Chapter 13

Sequences and Series of Functions

These notes are based on the notes *A Teacher's Guide to Calculus* by Dr. Louis Talman.

The treatment of power series that we find in most of today's elementary calculus texts overemphasizes the notion of pointwise convergence (see Definition 13.4) at the expense of the much more important notion of uniform convergence on compact sub-intervals (see Definitions 13.5 and 13.6).

On the other hand, the uses we encounter for power series all depend in fundamental ways on the facts that we get the right things when we differentiate or integrate them term-by-term (see Theorem 13.16). That these calculations work depends upon the fact that a power series always converges uniformly on the compact sub-intervals of the interior of its interval of convergence (see Theorem 13.11, Theorem 13.12, Definition 13.7, and Theorem 13.13). The presentation here will downplay the role of pointwise convergence in sequences and series and will discuss uniform convergence on compact sub-intervals.

13.1 Some Technical Preliminaries

We need to recall the definitions of the limit superior, and the limit inferior of an arbitrary sequence

Definition 13.1 *Let $\{a_k\}$, $k = 0, \dots, \infty$ be a sequence of real numbers. If for every positive number M there is a positive integer n such that $M < a_n$, we put the limit superior of the sequence $\{a_k\}$, written* $\limsup a_k$, *equal to ∞. If there is a positive number M such that $a_k < M$ for all non-negative integers k, we define*

$$\limsup a_k = \lim_{n\to\infty} \operatorname{lub}\{a_k \mid n \le k\}. \tag{13.1}$$

We define the limit inferior of the sequence $\{a_k\}$, written $\liminf a_k$, *by*

$$\liminf a_k = -\limsup(-a_k). \tag{13.2}$$

Theorem 13.1 *If* $\{a_k\}$, $k = 0, \ldots, \infty$ *is any sequence of real numbers, then* $\limsup a_k$ *exists (but may be* $\pm\infty$*).*

PROOF: It is clear that the limit superior of $\{a_k\}$ exists if the sequence is not bounded above, so assume that there is an upper bound M for the sequence.

Note that each of the sets $\{a_k \mid n \leq k\}$ that appears in 13.1 is bounded above (by M), so that all of the least upper bounds in 13.1 exist. For each nonnegative integer n, let us put $\ell_n = \text{lub}\{a_k \mid n \leq k\}$. Note also that, for every non-negative integer n,

$$\ell_n \geq \ell_{n+1}. \tag{13.3}$$

If the sequence $\{\ell_n\}$ has a lower bound, then it has a greatest lower bound L, and the intervals $[L, \ell_n]$ form a nested family whose intersection contains only L itself. It follows that L must therefore be $\lim_{n\to\infty} \ell_n = \lim a_k$. If, on the other hand, the sequence $\{\ell_n\}$ has no lower bound, then by 13.3 we must have $\limsup a_k = \lim_{n\to\infty} \ell_n = -\infty$. Thus, the limit superior of any sequence whatsoever existsprovided that we admit $\pm\infty$ as a possibility ∎

Corollary 13.1 *If* $\{a_k\}$ *is any sequence of real numbers, then* $\liminf a_k$ *exists (but may be* $\pm\infty$*).*

We will need some elementary properties of the limit superior.

Theorem 13.2 (Properties of Limit Superior) *) Let* $\{a_k\}$ *and* $\{b_k\}$ *be bounded sequences of real numbers.*

i) The real number $L = \limsup a_k$ *if and only if for every* $\epsilon > 0$*, both*

1. $a_k > L + \epsilon$ *for at most finitely many integers* k*, and*
2. $a_k > L - \epsilon$ *for infinitely many integers* k*.*

ii) If $\limsup a_k = L > 0$ *and* $\{b_k\}$ *is a sequence of positive numbers for which* $\lim_{k\to\infty} b_k = M > 0$*, then* $\limsup(a_k b_k) = LM$*.*

iii) If $\lim_{k\to\infty} a_k$ *exists, then* $\limsup a_k = \lim_{k\to\infty} a_k$*.*

iv)

$$\limsup(a_k + b_k) \leq \limsup a_k + \limsup b_k. \tag{13.4}$$

PROOF: Let C be a number, let $\epsilon > 0$ and assume that there are infinitely many integers k for which $a_k > C + \epsilon$, then $\text{lub}\{a_k \mid k > n\} > C + \epsilon$ for every positive integer n, therefore $C < C + \epsilon \leq L$. On the other hand, if C is a number for which there is an $\epsilon > 0$ for which there are only finitely many integers k such that $C - \epsilon < a_k$, then we can find an integer N so that $a_k < C - \epsilon$ for all $k \geq N$. But then $C - \epsilon$ is an upper bound for each of the sets $\{a_k \mid k > n\}$ for which $n > N$. It follows that $\operatorname{lub}\{a_k \mid k > n\} \leq C - \epsilon$ for all $n \geq N$. This means that $L \leq C - \epsilon < C$. Consequently, neither (a) nor (b) can fail for L.

If, now, C is a number which has the property that for every $\epsilon > 0$ there are at most finitely many integers k for which $C + \epsilon < a_k$, then for each $\epsilon > 0$, $\operatorname{lub}\{a_k \mid k > n\}$ is less than or equal to $C + \epsilon$ when n is sufficiently large. Consequently, $\lim_{n\to\infty} \operatorname{lub}\{a_k \mid k > n\} \leq C + \epsilon$. Because this is so for every positive ϵ, this means that $L \leq C$. If C also has the property that for every positive ϵ there are infinitely many non-negative integers k for which $C - \epsilon < a_k$, then for each $\epsilon > 0$ and each n, $C - \epsilon < \operatorname{lub}\{a_k \mid k > n\}$. From this it follows that $C - \epsilon \leq L$ for every $\epsilon > 0$, and thus that $C \leq L$. Thus, if C has both of properties (a) and (b), it follows that $L = C$. This establishes part 1 of the Lemma.

Now let $\{a_k\}$ and $\{b_k\}$ be as in part 2. Suppose that $\epsilon > 0$ is given. Take

$$\eta = \sqrt{\left(\frac{L+M}{2}\right)^2 + \epsilon} - \frac{L+M}{2}, \tag{13.5}$$

and note that $\eta > 0$. We can now apply the definition of limit to the sequence $\{b_k\}$ and part 1 of this theorem to the sequence $\{a_k\}$ to choose an N so large that both

$$b_k < M + \eta \tag{13.6}$$

and

$$\tag{13.7}$$

$$a_k < L + \eta \tag{13.8}$$

whenever $k > N$. But then for such k we have

$$a_k b_k < (L+\eta)(M+\eta) = LM + (L+M)\eta + \eta^2. \tag{13.9}$$

We note now that

$$(L+M)\eta + \eta^2 = (L+M)\sqrt{\left(\frac{L+M}{2}\right)^2 + \epsilon} - \frac{(L+M)^2}{2}$$

$$2 + \left(\sqrt{\left(\frac{L+M}{2}\right)^2 + \epsilon} - \frac{L+M}{2}\right)^2 \tag{13.10}$$

$$= (L+M)\sqrt{\left(\frac{L+M}{2}\right)^2 + \epsilon} - 2\left(\frac{L+M}{2}\right)^2$$

$$+ \left(\frac{L+M}{2}\right)^2 + \epsilon \tag{13.11}$$

$$- (L+M)\sqrt{\left(\frac{L+M}{2}\right)^2 + \epsilon} + \left(\frac{L+M}{2}\right)^2$$

$$= \epsilon, \tag{13.12}$$

and thus that

$$a_k b_k < LM + \epsilon \tag{13.13}$$

for all $k > N$. It follows that there are at most finitely many integers k for which $a_k b_k \geq LM + \epsilon$.

On the other hand, we may suppose that

$$\epsilon < \min\{L, M\}, \tag{13.14}$$

and put

$$\theta = \frac{L+M}{2} - \sqrt{\left(\frac{L+M}{2}\right)^2 - \epsilon}. \tag{13.15}$$

Then $\theta > 0$ and we can find a positive integer N such that $0 < M - \theta < b_k$ for all $k > N$. Moreover, by part 1 of this theorem, there are infinitely many non-negative integers k, and therefore infinitely many integers $k > N$, for which $0 < L - \theta < b_k$. A calculation similar to that above shows that $(L-\theta)(M-\theta) = LM - \epsilon$, and it follows that there are infinitely many nonnegative integers k such that $LM - \epsilon < a_k b_k$. We may now conclude that $\limsup a_k b_k = LM$, and we have established 2.

If $\lim_{k\to\infty} a_k = L$, then for any $\epsilon > 0$ we can choose a non-negative integer N such that $|a_k - L| < \epsilon$ for all $k \geq N$. Equivalently, $-\epsilon < a_k - L < \epsilon$, or $L - \epsilon < a_k < L + \epsilon$ when $k \geq N$. But then $L + \epsilon < a_k$ for at most those finitely many $k < N$ and $L - \epsilon < a_k$ for the infinitely many $k > N$. This proves 3.

For each positive integer n, let $A_n = \text{lub}\{a_k \mid k \geq n\}$; $B_n = \text{lub}\{b_k \mid k \geq n\}$. Then, given n, for every $k \geq n$ we must have $a_k + b_k \leq A_n + B_n$. Consequently,

$$\text{lub}\{a_k + b_k \mid k \leq n\} \leq A_n + B_n \tag{13.16}$$

for every n. The latter tends to $\limsup a_k + \limsup b_k$ as n goes to infinity. This establishes 4. ∎

The reader should note that the inequality 13.4 cannot be improved to equality. This can be easily seen to be the case using the example $a_k = (-1)^k$ and $b_k = (-1)^{k+1}$.

When the limit superior is infinite, statements analogous to those of Theorem 13.2 are true. It is left to the reader to formulate and prove them.

Theorem 13.3 (Properties of Limit Inferior) *Let $\{a_k\}$ and $\{b_k\}$ be bounded sequences of real numbers.*

i) The real number $C = \liminf a_k$ if and only if for every $\epsilon > 0$, both

1. $a_k < C - \epsilon$ for at most finitely many integers k, and

2. $a_k < C + \epsilon$ for infinitely many integers k.

ii) If $\liminf a_k = C > 0$ and $\{b_k\}$ is a sequence of positive numbers for which $\liminf_{k\to\infty} b_k = D > 0$, then $\liminf(a_k b_k) = CD$.

iii) If $lim_{k\to\infty} a_k$ *exists, then* $\liminf a_k = \lim_{k\to\infty} a_k$.

iv)

$$\liminf(a_k + b_k) \geq \liminf a_k + \liminf b_k. \tag{13.17}$$

For the sake of consistence and completeness, we will record some definitions here. They will not play a central role in what is to follow, but we will need to know some things about them.

Definition 13.2 *Let* $\{a_k\} = 0$ *be a sequence of real numbers, and consider the auxiliary sequence of partial sums*

$$s_n = \sum_{k=0}^{n} a_k \tag{13.18}$$

If $\lim_{n\to\infty} s_n$ *exists and equals a number* L*, we will say that the series* $\sum_{k=0}^{\infty} a_k$ *converges to* L*. Otherwise we will say that the series diverges. If the series* $\sum_{k=0}^{\infty} |a_k|$ *converges, we will say that the series* $\sum_{k=0}^{\infty} a_k$ *converges absolutely. A series which converges, but does not converge absolutely, is said to converge conditionally.*

The alternating harmonic series $\sum_{k=0}^{\infty} \frac{(-1)^k}{k}$ is a standard example, appearing in almost all elementary calculus texts, showing that it is possible for a series to converge conditionally. However, a series that converges absolutely must converge.

Theorem 13.4 *If the series* $\sum_{k=0}^{\infty} |a_k|$ *converges, then so does the series* $\sum_{k=0}^{\infty} a_k$.

Proof: For each non-negative integer n, let $s_n = \sum_{k=0}^{\infty} a_k$ and $S_n = \sum_{k=0}^{\infty} |a_k|$. For each n we have

$$0 \leq a_n + |a_n| \leq 2|a_n|, \tag{13.19}$$

so, by the Comparison Test, $\sum_{k=0}^{\infty}(a_k + |a_k|)$ is a convergent series. But

$$\sum_{k=0}^{\infty}(a_k + |a_k|) = s_n + S_n, \text{ and} \tag{13.20}$$

$$\sum_{k=0}^{\infty} |a_k| = S_n, \tag{13.21}$$

so that $\lim_{n\to\infty}(s_n + S_n)$ and $\lim_{n\to\infty} S_n$ both exist. Consequently,

$$\sum_{n=0}^{\infty} a_k = \lim_{n\to\infty} s_n \tag{13.22}$$
$$= \lim_{n\to\infty} (s_n + S_n - S_n) \tag{13.23}$$
$$= \lim_{n\to\infty} (s_n + S_n) - \lim_{n\to\infty} S_n \tag{13.24}$$

exists and the series $\sum_{k=0}^{\infty} a_k$ converges. ∎

13.2 What Geometric Series Can Tell Us

The well-known geometric series $\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \dots$ is archetypal for the behavior of power series, so we begin with a treatment of that series. You need to note an important notational convention — we will interpret x^0 to mean 1 even when $x = 0$. This standard misuse of notation allows us to write, e.g., $\sum_{k=0}^{\infty} x^k$ instead of the clumsier $1 + \sum_{k=1}^{\infty} x^k$. The convention is widely observed in discussion of power series, but rarely is it noted explicitly.

