Chapter 10

Poincaré Upper Half Plane Model

The next model of the hyperbolic plane that we will consider is also due to Henri Poincaré. We will be using the upper half plane, or $\{(x, y) \mid y > 0\}$. We will want to think of this with a different distance metric on it.

Let $\mathscr{H} = \{x + iy \mid y > 0\}$ together with the arclength element

$$ds = \frac{\sqrt{dx^2 + dy^2}}{y}.$$

Note that we have changed the arclength element for this model!!!

10.1 Vertical Lines

Let $\mathbf{x}(t) = (x(t), y(t))$ be a piecewise smooth parametrization of a curve between the points $\mathbf{x}(t_0)$ and $\mathbf{x}(t_1)$.

Recall that in order to find the length of a curve we break the curve into small pieces and approximate the curve by multiple line segments. In the limiting process we find that the Euclidean arclength element is $ds = \sqrt{dx^2 + dy^2}$. We then find the length of a curve by integrating the arclength over the parametrization of the curve.

$$s = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Now, we want to work in the Poincaré Half Plane model. In this case the length of this same curve would be

$$s_P = \int_{t_0}^{t_1} \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{y}\, dt.$$

Let's look at this for a vertical line segment from (x_0, y_0) to (x_0, y_1). We need to parameterize the curve, and then use the arclength element to find its length. Its parametrization is:

$$\mathbf{x}(t) = (x_0, y), \quad y \in [y_0, y_1].$$

The Poincaré arclength is then

$$s_P = \int_{t_0}^{t_1} \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{y}\, dt = \int_{t_0}^{t_1} \frac{1}{y}\, dy = \ln(y)|_{y_0}^{y_1} = \ln(y_1) - \ln(y_0) = \ln(y_1/y_0)$$

Now, consider any piecewise smooth curve $\mathbf{x}(t) = (x(t), y(t))$ starting at (x_0, y_0) and ending at (x_0, y_1). So this curves starts and ends at the same points as this vertical line segment. Suppose that $y(t)$ is an increasing function. This is reasonable. Now, we have

$$\begin{aligned} s &= \int_{t_0}^{t_1} \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{y}\, dt \\ &\geq \int_{t_0}^{t_1} \frac{\sqrt{\left(\frac{dy}{dt}\right)^2}}{y}\, dt \\ &\geq \int_{y(t_0)}^{y(t_1)} \frac{dy}{y} \\ &\geq \ln(y(t_1)) - \ln(y(t_0)). \end{aligned}$$

This means that this curve is longer than the vertical line segment which joins the two points. Therefore, the shortest path that joins these two points is a vertical (Euclidean) line segment. Thus, vertical (Euclidean) lines in the upper half plane are lines in the Poincaré model.

Let's find the distance from $(1,1)$ to $(1,0)$ which would be the distance to the real axis. Now, since $(1,0)$ is NOT a point of $\mathscr{H}$, we need to find $\lim_{\delta \to 0} d((1,1),(1,\delta))$. According to what we have above,

$$d_P((1,1),(1,\delta)) = \ln(1) - \ln(\delta) = -\ln(\delta).$$

Now, in the limit we find that

$$d_P((1,1),(1,0)) = \lim_{\delta \to 0} d_P((1,1),(1,\delta)) = \lim_{\delta \to 0} -\ln(\delta) = +\infty$$

This tells us that a vertical line has infinite extent in either direction.

10.2 Isometries

An isometry is a map that preserves distance, *i.e.*, a function $f\colon \mathscr{H}^2 \to \mathscr{H}^2$ is an isometry if

$$d(f(P), f(Q) = d(P,Q).$$

What are some isometries in the Euclidean plane? The usual isometries are translations (sometimes called parallel translations), reflections through a line, rotations about a point. Any others? Then, what are the isometries of $\mathscr{H}$?

The arclength element must be preserved under the action of any isometry. That is, a map

$$(u(x,y), v(x,y))$$

is an isometry if

$$\frac{du^2 + dv^2}{v^2} = \frac{dx^2 + dy^2}{y^2}.$$

Some maps will be obvious candidates for isometries and some will not.

Let's start with the following candidate:

$$T_a(x, y) = (u, v) = (x + a, y).$$

Now, clearly $du = dx$ and $dv = dy$, so

$$\frac{du^2 + dv^2}{v^2} = \frac{dx^2 + dy^2}{y^2}.$$

Thus, T_a is an isometry. What does it do? It translates the point a units in the horizontal direction. This is called the *horizontal translation by a*.

Let's try:

$$R_b(x, y) = (u, v) = (2b - x, y).$$

Again, $du = -dx$, $dv = dy$ and our arclength element is preserved. This isometry is a *reflection* through the vertical line $x = b$.

