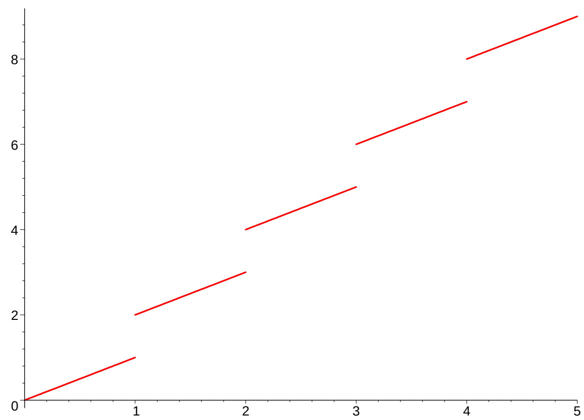
MATH 6102 — SPRING 2007
ASSIGNMENT 6
SOLUTIONS

February 26, 2007

1. *The symbol* $[x]$ *means the largest integer less than or equal to* x*. Find the following integral:*

$$\int_0^5 x + [x]\,dx.$$



Looking at the graph we see that this integral is the sum of the areas of a triangle and 4 trapezoids. Thus, the integral is

$$\int_0^5 x+[x]\,dx = \frac{1}{2}(1)(1)+\left(\frac{2+3}{2}\right)(1)+\left(\frac{4+5}{2}\right)(1)+\left(\frac{6+7}{2}\right)(1)+\left(\frac{8+9}{2}\right)(1) = \frac{1}{2}+\frac{5}{2}+\frac{9}{2}+\frac{13}{2}+\frac{17}{2} = \frac{45}{2}$$

2. *Is the function*

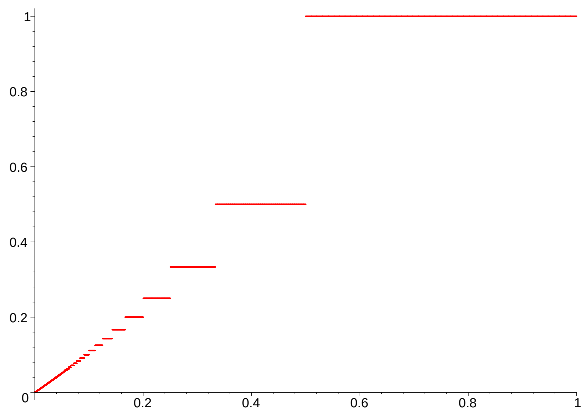
$$f(x) = \begin{cases} \dfrac{1}{\left[\dfrac{1}{x}\right]} & 0 < x \le 1 \\ 0 & x = 0 \text{ or } x > 1 \end{cases}$$

integrable on [0,2]? If so, what is it's integral? If not, why not?

The graph of this function is below:

The value of the function is

$$f(x) = \begin{cases} 1 & \frac{1}{2} < x \le 1 \\ \frac{1}{2} & \frac{1}{3} < x \le \frac{1}{2} \\ \frac{1}{3} & \frac{1}{4} < x \le \frac{1}{3} \\ \frac{1}{4} & \frac{1}{5} < x \le \frac{1}{4} \\ \vdots & \end{cases}$$



Thus, the integral of this function is

$$\begin{aligned}\int f(x)\,dx &= (1)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{3}-\frac{1}{4}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{4}-\frac{1}{5}\right) + \dots \\ &= \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \frac{\pi^2}{6} - 1\end{aligned}$$

Since this sum exists, the function is integrable.

3. *Find a function g such that*

(a) $\displaystyle\int_0^x tg(t)\,dt = x + x^2$

Use the Fundamental Theorem of Calculus to see that

$$\begin{aligned}\frac{d}{dx}\int_0^x tg(t)\,dt &= \frac{d}{dx}(x+x^2) \\ xg(x) &= 1 + 2x \\ g(x) &= \frac{1+2x}{x} = \frac{1}{x} + 2\end{aligned}$$

It is necessary that $g(0)$ be defined, though it may not have to be 0.

(b) $\displaystyle\int_0^{x^2} tg(t)\,dt = x + x^2$

Use the Fundamental Theorem of Calculus to see that

$$\begin{aligned}\frac{d}{dx}\int_0^{x^2} tg(t)\,dt &= \frac{d}{dx}(x+x^2)\\ x^2 g(x^2)\cdot 2x &= 1+2x\\ g(x^2) &= \frac{1+2x}{2x^3}\\ g(x) &= \frac{1+2\sqrt{x}}{2x^{3/2}}\end{aligned}$$

As above, it is necessary that $g(0)$ be defined, though it may not have to be 0.

4. *Find all functions f satisfying*

$$\int_0^x f = (f(x))^2 + C.$$

Use the Fundamental Theorem of Calculus again:

$$\begin{aligned}\int_0^x f &= (f(x))^2 + C\\ \frac{d}{dx}\int_0^x f &= \frac{d}{dx}\left[(f(x))^2 + C\right]\\ f(x) &= 2f(x)f'(x)\\ f(x)(1-2f'(x)) &= 0\end{aligned}$$

Thus, either $f(x)=0$, or $f'(x)=1/2$ and $f(x)=x/2$.

5. *Find $F'(x)$ if $F(x)=\int_0^x xf(t)\,dt$.*

Note that in integrating with respect to t, we can pull the x out of the integral, so that

$$F(x) = x\int_0^x f(t)\,dt$$

Therefore,

$$F'(x) = \frac{dx}{dx}\int_0^x f(t)\,dt + x\frac{d}{dx}\left(\int_0^x f(t)\,dt\right) = \int_0^x f(t)\,dt + xf(x).$$

6. *Prove that if h is continuous, f and g are differentiable, and*

$$F(x) = \int_{f(x)}^{g(x)} h(t)\,dt,$$

then $F'(x) = h(g(x))\cdot g'(x) - h(f(x))\cdot f'(x)$.

Let $a\in\mathbb{R}$ and $H(x)=\int_a^x h(t)\,dt$, then $H'(x)=h(x)$ and

$$\begin{aligned}F(x) &= \int_{f(x)}^{g(x)} h(t)\,dt\\ &= \int_{f(x)}^{a} h(t)\,dt + \int_a^{g(x)} h(t)\,dt\\ &= -H(f(x)) + H(g(x))\\ F'(x) &= -H'(f(x))\cdot f'(x) + H'(g(x))\cdot g'(x)\\ &= h(g(x))\cdot f'(x) - h(f(x))\cdot g(x)\end{aligned}$$