Definition 13.3 *For each non-negative integer n we let g_n be the polynomial function $g_n(x) = 1 + x + x^2 + \cdots + x^n = \sum_{k=0}^{n} x^k$. We understand the geometric series, $\sum_{k=0}^{\infty} x^k$ to mean $\lim_{n\to\infty} g_n(x)$ for those values of x for which the limit exists.*

Theorem 13.5 (Convergence of the Geometric Series) *When $|x| < 1$, we have*

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}. \tag{13.25}$$

If $|x| \geq 1$, the geometric series has no meaning.

Proof: For each $n \in \mathbb{N}$ and all $x \in \mathbb{R}$, we have

$$(1-x)g_n(x) = (1-x)(1 + x + \cdots + x^n) \tag{13.26}$$
$$= 1 - x^{n+1}. \tag{13.27}$$

Thus, when $x \neq 1$, we have
$$g_n(x) = \frac{1 - x^{n+1}}{1 - x}. \tag{13.28}$$
If $|x| < 1$, then $\lim_{n\to\infty} x^{n+1} = 0$, from which we now see that $\lim_{n\to\infty} g_n(x) = \dfrac{1}{1-x}$ for such x. If $|x| > 1$, then $\lim_{n\to\infty} x^{n+1}$ does not exist, and we can attach no meaning to $\lim_{n\to\infty} g_n(x)$ for such values of x. Now $g_n(-1) = 0$ for n odd, but $g_n(-1) = 1$ for n even. Consequently, $\lim_{n\to\infty} g_n(-1)$ does not exist. Finally, $g_n(1) = n + 1$, and so $\lim_{n\to\infty} g_n(1)$ does not exist. ∎

If this were all there was about geometric series, if would be unremarkable. However, the polynomials $g_n(x)$ converge to $\dfrac{1}{1-x}$ in a more interesting way than we have shown. This is one of the ways in which the geometric series is archetypal.

Theorem 13.6 (Convergence of the Geometric Series) *Let $a \in (0, 1)$ be arbitrary, and let $\epsilon > 0$ be given. There exists $N \in \mathbb{N}$ such that whenever $n > N$ we have*
$$\left| g_n(x) - \frac{1}{1-x} \right| < \epsilon \tag{13.29}$$
for all $x \in [-a, a]$.

Proof: Let $a \in (0, 1)$ and $\epsilon > 0$ be given. We know that $\lim_{n\to\infty} a^{n+1} = 0$, so we can choose $N \in \mathbb{N}$ so large that $a^{n+1} < (1 - a)\epsilon$ whenever $n \geq N$. If $-a \leq x \leq a$, we know that $|1 - x| = 1 - x \geq 1 - a$ and that $|x|^{n+1} \leq a^{n+1}$. Thus, we have, for all $n \geq N$, and for all $x \in [-a, a]$,
$$\left| g_n(x) - \frac{1}{1-x} \right| = \left| \frac{1 - x^{n+1}}{1-x} - \frac{1}{1-x} \right| \tag{13.30}$$
$$= \frac{|x|^{n+1}}{|1-x|} \tag{13.31}$$
$$\leq \frac{a^{n+1}}{1-a} < \epsilon. \tag{13.32}$$
which is what we needed to prove. ∎

Note that it is possible, with a calculator, to actually compute appropriate values of N for specific numerical values of a and ϵ. For example, let us find N so that $n > N$ implies that $\left| g_n(x) - \dfrac{1}{1-x} \right| < \dfrac{1}{100}$ for all $x \in \left[-\frac{99}{100}, \frac{99}{100}\right]$. Because $x \leq \frac{99}{100}$, it follows that $1 - x \geq \frac{1}{100}$. Thus, regardless of n and $x \in \left[-\frac{99}{100}, \frac{99}{100}\right]$, we always have
$$\frac{|x|^{n+1}}{|1-x|} \leq \left(\frac{99}{100}\right)^{n+1} \cdot 100. \tag{13.33}$$

We can force the left-hand side of this inequality to be less than $\frac{1}{100}$ by requiring that

$$\left(\frac{99}{100}\right)^{n+1} \cdot 100 < \frac{1}{100}, \tag{13.34}$$

or, equivalently, by requiring that

$$(n+1)(\log 99 - \log 100) < -\log 10000. \tag{13.35}$$

Taking into account the fact that $\log 99 - \log 100 < 0$, we find that we must make sure that

$$n > \frac{\log 100 + \log 99}{\log 100 - \log 99} \approx 915.4. \tag{13.36}$$

Thus, we should take $N = 915$ (or larger). Doing calculations like this one can be useful first steps for students to take on the long path to understanding what the fuss is all about.

The importance of Theorem 13.6 is that the polynomials $g_n(x)$ can be made to approximate the limit function $\dfrac{1}{1-x}$ **uniformly** well over arbitrary closed subintervals of the interval where they converge to that function. This kind of convergence has much to do with our reasons for studying power series in the first place.

Example 13.1 [Non-preservation of Integrals] For each $n \in \mathbb{N}$, let $f_n(x)$ be defined on $[0,1]$ by

$$f_n(x) = \begin{cases} n^2 x, & \text{if } 0 \le x \le \frac{1}{n} \\ 2n - n^2 x, & \text{if } \frac{1}{n} < x \le \frac{2}{n} \\ 0, & \text{otherwise.} \end{cases} \tag{13.37}$$

The graph of f_n consists of the union of (a) the line segment connecting the point $(0,0)$ to the point $\left(\dfrac{1}{n}, n\right)$, (b) the line segment connecting the point $\left(\dfrac{1}{n}, n\right)$ to the point $\left(\dfrac{2}{n}, 0\right)$, and (c) the line segment connecting the point $\left(\dfrac{2}{n}, 0\right)$ to the point $(1,0)$.

Thus, each f_n is continuous on $[0,1]$ and for each $n \in \mathbb{N}$ the integral of f_n over $[0,1]$ is the area of the isosceles triangle whose base is the interval $\left(0, \dfrac{2}{n}\right)$ and whose vertex lies at $\left(\dfrac{1}{n}, n\right)$. This area is 1 in every case, and so $\displaystyle\int_0^1 f_n(t)\,dt = 1$ for all n. Consequently,

$$\lim_{n\to\infty} \int_0^1 f_n(t)\,dt = 1.$$

But now observe that $\lim_{n\to\infty} f_n(0) = 0$ because $f_n(0) = 0$ for every $n \in \mathbb{N}$. On the other hand, if $0 < a \le 1$, we can choose $N \in \mathbb{N}$ so large that whenever $n > N$ we must have $a > \dfrac{2}{n}$. This means that $f_n(a) = 0$ for all sufficiently large n, and we conclude that $\lim_{n\to\infty} f_n(a) = 0$. Thus, for each $x \in [0,1]$ the sequence $\{f_n(x)\}$ converges to zero. But then $\displaystyle\int_0^1 \left(\lim_{n\to\infty} f_n(t)\right) dt = \int_0^1 0\, dt = 0$.

This example thus shows that, in general, we may **not** interchange computation of definite integrals with passage to a limit.

That is, we must ordinarily expect that

$$\lim_{n\to\infty} \int_a^b f_n(t)\, dt \neq \int_a^b \left(\lim_{n\to\infty} f_n(t)\right) dt.$$

Here is a first, and extremely important, consequence of Theorem 13.6: The way in which the functions g_n converge to the function $x \mapsto \dfrac{1}{1-x}$ in the interval $(-1,1)$ allows the interchange of limit on the one hand with integration on the other hand for this particular sequence of polynomial functions.

Theorem 13.7 (Integration of the Geometric Series) *Let $x \in (-1,1)$. Then*

$$\lim_{n\to\infty} \int_0^x g_n(t)\, dt = \int_0^x \frac{dt}{1-t}. \tag{13.38}$$

Proof: Let $\epsilon > 0$ be given. Assume that $x > 0$. The proof is similar for $x < 0$. Because $0 < x < 1$, we can find $a > 0$ so that $x < a < 1$. By Theorem 13.6, there is a positive integer N such that

$$\left| g_n(t) - \frac{1}{1-t} \right| < \frac{\epsilon}{a} \tag{13.39}$$

whenever $n > N$ and $-a < t < a$. For such n, we then have

$$\left| \int_0^x g_n(t)\, dt - \int_0^x \frac{dt}{1-t} \right| = \left| \int_0^x \left(g_n(t) - \frac{1}{1-t} \right) dt \right| \tag{13.40}$$

$$\leq \int_0^x \left| g_n(t) - \frac{1}{1-t} \right| dt \tag{13.41}$$

$$\leq \frac{\epsilon}{a} \int_0^x dt = \frac{\epsilon}{a} x < \epsilon, \tag{13.42}$$

and the conclusion of the theorem follows. ∎

13.3 Convergence of Sequences of Functions

Definition 13.4 *Let I be an interval in $\mathbb{R}$, and for each $n \in \mathbb{N}$, suppose that we are given a function f_n defined on I. If f is a function defined on I and if for each $x \in I$ it is the case that*

$$\lim_{n\to\infty} f_n(x) = f(x),$$

we will say that the sequence $\{f_n\}_{n=0}^{\infty}$ converges to the function f on I in the ***pointwise*** *sense, or, more simply, that f_n* converges pointwise *to f on I.*

Pointwise convergence is the most natural way to define convergence of a sequence of functions to another function. There is a price to pay, however. There are unpleasant deficiencies in this naïve approach to convergence. We saw in the example in the previous section that pointwise convergence does not preserve integrals.

The following example displays another problem with pointwise convergence.

Example 13.2 Let I be the closed unit interval $I = [0, 1]$, and for each $n \in \mathbb{N}$, define a function f_n on I by

$$f_n(x) = \begin{cases} 0, & \text{if } 0 \le x \le \frac{1}{2} \\ (n+1)\left(x - \frac{1}{2}\right), & \text{if } \frac{1}{2} < x < \frac{1}{2} + \frac{1}{n+1} \\ 1, & \text{if } \frac{1}{2} + \frac{1}{n+1} \le x \le 1 \end{cases} \tag{13.43}$$

For each n, the function f_n has the constant value 0 on the interval $[0, \frac{1}{2}]$ and the constant value 1 on the interval $[\frac{1}{2} + \frac{1}{n+1}, 1]$. The graph of the function f_n on the interval $[\frac{1}{2}, \frac{1}{2} + \frac{1}{n+1}]$ is the line segment from the point $(\frac{1}{2}, 0)$ and $(\frac{1}{2} + \frac{1}{n+1}, 1)$. Thus, each of the functions f_n is continuous on the interval $[0, 1]$.

Now if $0 \le x \le \frac{1}{2}$, then

$$\lim_{n\to\infty} f_n(x) = 0.$$

If $x > \frac{1}{2}$, then because

$$\lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{n+1}\right) = \frac{1}{2}$$

we can always choose $N \in \mathbb{N}$ large enough that $\frac{1}{2} + \frac{1}{N+1} < x$, and if $n \ge N$ we must then have $f_n(x) = 1$. Hence

$$\lim_{[} n \to \infty f_n(x) = 1$$

for every $x \in \left(\frac{1}{2}, 1\right]$. The sequence $\{f_n\}$ therefore converges pointwise on I to the function f whose values are given by

$$f(x) = \begin{cases} 0, & \text{if } 0 \le x \le \frac{1}{2} \\ 1, & \text{if } \frac{1}{2} < x \le 1. \end{cases} \tag{13.44}$$

We thus have a sequence of continuous functions on I which converges pointwise on I to a discontinuous function.

