Chapter 14

Series of Functions of Several Variables

The idea of sequences and series of functions can be extended to functions of several variables. For example,

$$\sum_{n=1}^{\infty}(xy)^n = xy + x^2y^2 + x^3y^3 + \cdots + x^ny^n + \ldots$$

is a series of functions of two variables of two variables. All of the other notions, such as uniform convergence, can be extended as well.

Theorem 14.1 (Weierstrauss M-Test) *Let* $\sum_{n=1}^{\infty} f_n(x)$ *be a series of functions all defined on a subset of* $U \subseteq \mathbb{R}$*. If there is a convergent series of constants* $\sum_{n=1}^{\infty} M_n$ *such that*

$$|f_n(x)| \leq M_n \text{ for all } x \in U,$$

then the series $\sum_{n=1}^{\infty} f_n(x)$ *converges absolutely for each* $x \in U$ *and is uniformly convergent on* U*.*

Now we can define *power series* in several variables. For two variables x, y such a power series is a series

$$\sum_{n=0}^{\infty} f_n(x,y) = f_0(x,y) + f_1(x,y) + \ldots$$

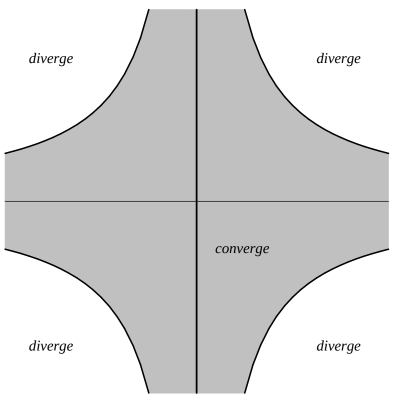
where

$$f_n(x,y) = c_{n,0}x^n + c_{n,1}x^{n-1}y + c_{n-2,2}x^{n-2}y^2 + \cdots + c_{n,n-1}xy^{n-1} + c_{n,n}y^n,$$

with each $c_{i,j}$ being a constant. What we have done is to collect all the terms of the *same degree*. Each f_n is called a *homogeneous polynomial of degree* n in x and y. Our first series is an example of this where

$$f_{2n}(x,y) = x^n y^n$$
$$f_{2n+1}(x,y) = 0.$$

This power series also is an example of how complicated the set on which the series converges might be. Since the series looks like a geometric series, we can, in fact, show that it converges for $|xy| < 1$.



If a function $f(x,y)$ can be represented by such a power series in a neighborhood of the origin,

$$f(x,y) = c_{0,0} + (c_{1,0}x + c_{1,1}y) + (c_{2,0}x^2 + c_{2,1}xy + c_{2,2}y^2) + \dots$$

then when we do a term-by-term differentiation, we find that

$$c_{0,0} = f(0,0) \qquad c_{1,0} = \frac{\partial f}{\partial x}(0,0)$$
$$c_{1,1} = \frac{\partial f}{\partial y}(0,0) \qquad c_{2,0} = \frac{1}{2!}\frac{\partial^2 f}{\partial x^2}(0,0)$$
$$c_{2,1} = \frac{2}{2!}\frac{\partial^2 f}{\partial x \partial y}(0,0) \qquad c_{2,2} = \frac{1}{2!}\frac{\partial^2 f}{\partial y^2}(0,0)$$

In general we can show that

$$f_n(x,y) = \frac{1}{n!}\left(\binom{n}{0}\frac{\partial^n f}{\partial x^n}x^n + \binom{n}{1}\frac{\partial^n f}{\partial^n - 1x\partial y}x^{n-1}y + \frac{1}{2!}\binom{n}{2}\frac{\partial^n f}{\partial^{n-2}x\partial^2 y}x^{n-2}y^2 + \dots \right.$$
$$\left. + \frac{1}{k!}\binom{n}{k}\frac{\partial^n f}{\partial^{n-k}x\partial^k y}x^{n-k}y^k + \dots + \dots + \binom{n}{n}\frac{\partial^n f}{\partial y^n}y^n\right),$$

with all the derivatives being evaluated at $(0,0)$. A series $\sum f_n(x,y)$, in which f_n are given by above is known as a Taylor series in x and y, about $(0,0)$, and the function $f(x,y)$ that it represents is called *analytic* in the region where it converges.

The Taylor series expansion about a general point (a,b) is obtained by translating from the origin:

$$f(x,y) = f(a,b) = \left[\frac{\partial f}{\partial x}(x-a) + \frac{\partial f}{\partial y}(y-b)\right]$$
$$+ \frac{1}{2!}\left[\frac{\partial^2 f}{\partial x^2}(x-a)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x-a)(y-b) + \frac{\partial^2 f}{\partial y^2}(y-b)^2\right]$$
$$+ \dots + \frac{1}{n!}\left[\frac{\partial^n f}{\partial x^n}(x-a)^n + \dots\right] + \dots,$$

where all the derivatives are evaluated at (a, b).

