Chapter 2

Three Little Theorems

2.1 Introductions

Theorem 2.1 *If f is continuous on $[a, b]$ and $f(a) < 0 < f(b)$, then there is some $c \in [a, b]$ so that $f(c) = 0$.*

Geometrically, this says that the graph of a continuous function which starts below the x-axis and ends above it, must cross it at some point.

The proof of this will rely on the Least Upper Bound Axiom. While we did prove this last semester, we will go through it again here.

PROOF: Let $A = \{x \in [a, b] \mid f \text{ is negative on } [a, x]\}$. We see that $A \neq \emptyset$ since $a \in A$. Since f is continuous and $f(a) < 0$ there exists a $\delta > 0$ so that $[a, a + \delta) \subset A$. To see this, you can take $\epsilon = (0 - f(a))/2$, then there is a $\delta > 0$ so that whenever $|x - a| < \delta$ we have that $|f(x) - f(a)| < \epsilon$, which will mean that $f(x) < 0$.

Now, b is an upper bound for A. Likewise, there is a $\delta > 0$ so that all points in $(b - \delta, b]$ are upper bounds for A. This is proven similarly to the above since $f(b) > 0$).

Thus, A has a least upper bound α so that $a < \alpha < b$. We want to show that $f(\alpha) = 0$.

Assume not. Then either $f(\alpha) < 0$ or $f(\alpha) > 0$. Let's begin by assuming that $f(\alpha) < 0$. There is a $\delta > 0$ so that $f(x) < 0$ for all $\alpha - \delta < x < \alpha + \delta$. Thus, there is some number $x_0 \in A$ which satisfies $\alpha - \delta < x_0 < \alpha$ otherwise α would not be the *least* upper bound of A. This means that f is negative for all points in $[a, x_0]$. If $x_1 \in (\alpha, \alpha + \delta)$, then $f(x_1) < 0$ and f is negative on the interval $[x_0, x_1]$. This means that f is negative on the union of the two intervals, which is $[a, x_1]$. Thus, $x_1 \in A$. This contradicts the fact that α is the least upper bound for A, so the assumption that $f(\alpha) < 0$ must be false.

If we assume that $f(\alpha) > 0$, then there is a number $\delta > 0$ so that $f(x) > 0$ on $(\alpha - \delta, \alpha + \delta)$. This would mean that there is a number $x_0 \in A$ satisfying $\alpha - \delta < x_0 < \alpha$. This would mean that f is negative on $[a, x_0]$ which would be impossible since $f(x_0) > 0$. Thus, this assumption also must be false and we are left with $f(\alpha) = 0$. ∎

Theorem 2.2 *If f is continuous on $[a, b]$, then f is bounded on $[a, b]$, i.e., there is some number M so that $f(x) < M$ for all $x \in [a, b]$.*

This says that the graph of a continuous function on a closed interval must lie below some line parallel to the x-axis.

The proof of this theorem is very much like the previous. Here one takes the set $A = \{x \in [a, b] \mid f \text{ is bounded above on } [a, x]\}$. Again, note that $A \neq \emptyset$ since $a \in A$. Also, note that A is bounded above by b, so that A has a least upper bound, α. The proof consists of showing that $\alpha = b$.

Theorem 2.3 *If f is continuous on $[a, b]$, then there is some number $y \in [a, b]$ so that $f(y) \geq f(x)$ for all $x \in [a, b]$.*

PROOF: We already know that f is bounded on $[a, b]$, which means that the set $\{f(x) \mid x \in [a, b]\}$ is bounded. This set is not empty, so it has a least upper bound α. Since $\alpha \geq f(x)$ for all $x \in [a, b]$ is suffices to show that $\alpha = f(y)$ for some $y \in [a, b]$.