Definition 13.5 *Let I be an interval in $\mathbb{R}$, and for each $n \in \mathbb{N}$, suppose that we are given a function f_n defined on I. If f is a function defined on I and if for every $\epsilon > 0$ it is possible to find $N \in \mathbb{N}$ with the property that whenever $x \in I$ and $n > N$ we have $|f_n(x) - f(x)| < \epsilon$, we will say that the sequence $\{f_n\}$* converges uniformly *to the function f on I.*

The difference between pointwise convergence and uniform convergence lies in the order in which certain quantifiers come in the formal expressions of the definitions. Pointwise convergence of $\{f_n\}$ on an interval I to a function f is expressed by the condition

$$(\forall x \in I)(\forall \epsilon > 0)(\exists N \in \mathbb{N})((n > N) \Rightarrow (|f_n(x) - f(x)| < \epsilon)),$$

while uniform convergence is expressed by

$$(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall x \in I)((n > N) \Rightarrow (|f_n(x) - f(x)| < \epsilon)).$$

Uniform convergence is not a usual topic in elementary calculus. We want to study it here because it is the key idea that allows us to do calculus with functions that we represent by power series. We see first that uniform convergence implies pointwise convergence. We will then show that neither of the pathologies of the previous two examples can arise when convergence is uniform.

Theorem 13.8 *Suppose that the sequence $\{f_n\}$ converges uniformly to a function f on an interval I. Then $\{f_n\}$ converges pointwise to f on I.*

Proof: Fix an arbitrary point $x_0 \in I$. We will show that $\lim_{n\to\infty} f_n(x_0) = f(x_0)$.

Let $\epsilon > 0$ be given. The sequence $\{f_n\}$ converges uniformly to f on I, so there is an $N \in \mathbb{N}$ such that for every $x \in I$ we have $|f_n(x) - f(x)| < \epsilon$ provided only that $n > N$. But if the inequality $|f_n(x) - f(x)| < \epsilon$ is true for every $x \in I$, then it is true for $x_\in I$, and we are done. ∎

Theorem 13.9 *Let the functions f_n be continuous on an interval I, and suppose that the sequence $\{f_n\}$ converges uniformly to a function f on I. Then f is continuous on I.*

Proof: Let $x_0 \in I$ be an arbitrary point and let $\epsilon > 0$ be given. By uniform convergence, we can find $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon/3$ whenever $n > N$ and $x \in I$. Thus $|f_{N+1}(x) - f(x)| < \epsilon/3$ for all $x \in I$.

By the continuity of f_{N+1} at x_0, we can find $\delta > 0$ so that $|f_{N+1}(x) - f_{N+1}(x_0)| < \epsilon/3$ provided only that $x \in I$ and $|x - x_0| < \delta$. Thus, if $x \in I$ and $|x - x_0| < \delta$, we have

$$\begin{aligned} |f(x) - f(x_0)| &= |f(x) - f_{N+1}(x) + f_{N+1}(x) - f_{N+1}(x_0) + f_{N+1}(x_0) - f(x_0)| \\ &\le |f(x) - f_{N+1}(x)| + |f_{N+1}(x) - f_{N+1}(x_0)| + |f_{N+1}(x_0) - f(x_0)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

This shows that f is continuous at x_0. Because we chose x_0 to be an arbitrary point in I, we have shown that f is continuous on I. ∎

On the basis of Theorem 13.9, we find that the sequence of functions of our last example cannot be uniformly convergent. Thus, by Theorem 13.8, uniform convergence is stronger than and strictly different from pointwise convergence.

We introduce a third notion of convergence for sequences of functions because it will be convenient in dealing with power series. It lies between the two notions given above: It is stronger than pointwise convergence but weaker than uniform convergence.

Definition 13.6 *Let I be an interval in $\mathbb{R}$, and for each $n \in \mathbb{N}$, suppose that we are given a function f_n defined on I. If f is a function defined on I and if the sequence $\{f_n\}$ converges uniformly to f on every closed subinterval of I having finite length, we will say that the sequence $\{f_n\}$ converges uniformly to f on the compact subintervals of I.*

If the interval I is itself closed and bounded, then uniform convergence on the compact subintervals of I is clearly the same as uniform convergence on I itself. However, if I is open at either end (or at both) or if I does not have finite length, the two kinds of convergence are not the same. In order to see that this is so, consider the sequence of functions defined in Example 13.2, but on the interval $J = \left(\frac{1}{2}, 1\right)$. If $[a, b]$ is a closed subinterval of J, then $\frac{1}{2} < a$, and we have seen that if n is sufficiently large then the function f_n takes on only the value 1 throughout $[a, b]$. Consequently, convergence is uniform on $[a, b]$ (because $|f_n(x) - 1| = 0 < \epsilon$ for any positive ϵ whatsoever and any $x \in [a, b]$ when n is large). But convergence is not uniform on J itself because whenever $0 < \epsilon < 1$ no matter what $N \in \mathbb{N}$ we choose, every function f_n takes on values close to zero for certain values of x near $\frac{1}{2}$. To see this, let $\epsilon \in (0, 1)$ and put $x_n = \dfrac{2 + n - \epsilon}{2 + 2n}$, $n \in \mathbb{N}$. It is easily checked that, for each $n \in \mathbb{N}$, $f_n(x_n) = (1 - \epsilon)/2$. Consequently,

$$|f_n(x_n) - 1| = \left|\frac{1 - \epsilon}{2} - 1\right| \tag{13.45}$$

$$= \frac{1 + \epsilon}{2} > \frac{\epsilon + \epsilon}{2} = \epsilon. \tag{13.46}$$

Theorem 13.10 *Let the functions f_n be continuous on an interval (a, b), and suppose that the sequence $\{f_n\}$ converges uniformly on compact subintervals of (a, b) to a function f on (a, b). (We specifically admit either or both of the possibilities $a = -\infty$ and $b = \infty$ here.) Then f is continuous on (a, b).*

PROOF: Let $x_0 \in (a, b)$ be arbitrary. Choose $\delta > 0$ so small that $a < x_0 - \delta$ and $x_0 + \delta < b$. Then $[x_0 - \delta, x_0 + \delta]$ is a closed interval of finite length, contained entirely in (a, b). By hypothesis, the sequence $\{f_n\}$ converges uniformly to f on $[x_0 - \delta, x_0 + \delta]$. By Theorem 13.9, f is continuous on $[x_0 - \delta, x_0 + \delta]$ and therefore continuous at x_0. ∎

We turn now to the relation between integrals and convergent sequences. As with continuity, we have seen that pointwise convergence is not enough. Our next goal is to show that uniform convergence is enough to rectify this situation.

Theorem 13.11 *Let $\{f_n\}$ be a sequence of functions, each continuous on the interval $[a, b]$, and converging uniformly on $[a, b]$ to a function f. Then*

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx. \tag{13.47}$$

PROOF: Note first that Theorem 13.9 and one of our previous theorems on integrals insure the existence of the integral on the right side.

Let $\epsilon > 0$ be given. All that we need to show is how to find $N \in \mathbb{N}$ so that $\displaystyle\int_a^b f_n(x)dx$ is within ϵ of $\int_a^b f(x)dx$ whenever $n > N$. The sequence $\{f_n\}$ converges uniformly to f on $[a, b]$, so we can find $N \in \mathbb{N}$ so that $n > N$ implies that

$$f_n(x) - f(x)| < \frac{\epsilon}{b-a}$$

for every $x \in [a, b]$. Now when $n > N$ we have

$$\left|\int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx\right| = \left|\int_a^b (f_n(x) - f(x))\,dx\right| \tag{13.48}$$

$$\leq \int_a^b |f_n(x) - f(x)|\,dx \tag{13.49}$$

$$< \int_a^b \left(\frac{\epsilon}{b-a}\right)\,dt = \epsilon, \tag{13.50}$$

and the proof is complete. ∎

Example 13.3 For each $n \in \mathbb{N}$, define f_n on $[0, 1]$ by $f_n(x) = \dfrac{\sin(2\pi nx)}{n}$.

For each $n \in \mathbb{N}$ and each $x \in [0, 1]$, we have

$$\begin{aligned} |fn(x)| &= \left|\frac{\sin(2\pi nx)}{n}\right| \\ &\leq \frac{1}{n}, \end{aligned}$$

so the sequence $\{f_n\}$ converges uniformly on $[0,1]$ to the constant function zero — whose derivative is the function that is constantly zero. Each of the functions f_n is differentiable on $[0,1]$, with

$$f_n'(x) = 2\pi\cos(2\pi n x).$$

Thus $f_n'(1/2)$ is sometimes 2π and sometimes -2π, so that the sequence $\{f_n'\}$ cannot converge to the zero function in any of the senses we have discussed.

This example shows that, in general, we may not interchange computation of derivatives with passage to a limit:

$$\lim_{n\to\infty}\left(\frac{d}{dx}f_n(x)\right) \neq \frac{d}{dx}\left(\lim_{n\to\infty} f_n(x)\right),$$

even when the sequence $\{f_n\}$ converges uniformly. However, uniform convergence is still the key, as we see in the following theorem.

Theorem 13.12 *Let $\{f_n\}$ be a sequence of functions defined and continuously differentiable on an interval (a,b). Suppose that the sequence $\{f_n'\}$ converges, uniformly on the compact subintervals of (a,b), to a function g. If there is a number $x_0 \in (a,b)$ such that $\lim_{n\to\infty} f_n(x_0)$ exists, then the sequence $\{f_n\}$ converges, uniformly on the compact subintervals of (a,b), to a function f such that $f' = g$.*

PROOF: By the Fundamental Theorem of Calculus we may write, for each $n \in \mathbb{N}$ and any $x \in (a,b)$,

$$f_n(x) = f_n(x_0) + \int_{x_0}^{x} f_n'(t)\,dt. \tag{13.51}$$

The interval whose endpoints are x_0 and x is a compact subinterval of (a,b), so, by hypothesis, the sequence $\{f_n'\}$ converges uniformly to g on that interval. Applying Theorem 13.11, we conclude that

$$\lim_{n\to\infty}\int_{x_0}^{x} f_n'(t)\,dt = \int_{x_0}^{x} g(t)\,dt.$$

Also by hypothesis, $\lim_{n\to\infty} f_n(x_0)$ exists. This defines a number $f(x_0)$. Applying 13.51, we find that $\lim_{n\to\infty} f_n(x)$ always exists. Thus, we define

$$\begin{aligned} f(x) &= \lim_{n\to\infty} f_n(x) &(13.52)\\ &= \lim_{n\to\infty}\left(f_n(x_0) + \int_{x_0}^{x} f_n'(t)\,dt\right) &(13.53)\\ &= f(x_0) + \int_{x_0}^{x} g(t)\,dt. &(13.54)\end{aligned}$$

It follows from the Fundamental Theorem of Calculus that $f'(t) = g(t)$ for all x in the open interval determined by x_0 and x. Because every point of (a, b) lies in such a compact subinterval, it follows that $f' = g$ throughout (a, b).

We have shown that $\{f_n\}$ converges pointwise on (a, b) to f. It remains to show that the sequence converges uniformly to f on the compact subintervals of (a, b). Let $a < \alpha < \beta < b$, and consider $|f_n(x) - f(x)|$ when $\alpha \le x \le \beta$. Let $\epsilon > 0$. The sequence $\{f'_n\}$ converges uniformly to f' on $[\alpha, \beta]$, so we can find $N \in \mathbb{N}$ such that

$$|f'_n(x) - f'(x)| \le \frac{\epsilon}{2(\beta - \alpha)}$$

whenever $x \in [\alpha, \beta]$ and $n > N$. We can choose N large enough that if $n > N$ then $|f_n(\alpha) - f(\alpha)| < \epsilon/2$ also. The Fundamental Theorem of Calculus now tells us that for $n > N$ and $x \in [\alpha, \beta]$,

$$|f_n(x) - f(x)| = \left| \left(f_n(\alpha) + \int_\alpha^x f'_n(u)\, du \right) - \left(f(\alpha) + \int_\alpha^x f(u)\, du \right) \right| \tag{13.55}$$

$$= \left| (f_n(\alpha) - f(\alpha)) + \int_\alpha^t (f'_n(u) - f'(u))\, du \right| \tag{13.56}$$

$$7 \le |f_n(\alpha) - f(\alpha)| + \int_\alpha^t |f'_n(u) - f'(u)|\, du \tag{13.57}$$

$$\le |f_n(\alpha) - f(\alpha)| + \int_\alpha^\beta |f'_n(u) - f'(u)|\, du \tag{13.58}$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2(\beta - \alpha)}(\beta - \alpha) = \epsilon. \tag{13.59}$$

This shows that $\{f_n\}$ converges to f uniformly on the compact subsets of (a, b). ∎

Definition 13.7 *For each non-negative integer n, let u_n be a function whose domain includes some interval I. We will say that the series $\sum_{k=1}^{\infty} u_k(x)$ converges uniformly on I (respectively, converges uniformly on the compact subsets of I) if the sequence of functions given on I by $s_n(x) = \sum_{k=1}^{n} u_k(x)$ converges uniformly on I (respectively, converges uniformly on the compact subsets of I.)*

13.4 Power Series

Power series are to functions what decimal expansions are to real numbers. It is a representation of a function that ***may*** give us some insight into the behavior of a function or make it easier to do computations.

