct two non-congruent, yet similar, triangles. In fact John Wallis attempted to prove the Parallel Postulate of Euclid by adding another postulate.

WALLIS' POSTULATE: *Given any triangle $\triangle ABC$ and given any segment DE. There exists a triangle $\triangle DEF$ having DE as one of its sides that is similar to $\triangle ABC$.*

However Wallis' Postulate is equivalent to Euclid's Parallel Postulate. Thus, we know that the negation of Wallis' Postulate must hold in hyperbolic geometry. That is, under certain circumstances similar triangles do not exist. We can prove a much stronger statement.

Theorem 4.14 (AAA Criterion) *In H^2 if $\angle A \cong \angle D$, $\angle B \cong \angle E$, and $\angle C \cong \angle F$, then $\triangle ABC \cong \triangle DEF$. That is, if two triangles are similar, then they are congruent.*

PROOF: Since $\angle BAC \cong \angle EDF$, there exists an isometry which sends D to A, the ray DE to the ray AB, and the ray DF to the ray AC. Let the image of E and F under this isometry be E' and F', respectively. If the two triangles are not congruent, then we may assume that $E' \neq B$ and that E' lies between A and B. Then BC and $E'F'$ cannot intersect by the Alternate Interior Angles Theorem. Then $BCE'F'$ forms a quadrilateral. The quadrilateral has the following angles:

$$\begin{aligned}
\angle E'BC &= \angle ABC \\
\angle F'CB &= \angle ACB \\
\angle BE'F' &= 180^\circ - \angle ABC \\
\angle CF'E' &= 180^\circ - \angle ACB
\end{aligned}$$

which sum to 360°. This contradiction leads us to the fact that $E' = B$ and $F' = C$ and the two triangles are congruent. ∎

As a consequence of Theorem 4.14 we shall see that in hyperbolic geometry a segment can be determined with the aid of an angle. For example, an angle of an equilateral triangle determines the length of a side uniquely. Thus in hyperbolic geometry there is an *absolute unit of length* as there is in elliptic geometry.