So (as usual) assume that $\alpha \neq f(y)$ for all $y \in [a, b]$. Define a new function g by

$$g(x) = \frac{1}{\alpha - f(x)}, \qquad x \in [a, b].$$

We see that g is defined and continuous on $[a, b]$, since the denominator is never 0. Since $\alpha = \text{lub}\{f(x) \mid x \in [a, b]\}$ for every $\epsilon > 0$ there is an $x \in [a, b]$ so that $\alpha - f(x) < \epsilon$. Therefore, for any $\epsilon > 0$ there is an $x \in [a, b]$ so that $g(x) > 1/\epsilon$. This means that g is not bounded on $[a, b]$, which contradicts the previous theorem. ∎

2.2 Consequences of the Three Little Theorems

Theorem 2.4 *If f is continuous on $[a, b]$ and $f(a) < c < f(b)$, then there exists a point $x \in [a, b]$ such that $f(x) = c$.*

PROOF: Let $g = f - c$. Then g is continuous and $g(a) < 0 < g(b)$. Thus by Theorem 2.1, there is an $x \in [a, b]$ so that $g(x) = 0$. This means that $f(x) = c$. ∎

Theorem 2.5 *If f is continuous on $[a, b]$ and $f(a) > c > f(b)$, then there exists a point $x \in [a, b]$ such that $f(x) = c$.*

PROOF: The function $-f$ is continuous on $[a, b]$ and $-f(a) < -c < -f(b)$. Thus, there is an $x \in [a, b]$ so that $-f(x) = -c$, from whence follows the result. ∎

Theorem 2.6 (Intermediate Value Theorem) *If f is a continuous function on an interval takes on two values, then it takes on every value in between.*

This is a slight generalization of Theorem 2.1.

Theorem 2.7 *If f is continuous on $[a, b]$, then f is bounded below on $[a, b]$.*

This is proven by considering the function $-f$ on the interval and using Theorem 2.2.

Note that this theorem and Theorem 2.2 prove that if f is continuous on a closed interval, then f is bounded on that interval, *i.e.*, there is a real number N so that $|f(x)| < N$ for all x in the closed interval.

Theorem 2.8 *If f is continuous on $[a, b]$, then there is some $y \in [a, b]$ such that $f(y) \leq f(x)$ for all $x \in [a, b]$.*

This is the analogue to Theorem 2.3. These two theorems show that a continuous function on a closed interval takes on its maximum and minimum values.

Theorem 2.9 *Every positive real number has a square root.*

PROOF: Consider the function $f(x) = x^2$, which is certainly continuous. If $\alpha \in \mathbb{R}^+$, then we need to show that this function takes on the value α.

There is a number $b > 0$ so that $f(b) > \alpha$. In fact, if $\alpha > 1$ we can take $b = \alpha$ while if $\alpha < 1$ we can take $b = 1$. Since $f(0) < \alpha < f(b)$, we can apply Theorem 2.4 to see that there is some number $x \in [0, b]$ so that $f(x) = x^2 = \alpha$. ∎

This same argument shows that a positive number has an nth root, for any natural number n. If n is odd, then we can show that *every* real number has an nth root. This statement is equivalent to saying that the equation

$$x^n - \alpha = 0$$

has a root if n is odd. We can generalize this as follows.

Theorem 2.10 *If n is odd, then any equation*

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$$

has a root.

PROOF: Let

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 = x^n \left(1 + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \cdots + \frac{a_0}{x^n}\right).$$

Recall that we have shown that

$$\left|\frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \cdots + \frac{a_0}{x^n}\right| \leq \frac{|a_{n-1}|}{|x|} + \frac{|a_{n-2}|}{|x^2|} + \cdots + \frac{|a_0|}{|x^n|}.$$