Theorem 13.13 *Let $\{a_k\}_{k=0}^{\infty}$ be a sequence of real numbers, and let a be any real number. There is a non-negative number R such that $\displaystyle\sum_{k=0}^{\infty} a_k(x-a)^k$ converges absolutely and uniformly on the compact subsets of the interval $(-R+a, R+a)$ and diverges on the complement of the interval $[-R+a, R+a]$.*

Proof: It suffices to prove the theorem for $a = 0$, the argument for non-zero a being exactly the same. Another way to think of this is to translat to 0 by replacing x with $t + a$.

Observe that the series $\sum_{k=0}^{\infty} a_k x^k$ clearly converges for $x = 0$. If it converges for no other real values of x, we take $R = 0$. If it converges for all positive real x, we take $R = \infty$. Otherwise, there is $x_1 \neq 0$ for which $\sum_{k=0}^{\infty} a_k x_1^k$ converges, and there is $x_2 \neq 0$ for which $\sum_{k=0}^{\infty} a_k x_2^k$ diverges. Note that $|x_2| < |x_1|$ is impossible. Indeed, we will show that if $-|x_1| < x < |x_1|$, then $\sum_{k=0}^{\infty} |a_k x^k|$ (and so also $\sum_{k=0}^{\infty} a_k x^k$ — see Theorem 13.4) converges. To see that this is so, first recall that the convergence of $\sum_{k=0}^{\infty} a_k x_1^k$ guarantees that $\lim_{n\to\infty} a_n x_1^n = 0$. This means there is a constant K with the property that $|a_n x_1^n| \leq K$ for all n. Consequently, whatever the nonnegative integer n may be, and whatever the real number x may be, we have, from $|x| < |x_1|$,

$$|a_n x^n| = |a_n x_1^n| \cdot \left|\frac{x}{x_1}\right|^n \tag{13.60}$$

$$\leq K \left|\frac{x}{x_1}\right|^n. \tag{13.61}$$

So if $|x| < |x_1|$, the series $\sum_{k=0}^{\infty} a_k x^k$ is dominated by the convergent geometric series $\sum_{k=0}^{\infty} K \left|\dfrac{x}{x_1}\right|^k$, and must, by the Comparison Test, converge absolutely. In particular, we must conclude that $|x_1| < |x_2|$. It follows from what we have just seen that the set S of all positive real numbers s for which $\sum_{k=0}^{\infty} a_k s^k$ converges is a non-empty set of real numbers which is bounded above (because the series does not converge for all positive real s). By the Least Upper Bound axiom, S must have a least upper bound. Let R be this least upper bound. Then $\sum_{k=0}^{\infty} a_k x^k$ converges for all $x \in (-R, R)$ and diverges for all $x \notin [-R, R]$. We put $f(x) = \displaystyle\sum_{k=0}^{\infty} a_k x^k$ for all $x \in (-R, R)$. It now remains to show that convergence is uniform on the compact subsets of $(-R, R)$.

We may assume that $R > 0$ — the assertion being vacuously true when $R = 0$. We will show that if $0 < T < R$, then $\displaystyle\sum_{k=0}^{\infty} a_k x^k$ converges uniformly on $[-T, T]$.

So, let $\epsilon > 0$ be given, and let $x \in [-T, T]$. Then $|x| \leq |T|$, and if n is a positive integer, we have

$$\left| f(x) - \sum +k = 0^n a_k x^k \right| = \left| \sum_{k=n+1}^{\infty} a_k x^k \right| \tag{13.62}$$

$$\leq \sum_{k=n+1}^{\infty} \left| a_k x^k \right| \tag{13.63}$$

$$\leq \sum_{k=n+1}^{\infty} |a_k| T^k. \tag{13.64}$$

But $\sum_{k=0}^{\infty} |a_k| T^k$ converges to some number L (because $0 < T < R$), so we can find a positive integer N with the property that

$$\begin{aligned} \sum_{k=n+1}^{\infty} |a_k| T^k &= \left| \sum_{k=n+1}^{\infty} |a_k| T^k \right| \\ &= \left| L - \sum_{k=0}^{n} |a_k| T^k \right| \\ &< \epsilon \end{aligned}$$

whenever $n > N$, and from this together with the above equation it follows that $\displaystyle\sum_{k=0}^{\infty} a_k x^k$ converges uniformly to $f(x)$ on $[-T, T]$. ∎

It now follows from the results of the previous section that term-by-term differentiation and integration of functions given by power series is legitimate — in regions where the differentiated series and the integrated series converge uniformly on compact sets. That this works as it does is the remarkable fact that underlies the usefulness of power series representations. This means then that our new task is to determine how it works. To accomplish this, given the series $\sum_{k=0}^{\infty} a_k x^k$, we must, first identify the number R, of Theorem 13.13, and study its properties.

Theorem 13.14 *[Extended Root Test] Let $\sum_{k=0}^{\infty} a_k$ be a series, and let*

$$L = \limsup \sqrt[k]{|a_k|}.$$

If $L < 1$, the series converges. If $L > 1$, the series diverges. If $L = 1$ the series may converge or diverge.

Proof: If $L < 1$, we can choose $\epsilon > 0$ so that $L + \epsilon < 1$. There are, then, only finitely many non-negative integers k for which $L + \epsilon < \sqrt[k]{a_k}$. Thus, there is a non-negative integer N such that $|a_k| < (L + \epsilon)^k$ for all $k \geq N$. But $\sum_{k=0}^{\infty} (L + \epsilon)^k$ is a convergent geometric series, because $L + \epsilon < 1$. By the Comparison Test, $\sum_{k=0}^{\infty} |a_k|$ must converge. This guarantees that $\sum_{k=0}^{\infty} a_k$ itself converges.

If $L > 1$, we choose $\epsilon > 0$ so that $L - \epsilon > 1$. There are then infinitely many non-negative integers k for which $L - \epsilon < \sqrt[k]{|a_k|}$, so we can choose a sequence of non-negative integers n_j, $j = 0, 1, \ldots$, such that $n_{j+1} > n_j$ for every j, and $1 < (L - \epsilon)n_j < |a_{n_j}|$ for every j. It follows that $\lim_{k\to\infty} a_k = 0$ is impossible, whence we conclude that $\sum_{k=0}^{\infty} a_k$ diverges.

To see that we can say nothing about the convergence of $\sum_{k=0}^{\infty} a_k$ when $L = 1$, we note that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if $p > 1$ and diverges if $p \le 1$. (This is a standard example which appears in every freshman calculus text.) However,

$$
\begin{aligned}
\lim_{k\to\infty} \sqrt[k]{\frac{1}{k^p}} &= \lim_{k\to\infty} \sqrt[k]{k^{-p}} \\
&= \lim_{k\to\infty} k^{-p/k} \\
&= \lim_{k\to\infty} \exp\left[-\frac{p \log k}{k}\right] \\
&= \exp\left[-p \lim_{k\to\infty} \frac{\log k}{k}\right] \\
&= \exp\left[-p \lim_{k\to\infty} \frac{(1/k)}{1}\right], \text{ by l'Hôpital's Rule,} \\
&= 1.
\end{aligned}
$$

∎

Now we can return to the number R of Theorem 13.13 and it properties.

Theorem 13.15 *[Radius of Convergence] Let $\sum_{k=0}^{\infty} a_k(x-a)^k$ be a power series, and put*

$$L = \limsup \sqrt[k]{|a_k|}.$$

If $L = 0$, put $R = 1$. If $L = 1$, put $R = 0$. Otherwise, put $R = 1/L$. So defined, R is the number whose existence and properties are assured by Theorem 13.13.

PROOF: We need only show that the series converges when $|x - a| < R$ and diverges when $|x - a| > R$. We apply the Extended Root Test:

$$
\begin{aligned}
\limsup \sqrt[k]{|a_k(x-a)^k|} &= \limsup \left(|x-a| \sqrt[k]{|a_k|}\right) \\
&= |x - a|L.
\end{aligned}
$$

and this is less than one precisely when $|x - a| < R$, greater than one precisely when $|x - a| > R$. ∎

Definition 13.8 *The number R of Theorem 13.15 is called the* radius of convergence *of the series $\sum_{k=0}^{\infty} a_k(x-a)^k$.*

We are now ready to describe the behavior of power series that makes them worthy objects of study.

Theorem 13.16 *[Fundamental Theorem of Power Series] Let f be a function given by a power series*

$$f(x) = \sum_{k=0}^{\infty} a_k(x-a)^k$$

on the interval $(a-R, a+R)$, where the positive number R is the radius of convergence for the series

$$\sum_{k=0}^{\infty} a_k(x-a)^k.$$

Then the series whose k-th term is $ka_k(x-a)^{k-1}$ also has radius of convergence R. Moreover f is differentiable (and therefore continuous) on $(a-R, a+R)$ with

$$f'(x) = \sum_{k=1}^{\infty} ka_k(x-a)^{k-1} \tag{13.65}$$

whenever $a-R < x < a+R$.

On that same interval let F be the function defined by

$$F(x) = \int_a^x f(t)\,dt.$$

Then F is given by

$$F(x) = \sum_{k=0}^{\infty} \frac{a_k}{k+1}(x-a)^{k+1}, \tag{13.66}$$

and the latter series also has radius of convergence R, so that the above representation is also valid on $(a-R, a+R)$.

Proof: Observe first that $\lim_{k\to\infty} \sqrt[k]{k} = 1$. Consequently, applying Theorem 13.15, we obtain R as the radius of convergence for the series of Equation 13.65 from the computation

$$\begin{aligned}\limsup \sqrt[k]{k|a_k|} &= \left(\lim_{k\to\infty} \sqrt[k]{k}\right)\left(\limsup \sqrt[k]{|a_k|}\right)\\ &= \limsup \sqrt[k]{|a_k|}\\ &= \frac{1}{R}.\end{aligned}$$

The series of Equation 13.65 therefore converges uniformly on the compact subsets of $(a-R, a+R)$. Because each of the sums $\sum_{k=1}^{n} k a_k (x-a)^{k-1}$ is the derivative of the corresponding sum $\sum_{k=1}^{n} a_k (x-a)^k$, an application of Theorem 13.12 assures us that Equation 13.65 is true for all $x \in (a-R, a+R)$.

The statements regarding the function F follow similarly from Theorem 13.11. ∎

Corollary 13.2 *Let f be a function given by a power series*

$$f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k$$

on the interval $(a-R, a+R)$, where the positive number R is the radius of convergence for the series. Then f possesses derivatives of all orders throughout the interval $(a-R, a+R)$.

Proof: f' is a function given by a power series on $(a-R, a+R)$, so, by Theorem 13.16, f'' must be as well. Continue inductively. ∎

Corollary 13.3 *[Taylor Coefficients] Let f be a function given by a power series*

$$f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k$$

on the interval $(a-R, a+R)$, where the positive number R is the radius of convergence for the series. Then for every integer $j = 0, 1, 2, \ldots$,

$$a_j = \frac{f^{(j)}(a)}{j!}.$$

Proof: Differentiate the series term-by-term j times and then set $x = a$. ∎

Corollary 13.4 *If*

$$f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k$$

and

$$f(x) = \sum_{k=0}^{\infty} b_k (x-a)^k,$$

each of the equalities holding on some open interval centered at $x = a$, then $a_k = b_k$ for every $k = 0, 1, 2, \ldots$.

It is clear from Corollary 13.2 that some functions — for example, those which do not possess derivatives of all orders — cannot be represented by power series. However, is it true that possession of derivatives of all orders is a necessary condition and sufficient for the existence of such a representation? The next example clarifies this.

Example 13.4 Let f be the function given by

$$f(x) = \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0 \\ 0, \text{if } x = 0 \end{cases} \tag{13.67}$$

Then f cannot be represented by a power series of the form $\displaystyle\sum_{k=0}^{\infty} a_k x^k$ on any non-trivial interval centered at $x = 0$.

To see this, note first that $\lim_{x\to 0} f(x) = 0$, so that f is continuous at $x = 0$. It is also clear that $f(x) > 0$ when $x \neq 0$.