We need to consider the following map:

$$\Phi(x, y) = (u, v) = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\right).$$

First, let's check that it is a Poincaré isometry. Let $r^2 = x^2 + y^2$. Then

$$\begin{aligned}
\frac{du^2 + dv^2}{v^2} &= \frac{r^4}{y^2}\left(\left(\frac{r^2dx - 2x^2dx - 2xydy}{r^4}\right)^2 + \left(\frac{r^2dy - 2xydx - 2y^2dy}{r^4}\right)^2\right) \\
&= \frac{1}{y^2}\left(\frac{((y^2 - x^2)dx - 2xydy)^2 - ((x^2 - y^2)dy - 2xydx)^2}{r^4}\right) \\
&= \frac{1}{r^4y^2}\left((x^4 - 2x^2y^2 + y^4 + 4x^2y^2)dx^2 - (2xy(y^2 - x^2) + 2xy(x^2 - y^2)dxdy + r^4dy^2\right) \\
&= \frac{dx^2 + dy^2}{y^2}
\end{aligned}$$

We will study this function further. It is called *inversion in the unit circle*.

10.3 Inversion in the Circle: Euclidean Considerations

We are building a tool that we will use in studying $\mathscr{H}$. This is a Euclidean tool, so we will be working in Euclidean geometry to prove results about this tool. There is more about inversions in the circle in Appendix A.

Let's look at this last isometry. We would like to understand what this function does. For each point (x, y), let $r^2 = x^2 + y^2$. This makes r the distance from the origin to (x, y). This function sends (x, y) to $(x/r^2, y/r^2)$. The distance from $\Phi(x, y) = (x/r^2, y/r^2)$ to the origin is $1/r^2$. Thus, if $r > 1$ then the image of the point is on the same ray, but its distance to the origin is now less than one. Likewise, if $r < 1$, then the image lies on the same ray but the image point lies at a distance greater than 1 from the origin. If $r = 1$, then $\Phi(x, y) = (x, y)$. Thus, Φ leaves the unit circle fixed and sends every point inside the unit circle outside the circle and every point outside the unit circle gets sent inside the unit circle. In other words, Φ turns the circle inside out.

What does Φ do to a line? What does it do to a circle? Let's see.

The image of a point P under inversion in a circle centered at O and with radius r is the point P' on the ray OP and such that

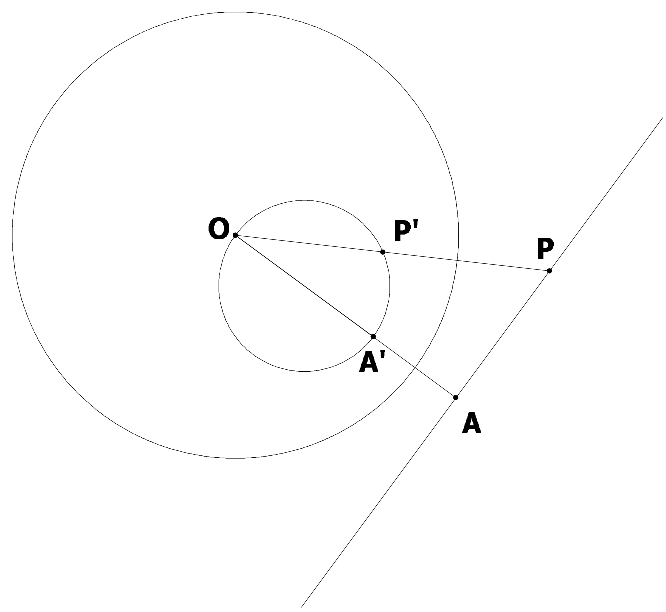
$$|OP'| = \frac{r^2}{|OP|}.$$

Lemma 10.1 *Let ℓ be a line which does not go through the origin O. The image of ℓ under inversion in the unit circle is a circle which goes through the origin O.*

PROOF: We will prove this for a line ℓ not intersecting the unit circle.

Let A be the foot of O on ℓ and let $|OA| = a$. Find A' on OA so that $|OA'| = 1/a$. Construct the circle with diameter OA'. We want to show that this circle is the image of ℓ under inversion.

Let $P \in \ell$ and let $|OP| = p$. Let P' be the intersection of the segment OP with the circle with diameter OA'. Let $|OP'| = x$. Now, look at the two triangles $\triangle OAP$ and $\triangle OP'A'$. These two Euclidean triangles are similar, so



$$\frac{|OP'|}{|OA'|} = \frac{|OA|}{|OP|}$$
$$\frac{x}{1/a} = \frac{a}{p}$$
$$x = \frac{1}{p}$$

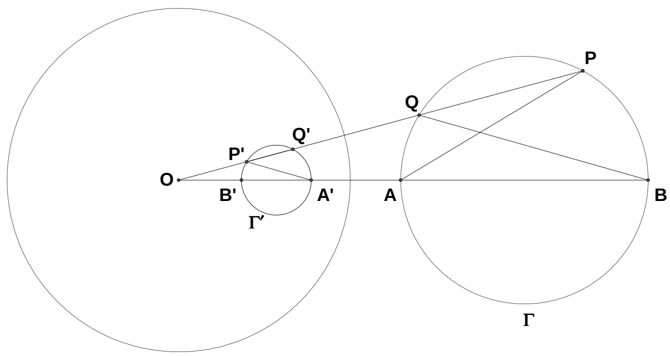
Therefore, P' is the image of P under inversion in the unit circle. ∎

Lemma 10.2 *Suppose Γ is a circle which does not go through the origin O. Then the image of Γ under inversion in the unit circle is a circle.*

PROOF: Again, we prove this for just one case: the case where Γ does not intersect the unit circle.