Choose x satisfying
$$x > \max\{1, 2n|a_{n-1}|, \ldots, 2n|a_0|\}.$$
Then $|x^k| > |x|$ and
$$\frac{|a_{n-k}|}{|x^k|} < \frac{|a_{n-k}|}{|x|} < \frac{|a_{n-k}|}{2n|a_{n-k}|} = \frac{1}{2n},$$
Therefore,
$$\left|\frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \cdots + \frac{a_0}{x^n}\right| \le \underbrace{\frac{1}{2n} + \cdots + \frac{1}{2n}}_{n \text{ terms}} = \frac{1}{2}.$$
Thus, we have shown that
$$-\frac{1}{2} \le \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \cdots + \frac{a_0}{x^n} \le \frac{1}{2},$$
Therefore
$$\frac{1}{2} \le 1 + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \cdots + \frac{a_0}{x^n}.$$
Thus, if we choose an $x_1 > 0$ which satisfies the above condition, then
$$\frac{x_1^n}{2} \le x_1^n \left(1 + \frac{a_{n-1}}{x_1} + \frac{a_{n-2}}{x_1^2} + \cdots + \frac{a_0}{x_1^n}\right) = f(x_1),$$
Thus, $f(x_1) > 0$.

Now, if we choose an $x_2 < 0$ satisfying the above condition, then $x_2^n < 0$ and
$$\frac{x_2^n}{2} \le x_2^n \left(1 + \frac{a_{n-1}}{x_2} + \frac{a_{n-2}}{x_2^2} + \cdots + \frac{a_0}{x_1^n}\right) = f(x_2),$$
Thus, $f(x_2) < 0$.

Now, you can apply Theorem 2.1 to f on the interval $[x_2, x_1]$ to find a root. ∎

We can't do exactly the same thing for even powers of x though, because we know that we have equations with solutions, such as $x^2 - 1 = 0$, and those that do not, such as $x^2 + 1 = 0$. We can do the following.

Theorem 2.11 *If n is even and $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then there is a number y such that $f(y) \le f(x)$ for all x.*

From this it follows that

Theorem 2.12 *Consider the equation*
$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = c,$$
and suppose that n is even. Then there exists a number m such that the equation has a solution for $c \ge m$ and no solution for $c < m$.

Chapter 3

Differentiation

This is meant to be a theoretical treatment of differentiation and all of its related concepts. These would have been covered in a standard Calculus course, but here we will endeavor to include proofs of the main results.

3.1 Basic Properties of the Derivative

Definition 3.1 *Let f be a real-valued function defined on an open interval containing the point a. We say the f is* differentiable at a*, or that f* has a derivative at a*, if the limit*

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists and is finite. We will write $f'(a)$ for the derivative of f at a:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

whenever this limit exists and is finite.

We will want to speak about f' as function in its own right. The domain of f' is the set of points at which f has a derivative, so $\operatorname{dom}(f') \subseteq \operatorname{dom} f$.

The algebra of derivatives turns out to be pretty simple.

Example 3.1 The derivative of $f(x) = x^2 - 1$ at $x = 3$ is given by

$$f'(3) = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} = \lim_{x \to 3} \frac{(x^2 - 1) - 8}{x - 3} = \lim_{x \to 3} x + 3 = 6.$$

It is not much more difficult to compute the derivative at $x = a$:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{(x^2 - 1) - (a^2 - 1)}{x - a} = \lim_{x \to a} x + a = 2a.$$

This computation is valid for all real numbers a, so the function $f'(x) = 2x$ is the derivative function of the function $f(x) = x^2 - 1$.

Example 3.2 Let $n \in \mathbb{Z}^+$, and let $f(x) = x^n$ for all $x \in \mathbb{R}$. Let $a \in \mathbb{R}$, then

$$f(x) - f(a) = x^n - a^n = (x-a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1}).$$

Therefore,

$$\frac{f(x)-f(a)}{x-a} = x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1},$$

for $x \neq a$. Thus,

$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \\ &= a^{n-1} + aa^{n-2} + a^2a^{n-3} + \cdots + a^{n-2}a + a^{n-1} = na^{n-1}. \end{aligned}$$

We want to find the derivative of the function $f(x) = \sin(x)$. This will require the following lemma.