Let's compute $f'(x)$. Clearly,

$$f'(x) = \frac{2e^{-1/x^2}}{x^3} \tag{13.68}$$

when $x \neq 0$. We need to compute $f'(0)$. In order to do this, it suffices to compute $\lim_{x\to 0^+} f'(x)$, for it is clear by symmetry that $\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^-} f'(x)$. Substituting $t = 1/x$ and applying l'Hôpital's Rule twice, we find

$$\begin{aligned} \lim_{x\to 0^+} \frac{2e^{-1/x^2}}{x^3} &= \lim_{t\to\infty} \frac{2t^3}{e^{t^2}} \\ &= \lim_{t\to\infty} \frac{6t^2}{2te^{t^2}} \\ &= \lim_{t\to\infty} \frac{3t}{e^{t^2}} \\ &= \lim_{t\to\infty} \frac{3}{2te^{t^2}} = 0. \end{aligned}$$

Therefore $f'(0) = 0$.

Now let us show inductively that for $x \neq 0$ we always have

$$f^{(k)}(x) = \frac{P(x)}{x^{n_k}} e^{-1/x^2},$$

where n_k is a certain positive integer, and $P(x)$ is a polynomial, which also depends on k, and whose degree is less than n_k. We have already seen that $f'(x)$ has this form when $x \neq 0$, so the statement is true for $k = 1$. Let us assume that this holds for a certain integer k and examine the form of $f^{(k+1)}(x)$. We have

$$\begin{aligned} f^{(k+1)}(x) &= \left(\frac{P'(x)x^{n_k} - P(x) \cdot n_k x^{n_k - 1}}{x^{2n_k}} + \frac{2P(x)}{x^{n_k} + 3} \right) e^{-1/x^2} \\ &= \left(\frac{P'(x)x^3 + (2 - n_k x^2)P(x)}{x^{n_k+3}} \right) e^{-1/x^2}, \end{aligned}$$

and we note that the numerator of the fraction in the latter expression is a polynomial whose degree is at most two more than the degree of $P(x)$, while the degree of the denominator is $n_k + 3$. The degree in the numerator having grown by at most two, while the degree in the denominator has grown by three, we see that $f^{(k+1)}(x)$ is also of the necessary form. This completes the induction.

Now if $P(x)$ is a polynomial of degree $m < n$, then

$$\begin{aligned} \lim_{x \to 0^+} \frac{P(x)}{x^n} e^{-1/x^2} &= \lim_{x \to 0^+} \left(P(x) \frac{e^{-1/x^2}}{x^n} \right) \\ &= \left(\lim_{x \to 0^+} P(x) \right) \cdot \left(\lim_{x \to 0^+} \frac{e^{-1/x^2}}{x^n} \right). \end{aligned}$$

Now the first of the two limits on the right side exists because $P(x)$ is a polynomial, and the second limit is zero by a l'Hôpital's Rule argument very similar to the one above. It therefore follows that

$$\lim_{x \to 0^+} \frac{P(x)}{x^n} e^{-1/x^2} = 0.$$

Similarly, the limit from the left is also zero. Thus we may conclude that $f^{(k)}(0) = 0$ for every k, and if it were the case that

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

on some non-trivial interval centered at $x = 0$, then we would have to have $a_k = f^{(k)}(0)/k! = 0$ for every $k = 0, 1, 2, \ldots$. However, if $a_k = 0$ for every k, then the series converges to the zero function, which is different from f.

13.5 The Algebra of Power Series

We have then that power series define functions and we have seen how to differentiate and integrate them. How do we add, subtract, multiply, and divide them?

Theorem 13.17 *Suppose that*

$$f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k$$

and

$$g(x) = \sum_{k=0}^{\infty} b_k (x-a)^k,$$

the first series having positive radius of convergence R_f and the second having positive radius of convergence R_g. Then

$$f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k)(x-c)^k,$$

and the radius of convergence of the latter series is

$$R \geq \min\{R_f, R_g\}.$$

PROOF: Note that, for each $n = 0, 1, \ldots$, we have

$$\sum_{k=0}^{n} (a_k + b_k)(x-a)^k = \sum_{k=0}^{n} a_k (x-c)^k + \sum_{k=0}^{n} b_k (x-a)^k.$$

If $|x-a| < R < \min\{R_f, R_g\}$, then both of the limits, as $n \to \infty$, on the right exist and so must the limit on the left. ∎

The statement of Theorem 13.17 cannot be improved. The series

$$\sum_{k=0}^{\infty} (-1)^k x^{2k}$$

and the series

$$\sum_{k=0}^{\infty} (-1)^k x^{2k+2}$$

both have radius of convergence 1, but their sum is the series whose first term is 1 and all of whose subsequent terms are zero. Thus it is possible that the radius of convergence for the sum substantially exceeds the smaller of the two given radii.

The obvious statement for the difference of two series is also true.

Theorem 13.18 *Suppose that*

$$f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k$$

and

$$g(x) = \sum_{k=0}^{\infty} b_k (x-a)^k,$$

the first series having positive radius of convergence R_f and the second having positive radius of convergence R_g. Then

$$f(x)g(x) = \sum_{k=0}^{\infty} c_k (x-a)^k,$$

where the numbers c_k are given by

$$c_k = \sum_{j=0}^{k} a_j b_{k-j},$$

and the radius of convergence of the latter series is

$$R \geq \min\{R_f, R_g\}.$$

Proof: It suffices to prove the theorem for $a = 0$, the proof for the more general case differing only in complexity of notation. Let $R = \min\{R_f, R_g\}$, and choose $x \in (-R, R)$. By Theorem 13.13, we may put

$$A(x) = \sum_{k=0}^{\infty} |a_k x^k|.$$

Now, let's examine the partial sums of the series on the right side of the product:

$$\begin{aligned}
\sum_{k=0}^{n} c_k x^k &= \sum_{k=0}^{n} \left(\left(\sum_{j=0}^{k} a_j b_{k-j} \right) \right) \\
&= \sum_{k=0}^{n} \sum_{j=0}^{k} (a_j x^j)(b_{k-j} x^{k-j}) \\
&= \sum_{j=0}^{n} \sum_{k=j}^{n} (a_j x^j)(b_{k-j} x^{k-j}) \\
&= \sum_{j=0}^{n} \left(a_j x^j \sum_{k=j}^{n} b_{k-j} x^{k-j} \right) \\
&= \sum_{j=0}^{n} \left(a_j x^j \sum_{k=0}^{n-j} b_k x^k \right) \\
&= \sum_{j=0}^{n} \left(a_j x^j \left(g(x) - \sum_{k=n-j+1}^{\infty} b_k x^k \right) \right) \\
&= g(x) \left(\sum_{j=0}^{n} a_j x^j \right) - \sum_{j=0}^{n} \left(a_j x^j \sum_{k=n-j+1}^{\infty} b_k x^k \right).
\end{aligned}$$

Now $g(x) \sum_{k=0}^{n} a_k x^k \to g(x) f(x)$ as $x \to \infty$, so we would like to show that the second of the two terms on the right side of the last equation above goes to zero as n goes to infinity.

Let $\epsilon > 0$ be given. We know that

$$\lim_{m \to \infty} \sum_{k=m}^{\infty} b_k x^k = 0,$$

because $\sum_{k=0}^{\infty} b_k x^k$ converges. Consequently, we can find $N \in \mathbb{N}$ sufficiently large that

$$\left| \sum_{k=m}^{\infty} b_k x^k \right| < \frac{\epsilon}{2(A(x) + 1)}$$

whenever $m > N$. Then, for $n > N$,

$$\left| \sum_{j=0}^{n} \left(a_j x^j \sum_{k=n-j+1}^{\infty} b_k x^k \right) \right|$$

$$\begin{aligned}
&\leq \left|\sum_{j=0}^{n-N-1}\left(a_jx^j\sum_{k=n-j+1}^{\infty}b_kx^k\right)\right| + \left|\sum_{j=n-N}^{n}\left(a_jx^j\sum_{k=n-j+1}^{\infty}b_kx^k\right)\right| \\
&< \frac{A(x)\epsilon}{2(A(x)+1)} + \left|\sum_{j=n-N}^{n}\left(a_jx^j\sum_{k=n-j+1}^{\infty}b_kx^k\right)\right| \\
&< \frac{\epsilon}{2} + \left|\sum_{j=n-N}^{n}\left(a_jx^j\sum_{k=n-j+1}^{\infty}b_kx^k\right)\right|. \qquad (13.69)
\end{aligned}$$

Note that the final expression in absolute values on the right side of the last equation involves just finitely many of the sums $\sum_{k=p}^{\infty} b_kx^k$, namely those for which $p = 1, 2, \ldots, N+1$, and that the sums involved are independent of what n may be. Consequently, there is a positive real number M such that

$$\left|\sum_{k=p}^{\infty} b_kx^k\right| \leq M$$

for each $p = 1, 2, \ldots, N+1$.

We can also find a positive integer P which has the property that

$$\begin{aligned}
\sum_{k=q+1}^{\infty}|a_kx^k| &= A(x) - \sum_{k=0}^{q}|a_kx^k| \\
&< \frac{\epsilon}{4M(N+1)}
\end{aligned}$$

whenever $q > P$. Consequently, as soon as $n > P + N$, we must have

$$\begin{aligned}
\sum_{k=n-N}^{n}|a_kx^k| &= \sum_{k=n-N+1}^{\infty}|a_kx^k| - \sum_{k=n+1}^{\infty}|a_kx^k| \\
&< \frac{\epsilon}{2M(N+1)}.
\end{aligned}$$

Combining the above inequalities, we find that we have

$$\begin{aligned}
\left|\sum_{j=n-N}^{n}\left(a_jx^j\sum_{k=n-j+1}^{\infty}b_kx^k\right)\right| &\leq \sum_{j=n-N}^{n}\left(|a_jx^j|\left|\sum_{k=n-j+1}^{\infty}b_kx^k\right|\right) \\
&< \sum_{j=n-N}^{n}\left(\frac{\epsilon}{2M(N+1)}\right)M = \frac{\epsilon}{2}.
\end{aligned}$$

We combine this last inequality with the inequality 13.69, and find that we have shown that when n is sufficiently large,

$$\left|\sum_{j=0}^{n}\left(a_jx^j\sum_{k=n-j+1}^{\infty}b_kx^k\right)\right| < \epsilon,$$

as desired. ∎

13.6 Taylor Polynomials

Our example of a power series that only converges when $x = 0$ raises the issue of just when the series
$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$
converges to $f(x)$ for all those values of x that lie in some non-trivial interval centered at $x = a$. In many cases, we can answer this question indirectly through the results of our previous sections. However, in other cases it is necessary to proceed more directly. The principal tool for doing so is the *Taylor Polynomial with Remainder.* The remainder can be represented in a variety of ways, and the strength of the hypotheses needed to support the desired conclusion can also vary.

We give look at several versions of the main result.

Theorem 13.19 *[Taylor Polynomial with General Remainder] Let $h > 0$ and let n be a non-negative integer. Suppose that f is a function defined and possessing derivatives of order up to and including $n+1$ throughout an interval $(a-h, a+h)$. Then for every $x \in (a-h, a+h)$ and every integer $m = 1, 2, \ldots, n+1$, there is a number ξ_m, lying in the interval whose end-points are a and x, such that*
$$f(x) = sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(\xi_m)}{m(n!)}(x-\xi_m)^{n+1-m}(x-a)^m.$$

PROOF: The conclusion is trivially true if $x = a$, so we may assume that $x \neq a$. Put
$$C = \frac{m!}{(x-a)^m}\left(f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k\right),$$
and define a function F by
$$F(t) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k - C\frac{(x-t)^m}{m!}.$$
The function F is a sum of functions that are continuous on the closed interval whose endpoints are a and x and that are differentiable on the interior of the same interval. In fact,
$$\begin{aligned} F'(t) &= -\sum_{k=0}^{n}\left(\frac{f^{(k+1)}(t)}{k!}(x-t)^k - \frac{kf^{(k)}(t)}{k!}(x-t)^{k-1}\right) + C\frac{(x-t)^{m-1}}{(m-1)!} \\ &= -\frac{f^{(n+1)}(t)}{n!}(x-t)^n + C\frac{(x-t)^{m-1}}{(m-1)!}, \end{aligned}$$
since the sum is a telescoping one.

