MATH 6102 — SPRING 2007
ASSIGNMENT 4

SOLUTIONS

February 12, 2007

1. *Let f be integrable on $[a,b]$, and suppose that g is a function on $[a,b]$ so that $f(x) = g(x)$ except for finitely many $x \in [a,b]$. Show that g is integrable and that $\int_a^b f = \int_a^b g$.*

   Let's use induction here. Assume that $f(x) = g(x)$ expect at one point $u \in [a,b]$. Let B be a bound for $|f|$ and $|g|$, $B > 0$. If $\epsilon > 0$ there is a partition P such that $U(f,P) - L(f,P) < \frac{\epsilon}{3}$. We may assume that $t_k - t_{k-1} < \frac{\epsilon}{12B}$ for all k. Since u belongs to at most two of the intervals $[t_{k-1}, t_k]$, we see that

   $$|U(g,P) - U(f,P)| \leq 2[B - (-B)] \cdot \max\{t_k - t_{k-1}\} < \frac{\epsilon}{3}.$$

   Likewise, $|L(g,P) - L(f,P)| < \frac{\epsilon}{3}$, so $U(g,P) - L(g,P) < \epsilon$. Hence g is integrable. The integrals agree since

   $$\int_a^b g \leq U(g,P) < U(f,P) + \frac{\epsilon}{3} < L(f,P) + \frac{2\epsilon}{3} \leq \int_a^b f + \frac{2\epsilon}{3}$$

   and similarly $\int_a^b g > \int_a^b f - \frac{2\epsilon}{3}$.

2. *Let f be integrable on $[a,b]$ and let $c \in [a,b]$. Prove that $\int_c^c f = 0$.*

   Any partition of $[c,c]$ must have $\text{mesh}(P) = 0$. Since the mesh has length 0 the integral must also be 0.

   Another way to see this is that

   $$\int_a^b f = \int_a^c f + \int_c^c f + \int_c^b f.$$

   Then subtracting off the equal parts from both sides leaves $\int_c^c f = 0$.

3. *Calculate $\lim_{x \to 0} \frac{1}{x} \int_0^x e^{t^2}\, dt$.*

   Let's use l'Hospital's Rule here:

   $$\begin{aligned} \lim_{x\to 0} \frac{\int_0^x e^{t^2}\, dt}{x} &= \lim_{x \to 0} \frac{e^{x^2}}{1} \\ &= 1 \end{aligned}$$

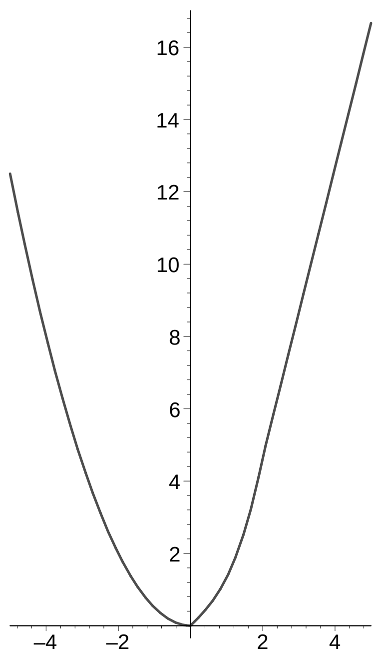
4. *Let f be defined as follows:*

   $$f(t) = \begin{cases} t & \text{for } t < 0, \\ t^2 + 1 & \text{for } 0 \leq t \leq 2, \\ 4 & \text{for } t > 2. \end{cases}$$

(a) *Determine the function $F(x) = \int_0^x f(t)\,dt$.*
Since we know that the function is piecewise continuous, then we can integrate it piecewise.

$$F(x) = \begin{cases} \int_0^x t\,dt & \text{for } t < 0, \\ \int_0^x t^2 + 1\,dt & \text{for } 0 \leq t \leq 2, \\ \int_0^x 4\,dt = \int_0^2 t^2 + 1\,dt + \int_2^x 4\,dt & \text{for } t > 2. \end{cases}$$
$$= \begin{cases} \frac{1}{2}x^2 & \text{for } x < 0, \\ \frac{1}{3}x^3 + x & \text{for } 0 \leq x \leq 2, \\ 4x - \frac{10}{3} & \text{for } x > 2. \end{cases}$$

(b) *Sketch F. Where is F continuous?*



F is continuous on $\mathbb{R}$. This we know from the Second Fundamental Theorem of Calculus.

(c) *Where is F differentiable? Find F' at all points of differentiability.*
$F(x)$ is differentiable when $f(x)$ is continuous. $f(x)$ is not continuous at $x = 0$ nor at $x = 2$. Thus, F is differentiable at all real numbers except $x = 0$ and $x = 2$, and $F'(x) = f(x)$.

5. *Let f be a continuous function on $\mathbb{R}$ and define*

$$F(x) = \int_{x-1}^{x+1} f(t)\,dt \quad \textit{for } x \in \mathbb{R}.$$

Show that F is differentiable on $\mathbb{R}$ and compute F'.

Let $F_0(x) = \int_0^{x+1} f(t)\,dt$. We know that F_0 is the composition of $G(x) = \int_0^x f(t)\,dt$ and $H(x) = x+1$, *i.e.*, $F_0(x) = G(H(x))$. G is differentiable by the Fundamental Theorem of Calculus and H is clearly differentiable. Thus, F_0 is differentiable.

Let $F_1(x) = \int_{x-1}^0 f(t)\,dt = -\int_0^{x-1} f(t)\,dt$. Now, just as above F_1 is differentiable. It is clear that $F = F_0 + F_1$ and

$$F'(x) = F_0'(x) + F_1'(x) = f(x+1) - f(x-1).$$

6. *Use the last example in the notes to show that*

$$\int_0^{1/2} \arcsin x\,dx = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1.$$

According to the last example, if we let $g(x) = \sin x$ then $g^{-1}(x) = \arcsin x$ and

$$\begin{aligned}
\int_a^b g(x)\,dx + \int_{g(a)}^{g(b)} g^{-1}(u)\,du &= b\cdot g(b) - a\cdot g(a) \\
\int_0^{\pi/6} \sin x\,dx + \int_0^{1/2} \arcsin x\,dx &= \frac{\pi}{6}\times\frac{1}{2} - 0\times 0 \\
\int_0^{1/2} \arcsin x\,dx &= \frac{\pi}{12} - \int_0^{\pi/6} \sin x\,dx \\
&= \frac{\pi}{12} + \cos x\big|_0^{\pi/6} \\
&= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1
\end{aligned}$$

7. *Let g be a strictly increasing continuous function mapping* $[0,1]$ *to* $[0,1]$. *Give a geometric argument showing*

$$\int_0^1 g(x)\,dx + \int_0^1 g^{-1}(u)\,du = 1.$$

If g is a strictly increasing continuous function on $[0,1]$, then we have that $g(0) < g(x) < g(1)$ for all $x \in [0,1]$. We know that

$$\int_0^1 g(x)\,dx + \int_{g(0)}^{g(1)} g^{-1}x\,dx = 1\cdot g(1) - 0\cdot g(0).$$

Since g is onto, we must have that $g(1) = 1$ and

$$\int_0^1 g(x)\,dx + \int_0^1 g^{-1}(u)\,du = 1.$$

Geometrically, the area under the curve $g^{-1}(x)$ is exactly the same as the area between the curve $y = g(x)$ and the y-axis. One way to see this is that we get g^{-1} by reflecting the graph of g across the line $y = x$. Since the graphs are both contained in the unit square, the sum of the two pieces has to be the area of the unit square, which is 1.