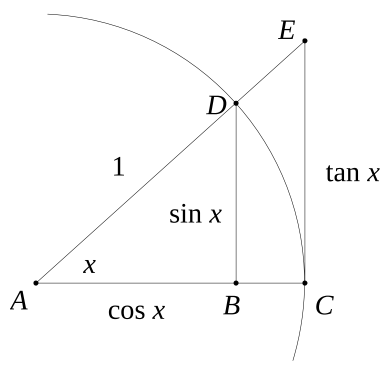

Figure 3.1:

Lemma 3.1

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

PROOF: In Figure 3.1 we are looking at a section of the unit circle. The area, A_I, of the inner triangle, $\triangle ABD$, has area $\frac{1}{2}\sin x \cos x$. The outer triangle, $\triangle ACE$, has area $A_O = \frac{1}{2}\tan x$. The sector of the circle defined by A, C, and D has area $A_S = \frac{x}{2}$. We clearly see that

$$A_I < A_S < A_O.$$

This means that

$$\begin{aligned} \frac{\sin x \cos x}{2} &< \frac{x}{2} < \frac{\tan x}{2} \\ \sin x \cos x &< x < \tan x \\ \cos x &< \frac{x}{\sin x} < \frac{1}{\cos x} \\ \cos x &< \frac{\sin x}{x} < \frac{1}{\cos x} \end{aligned}$$

Thus,

$$\lim_{x\to 0} \cos x \leq \lim_{x\to 0} \frac{\sin x}{x} \leq \lim_{x\to 0} \frac{1}{\cos x}$$

which leads to

$$1 \leq \lim_{x\to 0} \frac{\sin x}{x} \leq 1.$$

Therefore, we have that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

∎

Now, to find the derivative of the sine function we have:

$$\begin{aligned}
\lim_{x\to a}\frac{\sin x-\sin a}{x-a} &= \lim_{x\to a}\frac{2\sin(\frac{x-a}{2})\cos(\frac{x+a}{2})}{x-a}\\
&= \lim_{x\to a}\frac{\sin(\frac{x-a}{2})\cos(\frac{x+a}{2})}{\frac{x-a}{2}}\\
&= \lim_{x\to a}\frac{\sin(\frac{x-a}{2})}{\frac{x-a}{2}}\cos(\frac{x+a}{2})\\
&= \lim_{u\to 0}\frac{\sin(u)}{u}\lim_{u\to 0}\cos(u+a)\\
&= 1\cdot\cos a=\cos a
\end{aligned}$$

Thus, we get that the derivative function for $\sin x$ is $\cos x$.

We can show that the derivative function of the cosine function is $-\sin x$ in a similar manner.

$$\begin{aligned}
\lim_{x\to a}\frac{\cos x-\cos a}{x-a} &= \lim_{x\to a}\frac{-2\sin(\frac{x+a}{2})\sin(\frac{x-a}{2})}{x-a}\\
&= \lim_{x\to a}-\frac{\sin(\frac{x-a}{2})}{\frac{x-a}{2}}\sin(\frac{x+a}{2})\\
&= \lim_{x\to a}-\frac{\sin(\frac{x-a}{2})}{\frac{x-a}{2}}\lim_{x\to a}\sin(\frac{x+a}{2})\\
&= -1\cdot\sin a=-\sin a
\end{aligned}$$

One of the first things we should notice is that differentiability at a point implies continuity at a point.

Theorem 3.1 *If f is differentiable at $x=a$, then f is continuous at $x=a$.*

Proof: Since f is differentiable at $x=a$, we know that

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

exists and is finite. We need to show that $\lim_{x\to a} f(x)=f(a)$. We have

$$f(x)=(x-a)\frac{f(x)-f(a)}{x-a}+f(a)$$

for all $x \neq a$ in the domain of f. Taking the limit of both sides as $x \to a$ we get

$$\begin{aligned}
\lim_{x\to a} f(x) &= \lim_{x\to a}\left[(x-a)\frac{f(x)-f(a)}{x-a} + f(a)\right] \\
&= \lim_{x\to a}\left[(x-a)\frac{f(x)-f(a)}{x-a}\right] + \lim_{x\to a} f(a) \\
&= \lim_{x\to a}(x-a)\lim_{x\to a}\frac{f(x)-f(a)}{x-a} + f(a) \\
&= 0\cdot f'(a) + f(a) = f(a)
\end{aligned}$$

We are done. ∎

Note that the other direction is not true. What is a good counterexample?

