Chapter 2

Three Little Theorems

2.1 Introductions

Theorem 2.1 *If f is continuous on $[a, b]$ and $f(a) < 0 < f(b)$, then there is some $c \in [a, b]$ so that $f(c) = 0$.*

Geometrically, this says that the graph of a continuous function which starts below the x-axis and ends above it, must cross it at some point.

The proof of this will rely on the Least Upper Bound Axiom. While we did prove this last semester, we will go through it again here.

PROOF: Let $A = \{x \in [a, b] \mid f \text{ is negative on } [a, x]\}$. We see that $A \neq \emptyset$ since $a \in A$. Since f is continuous and $f(a) < 0$ there exists a $\delta > 0$ so that $[a, a + \delta) \subset A$. To see this, you can take $\epsilon = (0 - f(a))/2$, then there is a $\delta > 0$ so that whenever $|x - a| < \delta$ we have that $|f(x) - f(a)| < \epsilon$, which will mean that $f(x) < 0$.

Now, b is an upper bound for A. Likewise, there is a $\delta > 0$ so that all points in $(b - \delta, b]$ are upper bounds for A. This is proven similarly to the above since $f(b) > 0$).

Thus, A has a least upper bound α so that $a < \alpha < b$. We want to show that $f(\alpha) = 0$.

Assume not. Then either $f(\alpha) < 0$ or $f(\alpha) > 0$. Let's begin by assuming that $f(\alpha) < 0$. There is a $\delta > 0$ so that $f(x) < 0$ for all $\alpha - \delta < x < \alpha + \delta$. Thus, there is some number $x_0 \in A$ which satisfies $\alpha - \delta < x_0 < \alpha$ otherwise α would not be the *least* upper bound of A. This means that f is negative for all points in $[a, x_0]$. If $x_1 \in (\alpha, \alpha + \delta)$, then $f(x_1) < 0$ and f is negative on the interval $[x_0, x_1]$. This means that f is negative on the union of the two intervals, which is $[a, x_1]$. Thus, $x_1 \in A$. This contradicts the fact that α is the least upper bound for A, so the assumption that $f(\alpha) < 0$ must be false.

If we assume that $f(\alpha) > 0$, then there is a number $\delta > 0$ so that $f(x) > 0$ on $(\alpha - \delta, \alpha + \delta)$. This would mean that there is a number $x_0 \in A$ satisfying $\alpha - \delta < x_0 < \alpha$. This would mean that f is negative on $[a, x_0]$ which would be impossible since $f(x_0) > 0$. Thus, this assumption also must be false and we are left with $f(\alpha) = 0$. ∎

Theorem 2.2 *If f is continuous on $[a, b]$, then f is bounded on $[a, b]$, i.e., there is some number M so that $f(x) < M$ for all $x \in [a, b]$.*

This says that the graph of a continuous function on a closed interval must lie below some line parallel to the x-axis.

The proof of this theorem is very much like the previous. Here one takes the set $A = \{x \in [a, b] \mid f \text{ is bounded above on } [a, x]\}$. Again, note that $A \neq \emptyset$ since $a \in A$. Also, note that A is bounded above by b, so that A has a least upper bound, α. The proof consists of showing that $\alpha = b$.

Theorem 2.3 *If f is continuous on $[a, b]$, then there is some number $y \in [a, b]$ so that $f(y) \geq f(x)$ for all $x \in [a, b]$.*

Proof: We already know that f is bounded on $[a, b]$, which means that the set $\{f(x) \mid x \in [a, b]\}$ is bounded. This set is not empty, so it has a least upper bound α. Since $\alpha \geq f(x)$ for all $x \in [a, b]$ is suffices to show that $\alpha = f(y)$ for some $y \in [a, b]$.

So (as usual) assume that $\alpha \neq f(y)$ for all $y \in [a, b]$. Define a new function g by

$$g(x) = \frac{1}{\alpha - f(x)}, \qquad x \in [a, b].$$

We see that g is defined and continuous on $[a, b]$, since the denominator is never 0. Since $\alpha = \text{lub}\{f(x) \mid x \in [a, b]\}$ for every $\epsilon > 0$ there is an $x \in [a, b]$ so that $\alpha - f(x) < \epsilon$. Therefore, for any $\epsilon > 0$ there is an $x \in [a, b]$ so that $g(x) > 1/\epsilon$. This means that g is not bounded on $[a, b]$, which contradicts the previous theorem. ∎

2.2 Consequences of the Three Little Theorems

Theorem 2.4 *If f is continuous on $[a, b]$ and $f(a) < c < f(b)$, then there exists a point $x \in [a, b]$ such that $f(x) = c$.*

Proof: Let $g = f - c$. Then g is continuous and $g(a) < 0 < g(b)$. Thus by Theorem 2.1, there is an $x \in [a, b]$ so that $g(x) = 0$. This means that $f(x) = c$. ∎

Theorem 2.5 *If f is continuous on $[a, b]$ and $f(a) > c > f(b)$, then there exists a point $x \in [a, b]$ such that $f(x) = c$.*

Proof: The function $-f$ is continuous on $[a, b]$ and $-f(a) < -c < -f(b)$. Thus, there is an $x \in [a, b]$ so that $-f(x) = -c$, from whence follows the result. ∎

Theorem 2.6 (Intermediate Value Theorem) *If f is a continuous function on an interval takes on two values, then it takes on every value in between.*

This is a slight generalization of Theorem 2.1.

