MATH 6102 — SPRING 2007
ASSIGNMENT 2

SOLUTIONS

January 29, 2007

1. *Prove that there is a real number x so that $\sin x = x - 1$.*

   We need to find an interval on the real line so that the function $f(x) = \sin x - x + 1$ is negative at one endpoint and positive at the other endpoint. At $x = 0$, we see that $f(0) = \sin 0 - 0 + 1 = 1 > 0$. Since $\sin x$ is always between -1 and 1, we have that $0 \leq \sin x + 1 \leq 2$ for all x. If we take $x > 2$ then $\sin x + 1 - x < 0$. Let $x = \pi > 2$, because we know the exact value of $\sin \pi = 0$. Thus, $f(0) = \sin(0) - 0 + 1 = 1 > 0$ and $f(\pi) = \sin(\pi) - \pi + 1 = -\pi + 1 < 0$. Thus, by the Intermediate Value Theorem, there is some real number $c \in [0, \pi]$ so that $f(x) = 0$. Therefore, $\sin c = c - 1$.

2. *Suppose that f is continuous on $[a, b]$ and that $f(x)$ is always rational. What can be said about f?*

   Since f is continuous, if it takes on two different rational values, it must take on all values between those two rational numbers. Between any two rationals is an irrational, so if f took on two rational values it would also have to take on an irrational value. Since it is always rational, it cannot take on two rational values, it must take on only one. Thus, f is constant.

3. *Suppose that f is a continuous function on $[0, 1]$ and that $f([0, 1]) = [0, 1]$. Prove that $f(x) = x$ for some number x between 0 and 1.*

   We are given that f is onto, or that for any value $c \in [0, 1]$ there is an $x \in [0, 1]$ so that $f(x) = c$. Thus, there exists $x_0 \in [0, 1]$ so that $f(x_0) = 0$ and there exists $x_1 \in [0, 1]$ so that $f(x_1) = [0, 1]$. If $x_0 = 0$ or if $x_1 = 1$ we are done because then we would have $f(0) = 0$ or $f(1) = 1$.

   Let $h(x) = f(x) - x$. Then $h(x_0) = f(x_0) - x_0 = -x_0 < 0$ and $h(x_1) = f(x_1) - x_1 = 1 - x_1 > 0$. Therefore there is a number $c \in [0, 1]$ so that $h(c) = 0$. That means that $f(c) = c$, and we are done.

4. *If f is defined as above and g is also continuous on $[0, 1]$ and $g(0) = 0$, $g(1) = 1$ or $g(0) = 1$, $g(1) = 0$, then $f(x) = g(x)$ for some x.*

   Let $h(x) = g(x) - f(x)$. h is a continuous function on $[0, 1]$.

   Assume that $g(0) = 0$ and $g(1) = 1$. We may assume that $f(0) > 0$ and $f(1) < 1$, else we are done. Let $\alpha \in [0, 1]$ be the first value for which $g(x) = 1$. We know that $g(x) \in [0, 1]$ for all $x \in [0, \alpha]$ and $f(x) \in [0, 1]$ for all $x \in [0, \alpha]$. If $f(\alpha) = 1$, then $f(\alpha) = g(\alpha)$ and we have a point at which $f(x) = g(x)$ and we are done. So assume $f(\alpha) < 1$. Now, $h(0) = f(0) - g(0) = f(0) > 0$ and $h(\alpha) = f(\alpha) - 1 < 0$. Thus, there is a point in $[0, \alpha] \subset [0, 1]$ so that $0 = h(x) = g(x) - f(x)$, and we are done.

   If we assume that $g(0) = 1$ and $g(1) = 0$. Then let $h(x) = f(x) - g(x)$ and let β be the first place where $g(x) = 0$. The same analysis will apply to give you a point in $[0, \beta] \subset [0, 1]$ at which $h(x) = 0$ and $f(x) = g(x)$.