3.2 Rules of Differentiation

We now want to look at the basic rules of differentiation. We will start with the easy Sum and Difference Rules.

Theorem 3.2 *If f and g are both differentiable, then $f+g$ and $f-g$ are differentiable and*

$$(f \pm g)'(x) = f'(x) \pm g'(x).$$

Proof: We will show that $(f+g)'(a) = f'(a) + g'(a)$ for an arbitrary $a \in \operatorname{dom} f$.

$$\begin{aligned}
\lim_{x\to a}\frac{(f+g)(x)-(f+g)(a)}{x-a} &= \lim_{x\to a}\frac{f(x)+g(x)-(f(a)+g(a))}{x-a} \\
&= \lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a} + \frac{g(x)-g(a)}{x-a}\right] \\
&= f'(a) + g'(a)
\end{aligned}$$

The difference is proven in exactly the same way. ∎

This comes as no surprise.

Theorem 3.3 *If $c \in \mathbb{R}$ and f is the constant function given by $f(x) = c$ for all $x \in \mathbb{R}$, then $f'(x) = 0$ for all $x \in \mathbb{R}$.*

Proof: Simply compute the derivative:

$$\begin{aligned}
f'(a) &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \\
&= \lim_{x\to a}\frac{0}{x-a} = 0
\end{aligned}$$

and we are done. ∎

We need to compute three more general derivatives: the derivative of a product, the derivative of a quotient, and the derivative of a composition.

3.2.1 Product Rule

The *student product rule* would be what we would expect: $(fg)'(x) = f'(x)g'(x)$. This is ***not*** true. Simple counterexamples are numerous.

The real problem lies in the difference quotient. ***Note*** that

$$\frac{f(x)g(x) - f(a)g(a)}{x-a} \neq \frac{f(x)-f(a)}{x-a} \cdot \frac{g(x)-g(a)}{x-a}.$$

Instead, note that

$$\begin{aligned}\frac{f(x)g(x) - f(a)g(a)}{x-a} &= \frac{f(x)g(x) - f(x)g(a) + f(x)g(a) - f(a)g(a)}{x-a} \\ &= f(x)\frac{g(x)-g(a)}{x-a} + g(a)\frac{f(x)-f(a)}{x-a}\end{aligned}$$

Now, taking the limit as $x \to a$ gives us that

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

This is the Product Rule for derivatives.

Note that the following is an immediate corollary of the previous two rules.

Theorem 3.4

$$(cf)'(x) = c \cdot f'(x)$$

if f is differentiable and $c \in \mathbb{R}$.

3.2.2 Quotient Rule

Theorem 3.5 *If f and g are differentiable at $x = a$ and $g'(a) \neq 0$, then*

$$\left(\frac{f}{g}\right)'(a) = \frac{g(a)f'(a) - f(a)g'(a)}{(g(a))^2}.$$

PROOF: Again, we just have to rewrite the quotient appropriately. Since $g'(a) \neq 0$ and g is continuous at $x = a$, there is an open interval I containing a so that $g(x) \neq 0$ for $x \in I$. Thus, for $x \in I$, we can write

$$\begin{aligned}(f/g)(x) - (f/g)(a) &= \frac{f(x)}{g(x)} - \frac{f(a)}{g(a)} \\ &= \frac{g(a)f(x) - f(a)g(x)}{g(x)g(a)} \\ &= \frac{g(a)f(x) - g(a)f(a) + g(a)f(a) - f(a)g(x)}{g(x)g(a)}\end{aligned}$$

so

$$\frac{(f/g)(x)-(f/g)(a)}{x-a} = \left[g(a)\frac{f(x)-f(a)}{x-a} - f(a)\frac{g(x)-g(a)}{x-a}\right]\frac{1}{g(x)g(a)}$$

Taking the limits as $x \to a$ gives us the result. ∎

3.2.3 The Chain Rule

Theorem 3.6 *If f is differentiable at a and g is differentiable at $f(a)$, then the composite function $g \circ f$ is differentiable at a and $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$.*