Theorem 2.7 *If f is continuous on $[a, b]$, then f is bounded below on $[a, b]$.*

This is proven by considering the function $-f$ on the interval and using Theorem 2.2.

Note that this theorem and Theorem 2.2 prove that if f is continuous on a closed interval, then f is bounded on that interval, *i.e.*, there is a real number N so that $|f(x)| < N$ for all x in the closed interval.

Theorem 2.8 *If f is continuous on $[a, b]$, then there is some $y \in [a, b]$ such that $f(y) \leq f(x)$ for all $x \in [a, b]$.*

This is the analogue to Theorem 2.3. These two theorems show that a continuous function on a closed interval takes on its maximum and minimum values.

Theorem 2.9 *Every positive real number has a square root.*

Proof: Consider the function $f(x) = x^2$, which is certainly continuous. If $\alpha \in \mathbb{R}^+$, then we need to show that this function takes on the value α.

There is a number $b > 0$ so that $f(b) > \alpha$. In fact, if $\alpha > 1$ we can take $b = \alpha$ while if $\alpha < 1$ we can take $b = 1$. Since $f(0) < \alpha < f(b)$, we can apply Theorem 2.4 to see that there is some number $x \in [0, b]$ so that $f(x) = x^2 = \alpha$. ∎

This same argument shows that a positive number has an nth root, for any natural number n. If n is odd, then we can show that *every* real number has an nth root. This statement is equivalent to saying that the equation

$$x^n - \alpha = 0$$

has a root if n is odd. We can generalize this as follows.

Theorem 2.10 *If n is odd, then any equation*

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$$

has a root.

Proof: Let

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 = x^n \left(1 + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \cdots + \frac{a_0}{x^n}\right).$$

Recall that we have shown that

$$\left|\frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \cdots + \frac{a_0}{x^n}\right| \leq \frac{|a_{n-1}|}{|x|} + \frac{|a_{n-2}|}{|x^2|} + \cdots + \frac{|a_0|}{|x^n|}.$$

Choose x satisfying
$$x > \max\{1, 2n|a_{n-1}|, \ldots, 2n|a_0|\}.$$
Then $|x^k| > |x|$ and
$$\frac{|a_{n-k}|}{|x^k|} < \frac{|a_{n-k}|}{|x|} < \frac{|a_{n-k}|}{2n|a_{n-k}|} = \frac{1}{2n},$$
Therefore,
$$\left|\frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \cdots + \frac{a_0}{x^n}\right| \leq \underbrace{\frac{1}{2n} + \cdots + \frac{1}{2n}}_{n \text{ terms}} = \frac{1}{2}.$$

Thus, we have shown that
$$-\frac{1}{2} \leq \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \cdots + \frac{a_0}{x^n} \leq \frac{1}{2},$$
Therefore
$$\frac{1}{2} \leq 1 + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \cdots + \frac{a_0}{x^n}.$$
Thus, if we choose an $x_1 > 0$ which satisfies the above condition, then
$$\frac{x_1^n}{2} \leq x_1^n\left(1 + \frac{a_{n-1}}{x_1} + \frac{a_{n-2}}{x_1^2} + \cdots + \frac{a_0}{x_1^n}\right) = f(x_1),$$
Thus, $f(x_1) > 0$.

Now, if we choose an $x_2 < 0$ satisfying the above condition, then $x_2^n < 0$ and
$$\frac{x_2^n}{2} \leq x_2^n\left(1 + \frac{a_{n-1}}{x_2} + \frac{a_{n-2}}{x_2^2} + \cdots + \frac{a_0}{x_1^n}\right) = f(x_2),$$
Thus, $f(x_2) < 0$.

Now, you can apply Theorem 2.1 to f on the interval $[x_2, x_1]$ to find a root. ∎

We can't do exactly the same thing for even powers of x though, because we know that we have equations with solutions, such as $x^2 - 1 = 0$, and those that do not, such as $x^2 + 1 = 0$. We can do the following.

Theorem 2.11 *If n is even and $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then there is a number y such that $f(y) \leq f(x)$ for all x.*

From this it follows that

Theorem 2.12 *Consider the equation*
$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = c,$$
and suppose that n is even. Then there exists a number m such that the equation has a solution for $c \geq m$ and no solution for $c < m$.