Moreover, $F(a) = 0$ by the way we have chosen C, and $F(x) = 0$ by direct calculation. By Rolle's Theorem there is a number $\xi_m \in (a, x)$ such that $F'(\xi_m) = 0$. In particular, $\xi_m \neq x$, and substituting ξ_m for t in the last equation above yields

$$C = \frac{f^{(n+1)}(\xi_m)}{n!}(m-1)!(x-\xi_m)^{n+1-m}.$$

Substitution of the right side of this for C in gives the result, after some slight rearrangement. ∎

Corollary 13.5 *[Taylor Series with Lagrange Remainder] Let $h > 0$ and suppose that f is a function defined and possessing derivatives of order up to and including $n+1$ throughout the interval $(a-h, a+h)$. Then for every $x \in (a-h, a+h)$ there is a number ξ, lying in the interval whose end-points are a and x, such that*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}.$$

PROOF: Set $m = n + 1$ in the above theorem. ∎

Corollary 13.6 (Taylor Series with Cauchy Remainder) *Let $h > 0$ and suppose that f is a function defined and possessing derivatives of order up to and including $n+1$ throughout the interval $(a-h, a+h)$. Then for every $x \in (a-h, a+h)$ there is a number ξ, lying in the interval whose end-points are a and x, such that*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n(x-a).$$

PROOF: Set $m = 1$ in the above theorem. ∎

In many respects, the most useful form of the Taylor remainder is the integral form. The usefulness comes at a cost, but not a prohibitive one. (In practical terms, the cost is negligible.) We must assume that the derivative of order $n+1$ is continuous. We needed only the existence of the derivative of order $n+1$ in order to derive the forms above. The reader should note that we can reach the forms given above by way of the integral form of the remainder. However, we need the stronger hypothesis that $f^{(n+1)}$ is continuous in order to take this route to the other forms of the remainder.

Corollary 13.7 (Weak Taylor Series with General Remainder) *Suppose that f is a function defined and possessing continuous derivatives of order up to and including $n+1$ throughout an interval $(a-h, a+h)$, for some positive real number h. Then for every $x \in (a-h, a+h)$ and every integer $m = 1, 2, \ldots, n+1$, there is a number ξ_m, lying in the interval whose end-points are a and x, such that*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(\xi_m)}{m(n!)}(x-\xi_m)^{n+1} - m(x-a)^m.$$

Proof: Let us assume, for the moment, that $a < x$. We have

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n\,dt.$$

We take $F(t) = f^{(n+1)}(t)(x-t)^{n+1-m}$ and $\varphi(t) = (x-t)^{m-1}$. By hypothesis, F and φ are continuous on the interval $[a,x]$ and φ does not change sign in that interval. By the Mean Value Theorem for Integrals we may write, for a certain $\xi_m \in [a,x]$,

$$\begin{aligned}
\int_a^x f^{(n+1)}(t)(x-t)^n\,dt &= \int_a^x F(t)\varphi(t)\,dt \\
&= F(\xi_m)\int_a^x \varphi(t)\,dt \\
&= F(\xi_m)\int_a^x (x-t)^{m-1}\,dt \\
&= -\frac{F(\xi_m)}{m}(x-t)m\Big|_a^x \\
&= \frac{f^{(n+1)}(\xi_m}{m}(x-\xi_m)^{n+1-m}(x-a)^m.
\end{aligned}$$

The result follows.

When $x < a$, we have, by similar reasoning,

$$\begin{aligned}
\int_a^x f^{(n+1)}(t)(x-t)^n\,dt &= -\int_a^x F(t)\varphi(t)\,dt \\
&= -F(\xi_m)\int_x^a \varphi(t)\,dt \\
&= -F(\xi_m)\int_x^a (x-t)^{m-1}\,dt \\
&= \frac{F(\xi_m)}{m}(x-t)m\Big|_x^a \\
&= \frac{f^{(n+1)}(\xi_m}{m}(x-\xi_m)^{n+1-m}(x-a)^m.
\end{aligned}$$

∎

Theorem 13.20 (Taylor Series Representation Criterion) *Let f be a function defined and possessing derivatives of all orders in some open interval centered at $x = a$. If t is a point of this open interval, then*

$$f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(t-a)^k$$

if, and only if,

$$\lim_{n\to\infty} R_n(t) = 0,$$

where $R_n(t)$ denotes any of the remainders of the previous theorem or corollaries.

PROOF: The proof is immediate from the observations that

1. $f(t) = \displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(t)}{k!}(t-a)^k$ if and only if

$$\lim_{n\to\infty} \left(f(t) - \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(t-a)^k \right) = 0,$$

   and

2. $\displaystyle\left(f(t) - \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(t-a)^k \right) = R_n(t).$

∎

13.7 Power Series Representations

We begin with a useful lemma.

Theorem 13.21 *For any real number x,*

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0.$$

PROOF: For any real number x, we have

$$\lim_{n\to\infty} \left(\frac{\left(\dfrac{|x|^{n+1}}{(n+1)!}\right)}{\left(\dfrac{|x|^n}{n!}\right)} \right) = \lim_{n\to\infty} \frac{|x|}{n+1} = 0.$$

By the Ratio Test, the series $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges for each fixed value of x, and this implies our result. ∎

Definition 13.9 *If $R \in \mathbb{R}$ and n is a positive integer, we define*

$$\binom{r}{n} = \frac{1}{n!} \prod_{k=1}^{n} (r-k+1).$$

We also put

$$\binom{r}{0} = 1.$$

Theorem 13.22 *[The Binomial Series] For any real number r, we have*

$$(1+x)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^k$$

for all $x \in (-1,1)$.

PROOF: The proof here comes from Stewart's *Calculus: Concepts and Contexts*, Second Edition.

Put $f(x) = (1+x)^r$. For any non-negative integer k we see inductively that

$$f^{(k)}(x) = \left(\prod_{j=1}^{k} (r-j+1) \right) (1+x)^{r-k},$$

so that the k-th Maclaurin coefficient for $(1+x)^r$ is given by

$$\frac{f^{(k)}(0)}{k!} = \binom{m}{k}.$$

Thus if f is represented by a Maclaurin series, it must be the one that we have above. It remains only to show that this series does converge to $f(x)$ in the desired interval.

Let us apply the Ratio Test to this series. We obtain

$$\lim_{n\to\infty} \left[\frac{\left|\binom{r}{n+1} x^{n+1}\right|}{\left|\binom{r}{n} x^n\right|} \right] = |x| \lim_{n\to\infty} \left| \frac{r-n}{n} \right| = |x|.$$

We conclude that the series has radius of convergence $R = 1$, and that the interior of its interval of convergence is $(-1,1)$. Let us put

$$\psi(x) = \sum_{k=0}^{\infty} \binom{r}{k} x^k$$

throughout that open interval. By Theorem 13.16, we also have for $-1 < x < 1$,

$$\psi'(x) = \sum_{k=1}^{\infty} k \binom{r}{k} x^{k-1},$$

so that

$$\begin{aligned}
(1+x)\psi'(x) &= (1+x) \sum_{k=1}^{\infty} k \binom{r}{k} x^{k-1} \\
&= \sum_{k=1}^{\infty} k \binom{r}{k} x^{k-1} + \sum_{k=1}^{\infty} k \binom{r}{k} x^k \\
&= \sum_{k=0}^{\infty} (k+1) \binom{r}{k+1} x^k + \sum_{k=0}^{\infty} k \binom{r}{k} x^k \\
&= \sum_{k=0}^{\infty} \left[(k+1) \binom{r}{k+1} + k \binom{r}{k} \right] x^k.
\end{aligned}$$

But

$$\begin{aligned}(k+1)\binom{r}{k+1}+k\binom{r}{k} &= \frac{k+1}{(k+1)!}\prod_{j=1}^{k+1}(r-j+1)+\frac{k}{k!}\prod_{j=1}^{k}(r-j+1)\\ &= \frac{1}{k!}\left[(r-k)\prod_{j=1}^{k}(r-j+1)+k\prod_{j=1}^{k}(r-j+1)\right]\\ &= r\binom{r}{k},\end{aligned}$$

and so it follows that

$$(1+x)\psi'(x)=r\sum_{k=0}^{\infty}\binom{r}{k}x^k=r\psi(x).$$

We now find that

$$\begin{aligned}\frac{d}{dx}\left[(1+x)^{-r}\psi(x)\right] &= -r(1+x)^{-r-1}\psi(x)+(1+x)^{-r}\psi'(x)\\ &= -r(1+x)^{-r-1}\psi(x)+r(1+x)^{-r-1}\psi(x)\\ &= 0\end{aligned}$$

for all $x\in(-1,1)$. Thus, $(1+x)^{-r}\psi(x)$ is a constant function. Now $\psi(0)=1$, and so $(1+0)^{-r}\psi(0)=1$. Thus $\psi(x)=(1+x)^r$ for all $x\in(-1,1)$, as desired. ∎

Theorem 13.23 *We have*

$$\sin x=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}$$

for every $x\in\mathbb{R}$.

PROOF: Let $f(x)=\sin x$ for all real x. By Theorem 13.5, for each integer $n\geq 0$ we have

$$f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}x^k+R_n(x),$$

where

$$R_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}$$

for some ξ that lies between 0 and x. But for each non-negative integer q, $f^{(2q)}(x)=(-1)^q\sin x$ and $f^{(2q+1)}(x)=(-1)^q\cos x$, so that $f^{(2q)}(0)=0$ and $f^{(2q+1)}(0)=(-1)^q$. Moreover, $|f^{(n+1)}(\xi)|\leq 1$ for all n and all ξ lying between 0 and x, and from this we conclude that

$$|R_n(x)|\leq\frac{|x|^{n+1}}{(n+1)!}$$

for every non-negative integer n. Thus $\lim_{n\to\infty} R_n(x) = 0$. Applying Theorem **??**, we obtain

$$\sin x = \sum_{p=0}^{\infty} \frac{f^{(p)}(0)}{k!} x^p = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

∎

Observe that we have shown that the radius of convergence of the series on the right side is infinity.

Theorem 13.24 *We have*

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

for every $x \in \mathbb{R}$.

PROOF: Apply the Fundamental Theorem of Power Series to obtain

$$\cos x = \frac{d}{dx} \sin x = \frac{d}{dx} \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

where the radius of convergence of the latter series must be the same as that of the undifferentiated series, i.e., infinity. ∎

Theorem 13.25 *We have*

$$\exp(x) = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

for all $x \in \mathbb{R}$.

PROOF: Let $f(x) = e^x$ for all real x. By Theorem 13.5, for each integer $n \geq 0$ we have

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + R_n(x),$$

where

$$R_n(x) = \frac{f^{(n+1)}(\xi_n)}{(n+1)!} x^{n+1}$$

for some ξ_n between 0 and x. But $f^{(k)}(x) = e^x$ for all x and for all non-negative integers k. Thus $f^{(k)}(0) = 1$ for all k and $f^{(n+1)}(\xi_n) = e^{\xi_n}$ for all n. This means that

$$|R_n(x)| = \left| e^{\xi_n} \frac{x^{n+1}}{(n+1)!} \right| \leq e^{|x|} \frac{|x|^{n+1}}{(n+1)!}$$

and $\lim_{n\to\infty} R_n(x) = 0$. Substituting 1 for $f^{(k)}(0)$ throughout and passing to the limit gives us the desired result. ∎

Theorem 13.26 *We have*

$$\sinh x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

for all $x \in \mathbb{R}$.

PROOF: Because

$$\sinh x = \frac{1}{2}(e^x - e^{-x}),$$

we have, by Theorem 13.25 and Theorem 13.17,

$$\begin{aligned} \sinh x &= \frac{1}{2}\left(\sum_{j=0}^{\infty} \frac{x^j}{j!} - \sum_{j=0}^{\infty} (-1)^j \frac{x^j}{j!}\right) \\ &= \frac{1}{2}\sum_{j=0}^{\infty}\left(\frac{x^j}{j!} - (-1)^j \frac{xj}{j!}\right) \\ &= \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} \end{aligned}$$

for all $x \in \mathbb{R}$. ∎

Theorem 13.27 *We have*

$$\cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$$

for all $x \in \mathbb{R}$.

Theorem 13.28 *We have*

$$\log(1+x) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}$$

for all $x \in (-1, 1)$.

PROOF: Making use of the geometric series, we may write

$$\begin{aligned} \frac{d}{dx}\log(1+x) &= \frac{1}{1+x} = \frac{1}{1-(-x)} \\ &= \sum_{k=0}^{\infty} (-x)^k = \sum_{k=0}^{\infty} (-1)^k x^k \\ &= \sum_{k=1}^{\infty} (-1)^{k-1} x^{k-1}, \end{aligned}$$

the representation being correct when $|x| < 1$. By the Fundamental Theorem of Power Series, we have

$$\begin{aligned}
\log(1+x) &= \int_0^x \frac{dt}{1+t} \\
&= \int_0^x \left(\sum_{k=1}^{\infty} (-1)^{k-1} t^{k-1} \right) dt \\
&= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{t^k}{k} \Bigg|_0^x \\
&= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k},
\end{aligned}$$

the interior of the interval of convergence of the latter series being $(-1, 1)$. ∎

Theorem 13.29 *We have*

$$\arctan x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$$

for every $x \in (-1, 1)$.

