Chapter 10

Hypercycles and Horocycles

There is a curve peculiar to hyperbolic geometry, called the *horocycle*. Consider two limiting parallel lines, ℓ and m, with a common direction, say Ω. Let P be a point on one of these lines $P \in \ell$. If there exists a point $Q \in m$ such that the *singly asymptotic triangle*, $\triangle PQ\Omega$, has the property that

$$\angle PQ\Omega \cong \angle QP\Omega$$

then we say that Q *corresponds* to P. If the singly asymptotic triangle $\triangle PQ\Omega$ has the above property we shall say that it is *equiangular*. Note that it is obvious from the definition that if Q corresponds to P, then P corresponds to Q. The points P and Q are called a pair of *corresponding points*.

Theorem 10.1 *If points P and Q lie on two limiting parallel lines in the direction of the ideal point, Ω, they are corresponding points on these lines if and only if the perpendicular bisector of PQ is limiting parallel to the lines in the direction of Ω.*

Theorem 10.2 *Given any two limiting parallel lines, there exists a line each of whose points is equidistant from them. The line is limiting parallel to them in their common direction.*

PROOF: Let ℓ and m be limiting parallel lines with common direction Ω. Let $A \in \ell$ and $B \in m$. The bisector of $\angle BA\Omega$ in the singly asymptotic triangle $\triangle AB\Omega$ meets side $B\Omega$ in a point X and the bisector of $\angle AB\Omega$ meets side AX of the triangle $\triangle ABX$ in a point C. Thus the bisectors of the angles of the singly asymptotic triangle $\triangle AB\Omega$ meet in a point C. Drop perpendiculars from C to each of ℓ and m, say P and Q, respectively. By *Hypothesis-Angle* $\triangle CAP \cong \triangle CAM$ (M is the midpoint of AB) and $\triangle CBQ \cong \triangle CBM$. Thus, $CP \cong CM \cong CQ$. Thus, by *SAS* for singly asymptotic triangles, we have that

$$\triangle CP\Omega \cong \triangle CQ\Omega$$

and thus the angles at C are congruent. Now, consider the line $\overleftrightarrow{C\Omega}$ and let F be any point on it other than C. By *SAS* we have $\triangle CPF \cong \triangle CQF$. If S and T are the feet of F in ℓ and m, then we get that $\triangle PSF \cong \triangle QTF$ and $FS \cong FT$. Thus, every point on the line $C\Omega$ is equidistant from ℓ and m. ∎

This line is called the *equidistant line*.

Theorem 10.3 *Given any point on one of two limiting parallel lines, there is a unique point on the other which corresponds to it.*

Theorem 10.4 *If three points P, Q, and R lie on three parallels in the same direction so that P and Q are corresponding points on their parallels and Q and R are corresponding points on theirs, then P, Q, and R are noncollinear.*

Theorem 10.5 *If three points P, Q, and R lie on three parallels in the same direction so that P and Q are corresponding points on their parallels and Q and R are corresponding points on theirs, then P and R are corresponding points on their parallels.*

Consider any line ℓ, any point $P \in \ell$, and an ideal point in one direction of ℓ, say Ω. On each line parallel to ℓ in the direction Ω there is a unique point Q that corresponds to P. The set consisting of P and all such points Q is called a **horocycle**, or, more precisely, the horocycle determined by ℓ, P, and Ω. The lines parallel to ℓ in the direction Ω, together with ℓ, are called the *radii* of the horocycle. Since ℓ may be denoted by $P\Omega$, we may regard the horocycle as determined simply by P and Ω, and hence call it the *horocycle through P with direction* Ω, or in symbols, the horocycle (P, Ω).

All the points of this horocycle are mutually corresponding points by Theorem 10.5 , so the horocycle is equally well determined by any one of them and Ω. In other words if Q is any point of horocycle (P, Ω) other than P, then horocycle (Q, Ω) is the same as horocycle (P, Ω). If, however, P' is any point of ℓ other than P, then horocycle (P', Ω) is different from horocycle (P, Ω), even though they have the same direction and the same radii. Such horocycles, having the same direction and the same radii, are called *codirectional horocycles.*

There are analogies between horocycles and circles. We will mention a few.

Lemma 10.1 *There is a unique horocycle with a given direction which passes through a given point. (There is a unique circle with a given center which passes through a given point.)*

Lemma 10.2 *Two codirectional horocycles have no common point. (Two concentric circles have no common point.)*

Lemma 10.3 *A unique radius is associated with each point of a horocycle. (A unique radius is associated with each point of a circle.)*

A *tangent to a horocycle* at a point on the horocycle is defined to be the line through the point which is perpendicular to the radius associated with the point.

No line can meet a horocycle in more than two points. This is a consequence of the fact that no three points of a horocycle are collinear inasmuch as it is a set of mutually corresponding points, *cf.* Theorem 10.4.

Theorem 10.6 *The tangent at any point A of a horocycle meets the horocycle only in A. Every other line through A except the radius meets the horocycle in one further point B. If α is the acute angle between this line and the radius, then $d(A, B)$ is twice the segment which corresponds to α as angle of parallelism.*

PROOF: Let t be the tangent to the horocycle at A and let Ω be the direction of the horocycle. If t met the horocycle in another point B, we would have a singly asymptotic triangle with two right angles, since A and B are corresponding points. In fact the entire horocycle, except for A, lies on the same side of t, namely, the side containing the ray $A\Omega$.