Proof: Since g is differentiable at $f(a)$ it can be shown that $g \circ f$ is defined on some open interval containing a. Let

$$h(y) = \frac{g(y) - g(f(a))}{y - f(a)}$$

for $y \in \operatorname{dom} g$ and $y \neq f(a)$, and let $h(f(a)) = g'(f(a))$. Since $\lim_{y \to f(a)} h(y) = h(f(a))$, the function h is continuous at $f(a)$.

$$g(y) - g(f(a)) = h(y)[y - f(a)]$$

for all $y \in \operatorname{dom} g$ so

$$(g \circ f)(x) - (g \circ f)(a) = h(f(x))[f(x) - f(a)] \text{ for } x \in \operatorname{dom}(g \circ f).$$

Therefore

$$\frac{(g \circ f)(x) - (g \circ f)(a)}{x - a} = h(f(x))\frac{f(x) - f(a)}{x - a}$$

for $x \in \operatorname{dom}(g \circ f)$, $x \neq a$. Since $\lim_{x \to a} f(x) = f(a)$ and the function h is continuous at $f(a)$,

$$\lim_{x \to a} h(f(x)) = h(f(a)) = g'(f(a)).$$

The other limit in the above is $f'(a)$, so the result follows. ∎

3.2.4 Other Transcendental Derivatives

We should find the derivatives of the natural logarithm and the exponential functions. However, for a strictly rigorous treatment we will defer until we do integration. For now, we will deal with these functions as follows.

First, recall that

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e.$$

Now, we want to compute the derivative of the natural logarithm, $\ln x$ which is defined by $\ln a = b$ means $a = e^b$. We know that this function satisfies all of the usual properties of the logarithms, so we begin with a lemma.

Lemma 3.2 *If f is differentiable at $x = a$, then*

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

Then, if $f(x) = \ln x$ we have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\ln(x+h) - \ln x}{h} \\
&= \lim_{h\to 0} \frac{1}{h} \ln\left(\frac{x+h}{x}\right) \\
&= \lim_{h\to 0} \frac{1}{h} \ln\left(1 + \frac{h}{x}\right) \\
&= \lim_{h\to 0} \frac{1}{x}\frac{x}{h} \ln\left(1 + \frac{h}{x}\right) \\
&= \lim_{h\to 0} \frac{1}{x} \ln\left[\left(1 + \frac{h}{x}\right)^{x/h}\right] \\
&= \frac{1}{x} \lim_{t\to\infty} \ln\left(1 + \frac{1}{t}\right)^t \\
&= \frac{1}{x} \ln \lim_{t\to\infty} \left(1 + \frac{1}{t}\right)^t \\
&= \frac{1}{x} \ln(e) = \frac{1}{x}
\end{aligned}$$

Now, we made a number of assumptions here, such as the logarithm is a continuous function and others. Nonetheless, we are okay for now.

We have that the derivative of the natural logarithm of x is $1/x$. Now, to the exponential function. Here we will use the Chain Rule.

$$\begin{aligned}
\ln(e^x) &= x \\
\frac{d}{dx}\ln(e^x) &= \frac{d}{dx}x \\
\frac{1}{e^x}\frac{d}{dx}e^x &= 1 \\
\frac{d}{dx}e^x &= e^x
\end{aligned}$$

From these and the Chain Rule and the Product Rule the following are true:

$$\begin{aligned}
\frac{d}{dx}\sin x &= \cos x \\
\frac{d}{dx}\cos x &= -\sin x \\
\frac{d}{dx}\tan x &= \sec^2 x \\
\frac{d}{dx}\cot x &= -\csc^2 x \\
\frac{d}{dx}\sec x &= \sec x \tan x \\
\frac{d}{dx}\csc x &= -\csc x \cot x \\
\frac{d}{dx}e^x &= e^x \\
\frac{d}{dx}a^x &= a^x \ln a \\
\frac{d}{dx}\ln x &= \frac{1}{x} \\
\frac{d}{dx}\log_b x &= \frac{1}{x \ln b} \\
\frac{d}{dx}\sinh x &= \cosh x \\
\frac{d}{dx}\cosh x &= \sinh x
\end{aligned}$$

