ASSIGNMENT 8 *SOLUTIONS*

26-March-2007

1. *Find f_x and f_y if*

$$f(x,y) = \int_0^x te^{-t^2}dt.$$

The partial with respect to y is easy since f does not depend on y at all

$$f_y(x,y) = \frac{\partial f}{\partial y}(x,y) = 0.$$

To find the partial with respect to x, we only need to use the Fundamental Theorem of Calculus

$$f_x(x,y) = \frac{\partial f}{\partial x}(x,y) = \frac{d}{dx}\int_0^x te^{-t^2}dt = xe^{-x^2}.$$

2. *The function $h(x,y)$ is defined in terms of two differentiable real-valued functions f and g of one variable. For each of the following compute $\partial h/\partial x$ and $\partial h/\partial y$.*

(a) $h(x,y) = f(x) + g(y)$.

$$\frac{\partial h}{\partial x}(x,y) = f'(x) \qquad \frac{\partial h}{\partial y}(x,y) = g'(y)$$

(b) $h(x,y) = f(x)g(y)$.

$$\frac{\partial h}{\partial x}(x,y) = f'(x)g(y) \qquad \frac{\partial h}{\partial y}(x,y) = f(x)g'(y)$$

(c) $h(x,y) = f(x)/g(y)$.

$$\frac{\partial h}{\partial x}(x,y) = \frac{f'(x)}{g(y)} \qquad \frac{\partial h}{\partial y}(x,y) = -\frac{f(x)g'(y)}{([g(y)]^2}$$

(d) $h(x,y) = f(x)^{g(y)}$.

$$\frac{\partial h}{\partial x}(x,y) = g(y)f'(x)f(x)^{g(y)-1} \qquad \frac{\partial h}{\partial y}(x,y) = g'(y)f(x)^{g(y)}\log(f(x))$$

3. *Find the equation of the tangent plane to $f(x,y) = x^2 - xy + y^2/2 + 3$ at $\mathbf{a} = (3,2)$.*

According to our notes the tangent plane is given by

$$L_{(a,b)}(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b).$$

Now,

$$f(3,2) = 8$$

$$\frac{\partial f}{\partial x}(x,y) = 2x - y \qquad \frac{\partial f}{\partial y}(x,y) = -x + y$$

$$\frac{\partial f}{\partial x}(3,2) = 4 \qquad \frac{\partial f}{\partial y}(3,2) = -1$$

so

$$L_{(3,2)}(x,y) = 8 + 4(x-3) - (y-2) = 4x - y - 2.$$

4. *Find the derivative of the function* $\mathbf{F}(x,y,z) = (\log(x^2+y^2+z^2), 2xy+z, xy+yz+xz)$.

   This will be a matrix.

$$D\mathbf{F}(x,y,z) = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} \end{pmatrix} = \begin{pmatrix} \dfrac{2x}{x^2+y^2+z^2} & \dfrac{2y}{x^2+y^2+z^2} & \dfrac{2z}{x^2+y^2+z^2} \\ 2y & 2x & 1 \\ y+z & x+z & x+y \end{pmatrix}$$

5. *Show that the function* $\mathbf{F}(x,y) = (x+y^2, 2xy)$ *is differentiable at* $(0,0)$.

   According to our theorem we only need to show that the partials exist and are continuous.

$$\frac{\partial F_1}{\partial x} = 1 \qquad \frac{\partial F_1}{\partial y} = 2y$$
$$\frac{\partial F_2}{\partial x} = 2y \qquad \frac{\partial F_2}{\partial y} = 2x$$

   Each of these is a polynomial, so each is continuous at every point in the plane, thus the function $\mathbf{F}$ is differentiable.

6. *Assume that* g *is a differentiable function of two variables and let* $f(x,y) = g(x^2-y^2, y^2-x^2)$. *Prove that* $xf_x + yf_y = 0$.