PROOF: Again relying on the geometric series, we have

$$\begin{aligned}
\frac{d}{dx} arctan x &= \frac{1}{1+x^2} = \frac{1}{1-(-(x^2))} \\
&= \sum_{k=0}^{\infty} (-(x^2))^k = \sum_{k=0}^{\infty} (-1)^k x^{2k},
\end{aligned}$$

and the representation is valid when $|-(x^2)| < 1$, or when $|x| < 1$. Appealing once more to the Fundamental Theorem of Power Series, we find that

$$\begin{aligned}
\arctan x &= \int_0^x \frac{dt}{1+t^2} \\
&= \int_0^x \left(\sum_{k=0}^{\infty} (-1)^k t^{2k} \right) dt \\
&= \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+1}}{2k+1} \Bigg|_0^x \\
&= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}
\end{aligned}$$

the interior of the interval of convergence of the latter series being $(-1, 1)$. ∎

Theorem 13.30 *We have*

$$\arcsin x = \sum_{k=0}^{\infty} \frac{1}{2^{2k}} \binom{2k}{k} \frac{x^{2k+1}}{2k+1}$$

for every $x \in (-1, 1)$.

PROOF: We know that for all $x \in (-1, 1)$,

$$\begin{aligned} \frac{d}{dx} \arcsin x &= (1 - x^2)^{-1/2} \\ &= \sum_{k=0}^{\infty} (-1)^k \binom{-1/2}{k} x^{2k} \end{aligned}$$

But

$$\begin{aligned} \binom{-1/2}{k} &= \frac{(-1/2)(-3/2) \cdots ((1-2k)/2)}{k!} \\ &= (-1)^k \frac{1 \cdot 3 \cdots (2k-1)}{2^k k!} \\ &= (-1)^k \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots (2k-2)(2k-1)(2k)}{2^k k! (2 \cdot 4 \cdots 2k)} \\ &= (-1)^k \frac{(2k!)}{2^{2k} (k!)^2} \\ &= (-1)^k \frac{1}{2^{2k}} \binom{2k}{k}. \end{aligned}$$

Our formula follows if we combine this with the above and integrate the resulting series, term by term, from 0 to x. ∎

13.8 Other Convergence Tests

Theorem 13.31 *[Comparison of Ratios] Let* $\sum_{k=0}^{\infty} a_k$ *be a series of positive terms. If* $\sum_{k=0}^{\infty} c_k$ *is a convergent series of positive terms for which there is a positive integer* N *such that*

$$\frac{a_{k+1}}{a_k} \leq \frac{c_{k+1}}{c_k}$$

whenever $k \geq N$, *then* $\sum_{k=0}^{\infty} a_k$ *converges. On the other hand, if* $\sum_{k=0}^{\infty} d_k$ *is a divergent series of positive terms for which there is a positive integer* N *such that*

$$\frac{d_{k+1}}{d_k} \leq \frac{a_{k+1}}{a_k}$$

whenever $k \geq N$, *then* $\sum_{k=0}^{\infty} a_k$ *diverges.*

Proof: Let $\sum_{k=0}^{\infty} a_k$, $\sum_{k=0}^{\infty} c_k$, and N be as in the hypotheses. Then

$$\begin{aligned}\frac{a_{N+1}}{a_N} &\le \frac{c_{N+1}}{c_N}, \text{ so that}\\ a_{N+1} &\le \frac{a_N}{c_N} c_{N+1}.\end{aligned}$$

Now suppose inductively that there is a positive integer j for which

$$a_{N+j} \le \frac{a_N}{c_N} c_{N+j}.$$

By our hypothesis, we have

$$\frac{a_{N+j+1}}{a_{N+j}} \le \frac{c_{N+j+1}}{c_{N+j}},$$

from which it follows that

$$\begin{aligned}a_{N+j+1} &\le a_{N+j} \frac{c_{N+j+1}}{c_{N+j}}\\ &\le \left(\frac{a_N}{c_N} c_{N+j}\right) \frac{c_{N+j+1}}{c_{N+j}} \le \frac{a_N}{c_N} c_{N+j+1}.\end{aligned}$$

We conclude thus that the inequality

$$a_k \le \frac{a_N}{c_N} c_k$$

for all integers $k > N$. It now follows from the convergence of $\sum_{k=0}^{\infty} c_k$ and the Comparison Test that $\sum_{k=0}^{\infty} a_k$ converges.

Now let $\sum_{k=0}^{\infty} a_k$, $\sum_{k=0}^{\infty} d_k$, and N be as in the hypotheses. Then

$$\begin{aligned}\frac{a_{N+1}}{a_N} &\ge \frac{d_{N+1}}{d_N}, \text{ so that}\\ a_{N+1} &\ge \frac{a_N}{d_N} d_{N+1}.\end{aligned}$$

Suppose inductively that there is a positive integer j for which

$$a_{N+j} \ge \frac{a_N}{d_N} d_{N+j}.$$

By our hypothesis, we have

$$\frac{a_{N+j+1}}{a_{N+j}} \ge \frac{d_{N+j+1}}{d_{N+j}},$$

from which it follows that

$$\begin{aligned}a_{N+j+1} &\ge a_{N+j} \frac{d_{N+j+1}}{d_{N+j}}\\ &\ge \left(\frac{a_N}{d_N} d_{N+j}\right) \frac{d_{N+j+1}}{d_{N+j}} \ge \frac{a_N}{d_N} d_{N+j+1}.\end{aligned}$$

We conclude thus that the inequality

$$a_k \geq \frac{a_N}{d_N} d_k$$

is true for all integers $k > N$. Now because $\sum_{k=0}^{\infty} d_k$ diverges and the Comparison Test we see that $\sum_{k=0}^{\infty} a_k$ diverges. ∎

Theorem 13.32 *[Raabe's Test] Let $\sum_{k=0}^{\infty} a_k$ be a series of positive terms. If*

$$\limsup k \left(\frac{a_k}{a_{k+1}} - 1 \right) < 1,$$

then the series $\sum_{k=0}^{\infty} a_k$ diverges. If

$$\liminf k \left(\frac{a_k}{a_{k+1}} - 1 \right) > 1,$$

the series converges.

Proof: Let us begin by supposing that the first inequality holds. Then we can find an integer N so large that for every $k \geq N$ we have

$$\begin{aligned} k \left(\frac{a_k}{a_{k+1}} - 1 \right) &< 1, \text{ or} \\ \frac{a_k}{a_{k+1}} &< \frac{k+1}{k}, \text{ or} \\ \frac{a_{k+1}}{a_k} &> \frac{\left(\dfrac{1}{k+1} \right)}{\left(\frac{1}{k} \right)}. \end{aligned}$$

But $\sum_{k=1}^{\infty} 1/k$ is a divergent series, and it thus follows by the Comparison of Ratios Test that $\sum_{k=0}^{\infty} a_k$ diverges.

Now let us suppose on the other hand that the second inequality holds, and let L denote the limit inferior that appears in that inequality. Let us put

$$\begin{aligned} K &= 1 + \frac{2}{3}(L-1) \\ &= \frac{1+2L}{3}, \end{aligned}$$

and

$$\begin{aligned} H &= 1 + \frac{1}{3}(L-1) \\ &= \frac{2+L}{3}. \end{aligned}$$

Then $1 < H < K < L$. Thus, there is a positive integer N such that

$$k\left(\frac{a_k}{a_{k+1}} - 1\right) > K$$

whenever $k \geq N$. The condition $k \geq N$ therefore implies that

$$\begin{aligned} \frac{a_k}{a_{k+1}} - 1 &> \frac{K}{k}, \text{ or} \\ \frac{a_k}{a_{k+1}} &> 1 + \frac{K}{k}, \text{ or} \\ \frac{a_{k+1}}{a_k} &< \frac{1}{1+(K/k)}. \end{aligned}$$

Note that

$$\begin{aligned} \left(1+\frac{K}{k}\right)\left(1-\frac{H}{k}\right) &= 1 + \frac{K}{k} - \frac{H}{k} - \frac{HK}{k^2} \\ &= 1 + \frac{1}{k}\left[(K-H) - \frac{HK}{k}\right]. \end{aligned}$$

The number $K - H$ is positive, so when k is sufficiently large, the quantity $(K-H) - (HK/k)$ must be positive. Thus, when k is large enough we must have

$$\begin{aligned} \frac{a_{k+1}}{a_k} &< \frac{1}{1+(K/k)} \\ &< 1 - \frac{H}{k}. \end{aligned}$$

Now

$$\frac{d}{dx}x^H = Hx^{H-1},$$

and because $1 < H$,

$$Hx^{H-1} \leq Hk^{H-1}$$

for all $x \in (k-1, k)$. Therefore for each positive integer k we have $k^H - (k-1)^H \leq Hk^{H-1}$. Equivalently, when k is a positive integer and $H > 1$,

$$k^H - Hk^{H-1} \leq (k-1)^H.$$

Dividing the latter inequality through by k^H, we find that

$$1 - \frac{H}{k} \leq \frac{(k-1)^H}{k^H}.$$

Thus, we find that when k is sufficiently large we may write

$$\frac{a_{k+1}}{a_k} \leq \frac{(k-1)^H}{k^H} = \frac{\left(\frac{1}{k^H}\right)}{\left(\frac{1}{(k-1)^H}\right)}.$$

But $H > 1$, and so the series $\sum_{k=2}^{\infty} \frac{1}{(k-1)^H}$ is a convergent series. We complete the proof by appealing to the Comparison of Ratios Test, which assures us that $\sum_{k=0}^{\infty} a_k$ is convergent. ∎

Raabe's Test fails if

$$\liminf k\left(\frac{a_k}{a_{k+1}} - 1\right) \leq 1 \leq \limsup k\left(\frac{a_k}{a_{k+1}} - 1\right).$$

We can show an example of this even when equality holds in both cases so that the limit exists and is 1.

Consider the series

$$\sum_{k=2}^{\infty} \frac{1}{k(\log k)^p},$$

which the Integral Test shows to be divergent when $p = 1$ but convergent when $p > 1$. However, for this series, we have

$$\begin{aligned}
\lim_{k\to 0} k\left(\frac{a_k}{a_{k+1}} - 1\right) &= \lim_{k\to 0} k\left(\frac{(k+1)(\log(k+1))^p}{k(\log k)^p} - 1\right) \\
&= \lim_{k\to\infty}\left((k+1)\left(\frac{\log(k+1)}{\log k}\right)^p - k\right) \\
&= \lim_{k\to\infty}\left(k\left(\left(\frac{\log(k+1)}{\log k}\right)^p - 1\right) + \left(\frac{\log(k+1)}{\log k}\right)^p\right).
\end{aligned}$$

Now l'Hôpital's Rule gives

$$\begin{aligned}
\lim_{k\to\infty} \frac{\log(k+1)}{\log k} &= \lim_{k\to\infty} \frac{\left(\frac{1}{k+1}\right)}{\left(\frac{1}{k}\right)} \\
&= \lim_{k\to\infty} \frac{k}{k+1} = 1,
\end{aligned}$$

and from this it is immediate that

$$\lim_{k\to\infty}\left(\frac{\log(k+1)}{\log k}\right)^p = 1.$$

So, now we only have to evaluate

$$\lim_{k\to\infty} k\left(\left(\frac{\log(k+1)}{\log k}\right)^p - 1\right).$$

So substitute $t = 1/k$, $p = 1$, and we get

$$\begin{aligned}
\lim_{k\to\infty} k\left(\left(\frac{\log(k+1)}{\log k}\right)^p - 1\right) &= \lim_{t\to 0^+} \frac{1}{t}\left(\frac{\log(1/t+1)}{\log(1/t)} - 1\right) \\
&= \lim_{t\to 0^+} \frac{1}{t}\left(\frac{\log(1+t)-\log t}{-\log t} - 1\right) \\
&= \lim_{t\to 0^+} \frac{\log(1+t)}{-t\log t}
\end{aligned}$$

It is actually more convenient to examine the slightly more general limit

$$\begin{aligned}
\lim_{t\to 0} \frac{\log(1+t)}{-t\log|t|} &= \lim_{t\to 0} \frac{\left(t - \dfrac{t^2}{2} + \dfrac{t^3}{3} - \dfrac{t^4}{4} + \dots\right)}{-t\log|t|} \\
&= \lim_{t\to 0} \frac{\left(1 - \dfrac{t}{2} + \dfrac{t^2}{3} - \dfrac{t^3}{4} + \dots\right)}{-\log|t|} \\
&= 0.
\end{aligned}$$

Consequently,

$$\lim_{k\to\infty} k\left(\left(\frac{\log(k+1)}{\log k}\right) - 1\right) = 0.$$

However, if we take

$$u[t] = \begin{cases} \frac{\log(1+t)-\log|t|}{-\log|t|}, & \text{if } t \neq 0 \\ 1, & \text{if } t = 0, \end{cases}$$

we may also interpret the above limit as telling us that $u'(0) = 0$. But then under our substitution $t = 1/k$, we have

$$\begin{aligned}
\lim_{k\to\infty} k\left(\left(\frac{\log(k+1)}{\log k}\right)^p - 1\right) &= \lim_{t\to 0^+} \frac{(u(t))^p - (u(0))^p}{t} \\
&= p(u(0))^{p-1}u'(0), \text{ by the Chain Rule,} \\
&= 0.
\end{aligned}$$

It now follows that, whatever $p \geq 1$ may be, we always have

$$\lim_{k\to\infty} k\left(\frac{a_k}{a_{k+1}} - 1\right) = 1$$

when $a_k = \dfrac{1}{k(\log k)^p}$.