Recall that we had defined the hyperbolic sine and cosine by

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}.$$

Recall that the Lambert W-function is defined by $W(a) = b$ means that $a = b \cdot e^b$. Thus, to find the derivative of this function we will use the Product Rule and the Chain Rule.

$$\begin{aligned}
x &= W(x)e^{W(x)} \\
\frac{dx}{dx} &= \frac{d}{dx}\left(W(x)e^{W(x)}\right) \\
1 &= W'(x)e^{W(x)} + W(x)e^{W(x)}W'(x) \\
W'(x) &= \frac{1}{e^{W(x)}(1 + W(x))}
\end{aligned}$$

but $e^{W(x)} = x/W(x)$, so

$$W'(x) = \frac{W(x)}{x(1 + W(x))}, \ x \neq 0, -1/e$$

There are very few functions that we cannot differentiate using these theorems, if they have a derivative.

This technique leads us to a more general theorem.

Theorem 3.7 *Let f be a continuous one-to-one function defined on an interval and suppose that f is differentiable at $a = f^{-1}(b)$, with $f'(a) \neq 0$, then f^{-1} is differentiable at b and*

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}.$$

Proof: Let $b = f(a)$. Then

$$\lim_{h\to 0} \frac{f^{-1}(b+h) - f^{-1}(b)}{h} = \lim_{h\to 0} \frac{f^{-1}(b+h) - a}{h}.$$

Now, every number $b + h$ in the domain of f^{-1} can be written in the form $b + h = f(a + k)$ for a unique k. Then

$$\begin{aligned} \lim_{h\to 0} \frac{f^{-1}(b+h) - a}{h} &= \lim_{h\to 0} \frac{f^{-1}(f(a+k)) - a}{f(a+k) - b} \\ &= \lim_{h\to 0} \frac{k}{f(a+k) - f(a)} \end{aligned}$$

Now, since $b + h = f(a + k)$ we have

$$\begin{aligned} b + h &= f(a + k) \\ f^{-1}(b + h) &= a + k \\ k &= f^{-1}(b + h) - a = f^{-1}(b + h) - f^{-1}(b) \end{aligned}$$

The function f^{-1} is continuous at b, so we have that k approaches 0 as h approaches 0. Since

$$\lim_{k\to 0} \frac{f(a+k) - f(a)}{k} = f'(a) = f'(f^{-1}(b)) \neq 0,$$

we have that

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}.$$

∎

This leads to the derivatives of the inverse trigonometric functions. Let's consider the arctangent function. First, we will choose the tangent with domain $(-\pi/2, \pi/2)$. The range is $\mathbb{R}$ and the arctangent will have domain $\mathbb{R}$ and range $(-\pi/2, \pi/2)$. I will use the term arctan for the arctangent function.

$$\begin{aligned} \tan(\arctan x) &= x \\ \sec^2(\arctan x)\frac{d}{dx}\arctan x &= 1 \\ \frac{d}{dx}\arctan x &= \cos^2(\arctan x) \end{aligned}$$

However, because these are trigonometric functions, we know more. The arctangent of x is the angle whose tangent is x. The tangent of that angle can be found from a right triangle whose ratio of the opposite side to the adjacent side is $x : 1$. Thus, we can take the opposite side to have length x and the adjacent side to have length 1. That means that the hypotenuse is $\sqrt{1+x^2}$. Then the cosine of that angle is the ratio of the adjacent side to the hypotenuse, or $1 : \sqrt{1+x^2}$. Thus,

$$\cos(\arctan(x)) = \frac{1}{\sqrt{1+x^2}}$$

and therefore

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}.$$

Similar techniques show that

$$\begin{aligned} \frac{d}{dx}\arcsin x &= \frac{1}{\sqrt{1-x^2}} \\ \frac{d}{dx}\operatorname{arcsec} x &= \frac{1}{x\sqrt{x^2-1}} \end{aligned}$$