   Let $H(x,y) = (u(x,y), v(x,y))$ where $u(x,y) = x^2-y^2$ and $v(x,y) = y^2-x^2$. Then $f = g \circ H$ and by the Chain Rule

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial x} = 2xg_u - 2xg_v$$
$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial y} = -2yg_u + 2yg_v$$

   Therefore,

$$xf_y + yf_x = (-2xy + 2xy)(g_u + g_v) = 0.$$

7. *Let* $w = f(x,y,z)$, *where* $x = r\cos\theta$ *and* $y = r\sin\theta$. *Find* $\partial w/\partial r$, $\partial w/\partial\theta$, *and* $\partial w/\partial z$.

$$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r} = f_x\cos\theta + f_y\sin\theta$$
$$\frac{\partial w}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \theta} = -rf_x\sin\theta + rf_y\cos\theta$$
$$\frac{\partial w}{\partial z} = \frac{\partial f}{\partial z}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial z} = f_z$$

8. *Let $w = f(x, y, z)$, where $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, and $z = \rho \cos\phi$. Find $\partial w/\partial\rho$, $\partial w/\partial\phi$, and $\partial w/\partial\theta$.*

$$\frac{\partial w}{\partial \rho} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \rho} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \rho} = f_x \sin\phi\cos\theta + f_y \sin\phi\sin\theta + f_z\cos\phi$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \theta} = -f_x\rho\sin\phi\sin\theta + f_y\rho\sin\phi\cos\theta$$

$$\frac{\partial w}{\partial \phi} = \frac{\partial f}{\partial z}\frac{\partial x}{\partial \phi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \phi} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \phi} = f_x\rho\cos\phi\cos\theta + f_y\rho\cos\phi\sin\theta - f_z\rho\sin\phi$$

9. Find the directional derivative of $f(x, y) = \arctan(y/x)$ at $\mathbf{p} = (1, 1)$ in the direction of $\mathbf{v} = \mathbf{i} - 4\mathbf{j}$.

   The unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \mathbf{v}/\|\mathbf{v}\| = (1/\sqrt{17}, -4/\sqrt{17})$. The gradient is

$$\nabla f(x, y) = \left(-\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right)$$

   The gradient at $(1, 1)$ is

$$\nabla f(1, 1) = (-1/2, 1/2)$$

   and $D_u f(1, 1) = -5/2\sqrt{17}$.

10. *Let f and g be differentiable functions and let a, b, n be constants. Prove two of the following identities.*

   (a) $\nabla(af \pm bg) = a\nabla f \pm b\nabla g$.

$$\nabla(af \pm bg) = \langle\frac{\partial(af \pm bg)}{\partial x_i}\rangle = \langle af_{x_i} \pm bg_{x_i}\rangle = a\langle f_{x_i}\rangle \pm b\langle g_{x_i}\rangle = a\nabla f \pm b\nabla g$$

   (b) $\nabla(fg) = g\nabla f + f\nabla g$.

$$\nabla(fg) = \langle\frac{\partial fg}{\partial x_i}\rangle = \langle f_{x_i}g + fg_{x_i}\rangle = g\langle f_{x_i}\rangle + f\langle g_{x_i}\rangle = g\nabla f + f\nabla g$$

   (c) $\nabla(f/g) = (g\nabla f - f\nabla g)/g^2$, at all points where g is not zero.

$$\nabla(f/g) = \langle\frac{\partial f/g)}{\partial x_i}\rangle = \langle\frac{gf_{x_i} - fg_{x_i}}{g^2}\rangle = \frac{g\langle f_{x_i}\rangle - f\langle g_{x_i}\rangle}{g^2} = \frac{(g\nabla f - f\nabla g)}{g^2}$$

   (d) $\nabla f^n = nf^{n-1}\nabla f$.

$$\nabla f^n = \langle\frac{\partial f^n}{\partial x_i}\rangle = \langle nf^{n-1}f_{x_i}\rangle = nf^{n-1}\langle f_{x_i}\rangle = nf^{n-1}\nabla f$$