Theorem 13.33 *[Generalized Raabe Test] Let $\sum_{k=0}^{\infty} a_k$ be a series of positive terms. If*

$$\limsup \log k \left(k \left(\frac{a_k}{a_{k+1}} - 1 \right) - 1 \right) < 1,$$

then $\sum_{k=0}^{\infty} a_k$ diverges, while if

$$\liminf \log k \left(k \left(\frac{a_k}{a_{k+1}} - 1 \right) - 1 \right) > 1$$

the series converges.

Proof: Let us begin by supposing that the first inequality is true. Then we can find an integer N so large that for every $k \geq N$ we have

$$\begin{aligned} \log k \left(k \left(\frac{a_k}{a_{k+1}} - 1 \right) - 1 \right) &< 1, \text{ or} \\ k \left(\frac{a_k}{a_{k+1}} - 1 \right) &< \frac{1 + \log k}{\log k}, \text{ or} \\ \frac{a_k}{a_{k+1}} &< \frac{1 + \log k + k \log k}{k \log k}. \end{aligned}$$

If $f(x) = x \log x$, then $f'(x) = 1 + \log x$, and the latter is a non-negative, increasing function on $[k, k+1]$ when k is a positive integer. Thus

$$\begin{aligned} 1 + \log k &< (k+1)\log(k+1) - k \log k, \text{ or} \\ 1 + \log k + k \log k &< (k+1)\log(k+1) \end{aligned}$$

for each positive integer k. This now implies that,

$$\frac{a_k}{a_{k+1}} < \frac{(k+1)\log(k+1)}{k \log k},$$

at least when $k \geq N$. This latter is equivalent to

$$\frac{a_{k+1}}{a_k} > \frac{\left(\frac{1}{(k+1)\log(k+1)} \right)}{\left(\frac{1}{k \log k} \right)},$$

But $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k \log k}$ is a divergent series, and it thus follows by the Comparison of Ratios Test that $\sum_{k=0}^{\infty} a_k$ diverges.

Now let us suppose on the other hand that the second inequality holds, and let L designate the value of the limit inferior that appears in that inequality. Let us put

$$\begin{aligned} K &= 1 + \frac{2}{3}(L-1) \\ &= \frac{1+2L}{3}, \end{aligned}$$

and

$$\begin{aligned} H &= \frac{1+K}{2} \\ &= \frac{2+L}{3}. \end{aligned}$$

Then $1 < H < K < L$. Thus, there is a positive integer N such that

$$\log k \left(k \left(\frac{a_k}{a_{k+1}} - 1 \right) - 1 \right) > K$$

whenever $k \leq N$. The condition $k \geq N$ therefore implies that

$$\begin{aligned} k \left(\frac{a_k}{a_{k+1}} - 1 \right) - 1 &> \frac{K}{\log k}, \text{ or} \\ \frac{a_k}{a_{k+1}} - 1 &> \frac{K}{k \log k} + \frac{1}{k}, \text{ or} \\ \frac{a_k}{a_{k+1}} &> 1 + \frac{K}{k \log k} + \frac{1}{k}, \text{ or} \\ \frac{a_{k+1}}{a_k} &< \frac{1}{1 + [K/(k \log k)] + (1/k)}. \end{aligned}$$

Note now from the definition of H that

$$\begin{aligned} &\left(1 + \frac{1}{k} + \frac{K}{k \log k}\right)\left(1 - \frac{1}{k} - \frac{H}{k \log k}\right) - 1 \\ &\qquad = \frac{[(K-1)k - 2\log k - (1+3K)]\log k - K - K^2}{2k^2 (\log k)^2}. \end{aligned} \tag{13.70}$$

Because $K - 1 > 0$, the numerator of the above fraction grows without bound as $k \to \infty$. This means that for sufficiently large k we have

$$\left(1 + \frac{1}{k} + \frac{K}{k \log k}\right)\left(1 - \frac{1}{k} - \frac{H}{k \log k}\right) > 1.$$

From this we get

$$\begin{aligned} \frac{1}{1 + (K/(k \log k)) + (1/k)} &< 1 - \frac{1}{k} - \frac{H}{k \log k} \\ &< 1 - \frac{1}{k+1} - \frac{H}{(k+1)\log(k+1)}. \end{aligned}$$

This all now implies that

$$\frac{a_{k+1}}{a_k} < 1 - \frac{1}{k+1} - \frac{H}{(k+1)\log(k+1)}$$

for all sufficiently large integers k.

Now let $\varphi(x) = x(\log x)^H$. Then

$$\varphi'(x) = (\log x)^H + H(\log x)^{H-1},$$

and then if k is a positive integer, then

$$(k+1)(\log(k+1))^H - k(\log k)^H \leq (\log(k+1))^H + H(\log(k+1))^{H-1}.$$

This is equivalent to

$$(k+1)(\log(k+1))^H - (\log(k+1))^H - H(\log(k+1))^{H-1} \leq k(\log k)^H,$$

which, in its turn, is equivalent to

$$1 - \frac{1}{k+1} - \frac{H}{(k+1)\log(k+1)} \leq \frac{k(\log k)^H}{(k+1)(\log(k+1))^H}.$$

This now implies that

$$\frac{a_{k+1}}{a_k} \leq \frac{k(\log k)^H}{(k+1)(\log(k+1))^H} = \frac{\left(frac1(k+1)(\log(k+1))^H\right)}{\left(\frac{1}{k(\log k)^H}\right)}$$

for all sufficiently large integers k. Because $H > 1$, the series $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\log k)^H}$ converges, and the Comparison of Ratios Test now guarantees that $\sum_{k=0}^{\infty} a_k$ converges as well. ∎

Theorem 13.34 *[Gauss' Test] Let $\sum_{k=0}^{\infty} a_k$ be a series of positive terms, and suppose that there are polynomials*

$$p(x) = x^n + b_1 x^{n-1} + \cdots + b_n$$
$$q(x) = x^n + c_1 x^{n-1} + \cdots + c_n$$

such that for every positive integer k

$$\frac{a_{k+1}}{a_k} = \frac{p(k)}{q(k)}. \tag{13.71}$$

Then $\sum_{k=0}^{\infty} a_k$ diverges if $c_1 - b_1 \leq 1$ and $\sum_{k=0}^{\infty} a_k$ converges if $c_1 - b_1 > 1$.

PROOF: We will apply Raabe's Test. We have

$$\begin{aligned}
k\left(\frac{a_k}{a_{k+1}}-1\right) &= k\left(\frac{q(k)}{p(k)}-1\right)\\
&= k\left(\frac{(k^n+c_1k^{n-1}+\cdots+c_n)-(k^n+b_1k^{n-1}+\cdots+b_n)}{k^n+b_1k^{n-1}+\cdots+b_n}\right)\\
&= k\left(\frac{(c_1-b_1)k^{n-1}+(c_2-b_2)k^{n-2}+\cdots+(c_n-b_n)}{k^n+b_1k^{n-1}+\cdots+b_n}\right)\\
&= \frac{(c_1-b_1)k^{n}+(c_2-b_2)k^{n-1}+\cdots+(c_n-b_n)k}{k^n+b_1k_{n-1}+\cdots+b_n},
\end{aligned}$$

As a consequence of this

$$\lim_{k\to\infty} k\left(\frac{a_k}{a_{k+1}}-1\right) = c_1-b_1.$$

It now follows that $\sum_{k=0}^{\infty} a_k$ diverges if $c_1-b_1<1$ and converges if $c_1-b_1>1$.

If it should be that $c_1-b_1=1$, we apply the Generalized Raabe Test:

$$\begin{aligned}
\lim_{k\to\infty}\log k\left(k\left(\frac{a_k}{a_{k+1}}-1\right)-1\right) &= \lim_{k\to\infty}\log k\left(\frac{k^n+(c_2-b_2)k^{n-1}+\cdots+(c_n-b_n)k}{k^n+b_1k^{n-1}+\cdots+b_n}-1\right)\\
&= \lim_{k\to\infty}\frac{\log k}{k}\left(\frac{(c_2-b_2-b_1)k^n+\cdots+(c_n-b_n-b_{n-1})k-b_n}{k^n+b_1k^{n-1}+\cdots+b_n}\right)\\
&= 0\cdot(c_2-b_2-b_1)=0<1.
\end{aligned}$$

∎

Theorem 13.35 *[Kummer's Test] Let $\sum_{k=0}^{\infty} a_k$ be a series of positive terms, and let $\{c_k\}$ be a sequence of positive numbers. If*

$$\limsup\left(c_k\frac{a_k}{a_{k+1}}-c_{k+1}\right)>0,$$

then $\sum_{k=0}^{\infty} a_k$ converges. If $\displaystyle\sum_{k=0}^{\infty}\frac{1}{c_k}$ diverges and

$$\liminf\left(c_k\frac{a_k}{a_{k+1}}-c_{k+1}\right)<0,$$

then $\sum_{k=0}^{\infty} a_k$ diverges.

PROOF: Suppose that the first inequality holds, and let L be the limit superior that appears on the left side of that inequality. Choose a positive number $r<L$. We can then find a positive integer N sufficiently large that

$$c_k\frac{a_k}{a_{k+1}}-c_{k+1}>r$$

for all $k > N$. Equivalently, for every positive integer j we must have the sequence of inequalities:

$$\left.\begin{array}{ccccc} a_N c_N & - & a_{N+1}c_{N+1} & > & a_{N+1}r \\ a_{N+1}c_{N+1} & - & a_{N+2}c_{N+2} & > & a_{N+2}r \\ a_{N+2}c_{N+2} & - & a_{N+3}c_{N+3} & > & a_{N+3}r \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{N+j-1}c_{N+j-1} & - & a_{N+j}c_{N+j} & > & a_{N+j}r \end{array}\right\}$$

Summing this sequence of inequalities, and noting the telescoping effect on the left side of the sum, we obtain, for every $j \geq 1$, the inequality

$$a_N c_N - a_{N+j}c_{N+j} > r\sum_{i=1}^{j} a_{N+i}.$$

Thus, putting $s_n = \sum_{k=1}^{n} a_k$, we have

$$rs_{N+j} - rs_N < a_n c_N - a_{N+j}c_{N+j}.$$

This can be rewritten to obtain

$$\begin{aligned} rs_{N+j} &< rs_N + a_N c_N - a_{N+j}c_{N+j} \\ &< rs_N + a_N c_N \end{aligned}$$

for every positive integer j. Consequently, the partial sums s_n, $n > N$, are all bounded by the fixed number $s_N + a_N c_N/r$. However, because all of the numbers a_k are positive, the sequence $\{s_n\}$ is monotonically increasing and it now follows that $\lim_{n\to\infty} s_n$ exists — i.e, that $\sum_{k=0}^{\infty} a_k$ converges.

Suppose now that $\sum_{k=0}^{\infty} 1/c_k$ diverges, and that the second inequality holds. We can then find a positive integer N so large that

$$c_k \frac{a_k}{a_{k+1}} - c_{k+1} < 0$$

whenever $k \geq N$. Consequently

$$a_N c_N < a_{N+1}c_{N+1} < a_{N+2}c_{N+2} < \cdots < a_j c_j$$

for any integer $j > N$. But this means that whenever $j > N$ we must have

$$a_j > \frac{a_N c_N}{c_j}.$$

The series $\sum_{j=0}^{\infty} \dfrac{a_N c_N}{c_j}$ being a divergent series, it now follows from the Comparison Test that the series $\sum_{k=0}^{\infty} a_k$ diverges. ∎

Kummer's Test is a powerful test, in spite of its seeming simplicity. We could have based a proof of Raabe's Test on Kummer's Test, taking $c_k = k$. We could also have based a proof of the Generalized Raabe Test on Kummer's Test, taking $c_k = k \log k$. In the case of Raabe's Test and its generalization, we find that we must establish rather non-trivial limits. In circumstances involving specific series, we find that the difficulty of inventing the sequence $\{c_k\}$ required in its hypotheses extorts stee[payment for the use of Kummer's Test.