There is a notation that is often used, called *the nth differential* $d^n f$ of the function $f(x, y)$:

$$\begin{aligned} d^n f &= \frac{\partial^n f}{\partial x^n}(x-a)^n + \dots \\ &= \sum_{k=0}^{n} \binom{n}{k} \frac{\partial^n f}{\partial x^k \partial y^{n-k}}(a,b)(x-a)^k(y-b)^{n-k}. \end{aligned}$$

To indicate the dependence of $d^n f$ on the numbers a and b and the differences $x - a$ and $y - b$, we write

$$d^n f = d^n f(a, b; x-a, y-b).$$

The series can then be written more concisely:

$$\begin{aligned} f(x,y) = f(a,b) &+ df(a,b;x-a,y-b) \\ &+ \frac{1}{2!}d^2 f(a,b;x-a,y-b) + \dots + \frac{1}{n!}d^n f(a,b;x-a,y-b) + \dots \end{aligned}$$

14.1 Taylor's Formula for Functions of Several Variables

There is a Taylor's formula with remainder for functions of several variables:

$$\begin{aligned} f(x,y) = f(a,b) &+ df(a,b;x-a,y-b) + \dots + \frac{1}{n!}d^n f(a,b;x-a,y-b) \\ &+ \frac{1}{(n+1)!}d^{n+1} f(x^*,y^*;x-a,y-b); \\ &x^* = a + t^*(x-a), \quad y^* = b + t^*(y-b), \quad 0 < t^* < 1. \end{aligned}$$

The point (x^*, y^*) lies on the line segment between (a, b) and (x, y). For $n = 1$, the formula becomes

$$f(x,y) = f(a,b) + (x-a)f_x(x^*,y^*) + (y-b)f_y(x^*,y^*).$$

This is known as the Mean Value Theorem for a function of two variables.

Example 14.1 Lets find the power series expansion for $e^{x^2-y^2}$.

There are at least two ways we can do this. The first is the direct method.

$$f(x,y) = c_{0,0} + (c_{1,0}x + c_{1,1}y) + (c_{2,0}x^2 + c_{2,1}xy + c_{2,2}y^2) + \dots$$

and we know that

$$
\begin{aligned}
c_{0,0} &= f(0,0) = 1 \\
c_{1,0} &= f_x(0,0) = 2xe^{x^2-y^2}\Big|_{x=0,y=0} = 0 \\
c_{1,1} &= f_y(0,0) = -2ye^{x^2-y^2}\Big|_{x=0,y=0} = 0 \\
c_{2,0} &= \frac{1}{2!} f_{xx}(0,0) = \frac{1}{2!}\ (2+4x^2)e^{x^2-y^2}\Big|_{x=0,y=0} = 1 \\
c_{2,1} &= \frac{2}{2!} f_{xy}(0,0) = \frac{2}{2!}\ (-4xy)e^{x^2-y^2}\Big|_{x=0,y=0} = 0 \\
c_{2,2} &= \frac{1}{2!} f_{yy}(0,0) = \frac{1}{2!}\ (-2+4y^2)e^{x^2-y^2}\Big|_{x=0,y=0} = -1 \\
c_{3,k} &= 0 \\
c_{4,0} &= \frac{1}{4!} f_{xxxx}(0,0) = \frac{1}{4!}\ (12+48x^2+16x^4)e^{x^2-y^2}\Big|_{x=0,y=0} = \frac{1}{2} \\
c_{4,1} &= \frac{4}{4!} f_{xxxy}(0,0) = \frac{4}{4!}\ (-24xy-16x^3y)e^{x^2-y^2}\Big|_{x=0,y=0} = 0 \\
c_{4,2} &= \frac{6}{4!} f_{xxyy}(0,0) = \frac{6}{4!}\ (-4+8y^2-8x^2+16x^2y^2)e^{x^2-y^2}\Big|_{x=0,y=0} = -1 \\
c_{4,3} &= \frac{4}{4!} f_{xyyy}(0,0) = \frac{4}{4!}\ (24xy-16xy^3)e^{x^2-y^2}\Big|_{x=0,y=0} = 0 \\
c_{4,4} &= \frac{1}{4!} f_{yyyy}(0,0) = \frac{1}{4!}\ (12-48y^2+16y^4)e^{x^2-y^2}\Big|_{x=0,y=0} = \frac{1}{2}
\end{aligned}
$$

This gives us that

$$
\begin{aligned}
e^{x^2-y^2} &= 1 + x^2 - y^2 + \frac{1}{2}x^4 - x^2y^2 + \frac{1}{2}y^4 + \dots \\
&= 1 + (x^2-y^2) + \frac{1}{2!}(x^2-y^2)^2 + \dots
\end{aligned}
$$

It looks as if there is a pattern. Can we see it better?

We know that

$$e^x = 1 + x + \frac{1}{2!}x^2 + \dots + \frac{1}{n!}x^n + \dots$$

Substituting $x^2 - y^2$ for x gives us:

$$e^{x^2-y^2} = 1 + (x^2-y^2) + \frac{1}{2!}(x^2-y^2)^2 + \dots + \frac{1}{n!}(x^2-y^2)^n + \dots$$

and this converges for all x, y.

What about something like $e^x \sin y$? Here, we expect that we can take the series for e^x and $\sin y$ and multiply them together.

$$\begin{aligned}
e^x \sin y &= \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6!}x^3 + \dots\right)\left(y - \frac{1}{3!}y^3 + \frac{1}{5!}y^5 + \dots\right) \\
&= y + xy + \frac{1}{2}x^2 y - \frac{1}{3!}y^3 + \frac{1}{3!}x^3 y - \frac{1}{3!}xy^3 + \frac{1}{24}x^4 y - \frac{1}{12}x^2 y^3 + \frac{1}{120}y^5 + \dots \\
&= y + \frac{1}{2!}(2xy) + \frac{1}{3!}(3x^2 y - y^3) + \frac{1}{4!}(4x^3 y - 4xy^3) + \frac{1}{5!}(5x^4 y - 20x^2 y^3 + y^5) + \dots
\end{aligned}$$

Of course, we can use the previous results and check the coefficients from the derivatives. We will get the same result.