Problems for the Final Exam
MATH 6102 - Spring 2007

1. Define a function f on the real line by $f(x) = x + \lfloor x^2 \rfloor - \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. Where is f not continuous? Explain your answer.

2. Find the radius of convergence for the power series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^n}$.

3. A function f satisfies the two conditions

$$f'(x) = 1 + (f(x))^{10} \text{ and } f(0) = 1.$$

   Find the first four terms in the Taylor series expansion of f about $x = 0$.

4. Find the Taylor series and the radius of convergence for

$$f(x) = \int_0^x \frac{dt}{\sqrt{1+t^2}} \text{ about } x = 0.$$

5. If $f(1) = 1$ and $f'(1) = 2$, compute

$$\lim_{x \to 1} \frac{[f(x)]^2 - 1}{x^2 - 1}.$$

6. If $a, b, c, d \in \mathbb{R}$ find

$$\lim_{x \to 0} \frac{\sin ax \sin bx}{\sin cx \sin dx}.$$

7. Suppose that f is differentiable with derivative $f'(x) = (1 + x^3)^{-1/2}$. Show that $g = f^{-1}$ satisfies $g''(x) = \frac{3}{2}[g(x)]^2$. **DO NOT FIND $f(x)$.**

8. Find f^{-1} for each of the following:

   (a) $f(x) = x + \lfloor x \rfloor$.

   (b) $f(0.a_1 a_2 a_3 \ldots) = 0.a_2 a_1 a_3 \ldots$ for all numbers between 0 and 1.

9. Suppose that f and g are two differentiable functions which satisfy $fg' - f'g = 0$. Prove that if a and b are adjacent zeros of f[1], and $g(a)$ and $g(b)$ are not both 0, then $g(x) = 0$ for some x between a and b.
   HINT: Derive a contradiction from the assumption that $g(x) \neq 0$ for all x between a and b.

[1] This means that $f(a) = f(b) = 0$ and $f(x) \neq 0$ for all $x \in (a, b)$.

10. If f is three times differentiable and $f'(x) \neq 0$, the *Schwarzian derivative* of f at x is defined to be

$$\mathscr{D}f(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

   (a) Show that

$$\mathscr{D}(f \circ g) = [\mathscr{D}f \circ g] \cdot g'^2 + \mathscr{D}g.$$

   (b) Show that if $f(x) = \dfrac{ax+b}{cx+d}$, with $ad - bc \neq 0$, then $\mathscr{D}f = 0$. Show then that in this case $\mathscr{D}(f \circ g) = \mathscr{D}g$.

11. Find f' in terms of g' if

   (a) $f(x) = g(x + g(a))$.
   (b) $f(x) = g(x \cdot g(a))$
   (c) $f(x) = g(x + g(x))$
   (d) $f(x) = g(x)(x - a)$
   (e) $f(x) = g(a)(x - a)$
   (f) $f(x + 3) = g(x^2)$

12. Find $f'(x)$ if

$$f(x) = \sin\left(\frac{x}{x - \sin\left(\dfrac{x}{x - \sin x}\right)}\right).$$

13. Find $f'(x)$ if $f(x) = g(t + x)$, and if $f(t) = g(t + x)$.

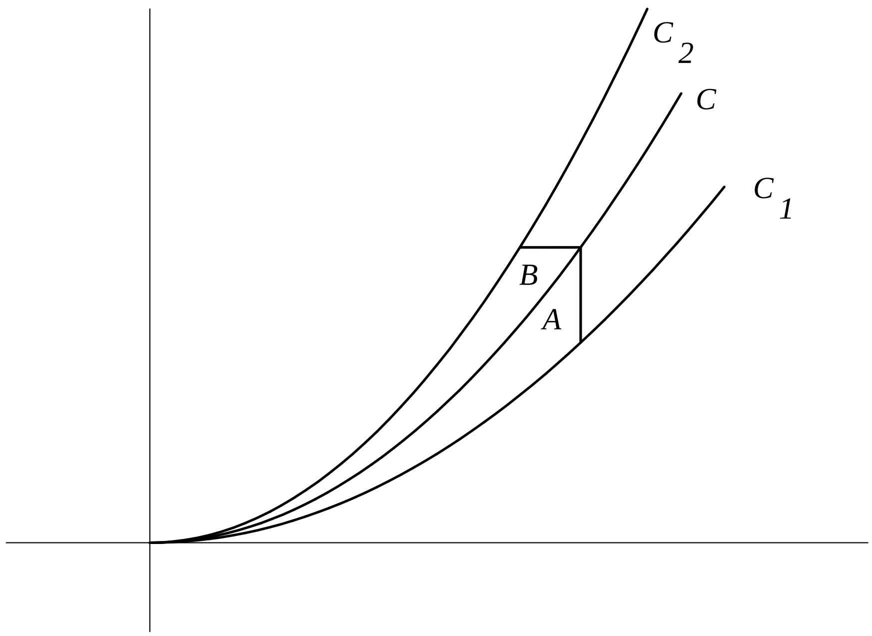
14. Find $(f^{-1})'(0)$ if

$$f(x) = \int_0^x 1 + \sin(\sin(t))\, dt.$$

15. Prove that if f is continuous, then

$$\int_0^x f(u)(x - u)\, du = \int_0^x \left(\int_0^u f(t)\, dt\right) du.$$

   Hint: Differentiate both sides.



16. Let C_1, C, and C_2 be curves passing through the origin, as shown in the figure above. Each point on C can be joined to a point of C_1 with a vertical line segment and to a point of C_2 with a horizontal line segment. We will say that C *bisects* C_1 and C_2 if the regions A and B have equal areas for every point on C.

    (a) If C_1 is the graph of $f(x) = x^2$, $x \geq 0$ and C is the graph of $f(x) = 2x^2$, $x \geq 0$, find C_2 so that C bisects C_1 and C_2.

    (b) Find C_2 if C_1 is the graph of x^m, and C is the graph of $f(x) = cx^m$ for some $c > 1$.