NOTE: You can show that $f(x) = x$ for some $x \in [0, 1]$ and $g(x) = x$ for some $x \in [0, 1]$. However, these do not have to be the same real number. Consider the following functions:

$$g(x) = \cos\left(\frac{\pi x}{2}\right) \qquad \text{and } g(x) = x \text{ when } x = 0.5946$$

$$f(x) = x - \frac{\sin(4\pi x)}{6} \qquad \text{and } f(x) = x \text{ when } x = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$$

$$f(x) = g(x) \text{ when } x = 0.5504$$

5. *The number a is called a **double root** of the polynomial function f if $f(x) = (x-a)^2 g(x)$ for some polynomial function g. Prove that a is a double root of f if and only if a is a root of f and f'.*

   Assume that a is a double root of f. Then $f(x) = (x-a)^2 g(x)$ and $f'(x) = 2(x-a)\,g(x) + (x-a)^2 g'(x) = (x-a)[2g(x) + (x-a)g'(x)]$ and a is a root of f'.

   Assume that a is a root of f and f'. That means that $f(x) = (x-a)g(x)$ and $f'(x) = (x-a)h(x)$ where g and h are polynomial functions.

$$\begin{aligned} f'(x) &= g(x) + (x-a)g'(x) \\ (x-a)h(x) &= g(x) + (x-a)g'(x) \\ g(x) &= (x-a)h(x) - (x-a)g'(x) \\ &= (x-a)[h(x) - g'(x)] \end{aligned}$$

   Since h and g are polynomial functions, then so is $h - g'$ and we have that $g(x) = (x-a)p(x)$, which implies that $f(x) = (x-a)^2 p(x)$ and a is a double root of f.

6. *Prove that it is impossible to write $x = f(x)g(x)$ where f and g are differentiable and $f(0) = g(0) = 0$.*

   Assume that we can, *i.e.*, there exist differentiable functions f and g so that $x = f(x)g(x)$ and $f(0) = 0 = g(0)$. Then

$$\begin{aligned} x &= f(x)g(x) \\ 1 &= f(x)g'(x) + f'(x)g(x) \text{ for all } x. \end{aligned}$$

   Thus, for $x = 0$

$$\begin{aligned} 1 &= f(0)g'(0) + f'(0)g(0) \\ 1 &= 0 \end{aligned}$$

   This contradiction indicates, then, that no two such functions exist.

7. *Suppose that f is differentiable at a. Prove that*

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a-h)}{2h}.$$

If f is differentiable at a, then

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

Replacing a by $a - h$ does not change the limit, so

$$f'(a) = \lim_{h\to 0} \frac{f(a) - f(a-h)}{h}$$

so

$$\begin{aligned}
2f'(a) = f'(a) + f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} + \lim_{h\to 0} \frac{f(a) - f(a-h)}{h} \\
&= \lim_{h\to 0} \left[\frac{f(a+h) - f(a)}{h} + \frac{f(a) - f(a-h)}{h} \right] \\
2f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a-h)}{h} \\
f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a-h)}{2h}
\end{aligned}$$

8. *Find the first derivative of each of the following:*

   (a) $\sin(\cos(x))$

   $$\frac{d}{dx} \sin(\cos(x)) = -\cos(\cos(x)) \sin(x).$$

   (b) $\sin\left(\frac{\cos x}{x}\right)$

   $$\frac{d}{dx} \sin\left(\frac{\cos x}{x}\right) = -\cos\left(\frac{\cos x}{x}\right)\left(\frac{\sin x}{x} + \frac{\cos x}{x^2}\right).$$

   (c) $\frac{\sin(\cos x)}{x}$

   $$\frac{d}{dx} \frac{\sin(\cos x)}{x} = -\frac{\cos(\cos x)\sin x}{x} - \frac{\sin(\cos x)}{x^2}.$$

9. *Find* $f(f'(x))$ *if* $f(x) = \frac{1}{x}$.

   $f'(x) = -1/x^2$, so $f(f'(x)) = 1/(-1/x^2) = -x^2$.