Let k be any line through A other than the tangent or radius. We need to show that k meets the horocycle in some other point. Let α be the acute angle between k and the ray $A\Omega$. Let C be the point of k, on the side of t containing the horocycle, such that AC is a segment corresponding to α as angle of parallelism. (RECALL: $e^{-d} = \tan(\alpha/2)$). The line perpendicular to k at C is then parallel to $A\Omega$ in the direction Ω. Let B be the point of k such that C is the midpoint of AB. The singly asymptotic triangles $\triangle AC\Omega$ and $\triangle BC\Omega$ are congruent. Hence $\angle CB\Omega = \alpha$, B corresponds to A, and $B \in (A, \Omega)$. ∎

A *chord* of a horocycle is a segment joining two points of the horocycle.

Theorem 10.7 *The line which bisects a chord of a horocycle at right angles is a radius of the horocycle.*

We can visualize a **horocycle** in the Poincaré model as follows. Let ℓ be the diameter of the unit disk whose interior represents the hyperbolic plane, and let O be the origin. It is a fact that the *hyperbolic circle* with hyperbolic center P is represented by a Euclidean circle whose Euclidean center R lies between P and A.

As P recedes from A towards the ideal point Ω, R is pulled up to the Euclidean midpoint of ΩA, so that the horocycle (A, Ω) is a Euclidean circle tangent to the unit disk at Ω and tangent to ℓ at A. It can be shown that **all** horocycles are represented in the Poincaré model by Euclidean circles inside the unit disk and tangent to boundary circle. For the Poincaré upper half plane model, our horocycles will be circles that are tangent to the x-axis.

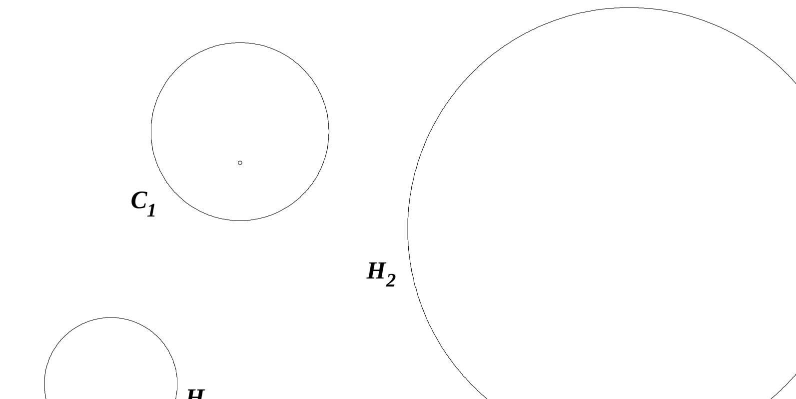

Figure 10.1: H_1 and H_2 are horocycles in the Poincaré model, C_1 is a circle

Another curve found specifically in the hyperbolic plane and nowhere else is the *equidistant curve*, or *hypercycle*. Given a line ℓ and a point P not on ℓ, consider the set of all points Q on one side of ℓ so that the perpendicular distance from Q to ℓ is the same as the perpendicular distance from P to ℓ.

The line ℓ is called the *axis*, or *base line*, and the common length of the perpendicular segments is called the *distance*. The perpendicular segments defining the hypercycle are called its *radii*. The following statements about hypercycles are analogous to statements about regular Euclidean circles.

1. Hypercycles with equal distances are congruent, those with unequal distances are not. (Circles with equal radii are congruent, those with unequal radii are not.)
2. A line cannot cut a hypercycle in more than two points.
3. If a line cuts a hypercycle in one point, it will cut it in a second unless it is tangent to the curve or parallel to it base line.
4. A tangent line to a hypercycle is defined to be the line perpendicular to the radius at that point. Since the tangent line and the base line have a common perpendicular, they must be hyperparallel. This perpendicular segment is the shortest distance between the two lines. Thus, each point on the *tangent line* must be at a greater perpendicular distance from the base line than the corresponding point on the hypercycle. Thus, the hypercycle can intersect the hypercycle in only one point.
5. A line perpendicular to a chord of a hypercycle at its midpoint is a radius and it bisects the arc subtended by the chord.
6. Two hypercycles intersect in at most two points.
7. No three points of a hypercycle are collinear.

In the Poincaré disk model let Ω and Λ be the ideal end points of ℓ. It can be shown that the hypercycle to ℓ through Ω is represented by the arc of the Euclidean circle passing through A, B, and Ω. This curve is orthogonal to all Poincaré lines perpendicular to the line ℓ. In the Poincaré upper half plane model, the hypercycle will be represented by an arc of a Euclidean circle passing through A, B, and Ω.

In the Poincaré disk model, $\mathscr{D}$, a Euclidean circle represents:

(a) a *hyperbolic circle* if it is entirely inside the unit disk;
(b) a *horocycle* if it is inside the unit disk except for one point where it is tangent to the unit disk;
(c) an *equidistant curve* if it cuts the unit disk non-orthogonally in two points;
(d) a *hyperbolic line* if it cuts the unit disk orthogonally.

A similar situation is true for the Poincaré upper half plane model, $\mathscr{H}$.

It follows that in the hyperbolic plane three non-collinear points lie either on a circle, a horocycle, or a hypercycle accordingly, as the perpendicular bisectors of the triangle are *concurrent* in an ordinary point, an ideal point, or an ultra-ideal point.