Let the line through O and the center of Γ intersect Γ at points A and B. Let $|OA| = a$ and $|OB| = b$. Let Γ' be the image of Γ under dilation by the factor $1/ab$. This dilation is $\Delta\colon (x, y) \mapsto (x/ab, y/ab)$.

Let B' and A' be the images of A and B, respectively, under this dilation, *i.e.* $\Delta(A) = B'$ and $\Delta(B) = A'$. Then $|OA'| = (1/ab)b = 1/a$ and $|OB'| = (1/ab)a = 1/b$. Thus, A' is the image of A under inversion in the unit circle. Likewise, B' is the image of B. Let ℓ' be an arbitrary ra through O which intersects Γ at P and Q. Let Q' and P' be the images of P and Q, respectively, under the dilation, Δ.



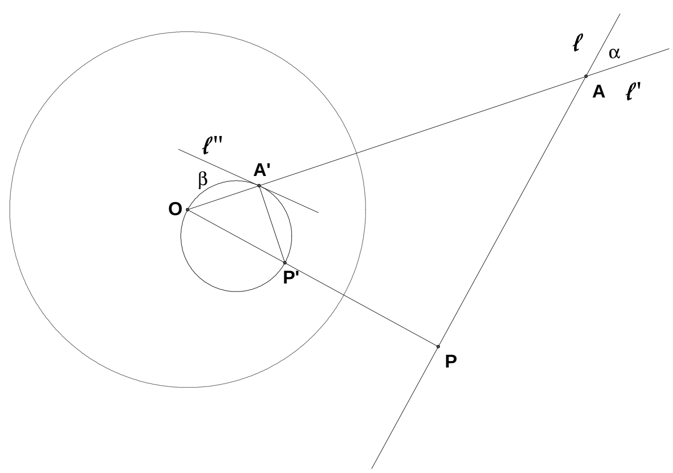
Now, $\triangle OA'P' \sim \triangle OBQ$, since one is the dilation of the other. Note that $\angle QBA \cong \angle QPA$ by the Star Trek lemma, and hence $\triangle OBQ \sim \triangle OPA$. Thus, $\triangle OA'P' \sim \triangle OPA$. From this it follows that

$$\frac{|OA'|}{|OP|} = \frac{|OP'|}{|OA|}$$
$$\frac{1/a}{|OP|} = \frac{|OP'|}{a}$$
$$|OP'| = \frac{1}{|OP|}$$

Thus, P' is the image of P under inversion, and Γ' is the image of Γ under inversion. ∎

Lemma 10.3 *Inversions preserve angles.*

Proof: We will just consider the case of an angle α created by the intersection of a line ℓ not intersecting the unit circle, and a line ℓ' through O.



Let A be the vertex of the angle α. Let P be the foot of O in ℓ. Let P' be the image of P under inversion. Then the image of ℓ is a circle Γ whose diameter is OP'. The image of A is $A' = \Gamma \bigcap \ell'$. Let ℓ'' be the tangent to Γ at A'. Then β, the angle formed by ℓ' and ℓ'' at A' is the image of α under inversion. We need to show that $\alpha \cong \beta$.

First, $\triangle OAP \sim \triangle OP'A'$, since they are both right triangles and share the angle O. Thus, $\angle A'P'O \cong \angle OAP \cong \alpha$. By the tangential case of the Star Trek lemma, $\beta \cong \angle A'P'O$. Thus, $\alpha \cong \beta$. ∎

10.4 Lines in the Poincaré Half Plane

From what we have just seen we can now prove the following.

Lemma 10.4 *Lines in the Poincaré upper half plane model are (Euclidean) lines and (Euclidean) half circles that are perpendicular to the x-axis.*

Proof: Let P and Q be points in $\mathscr{H}$ not on the same vertical line. Let Γ be the circle through P and Q whose center lies on the x-axis. Let Γ intersect the x-axis at M and N. Now consider the mapping φ which is the composition of a horizontal translation by $-M$ followed by inversion in the unit circle. This map φ is an isometry because it is the composition of two isometries. Note that M is first sent to O and then to ∞ by inversion. Thus, the image of Γ is a (Euclidean) line. Since the center of the circle is on the real axis, the circle intersects the axis at right angles. Since inversion preserves angles, the image of Γ is a vertical (Euclidean) line. Since vertical lines are lines in the model, and isometries preserve arclength, it follows that Γ is a line through P and Q. ∎

PROBLEM: Let $P = 4 + 4i$ and $Q = 5 + 3i$. We want to find M, N, and the distance from P to Q.

First we need to find Γ. We need to find the perpendicular bisector of the segment PQ and then find where this intersects the real axis. The midpoint of PQ is the point $(9+7i)/2$, or $(9/2, 7/2)$. The equation of the line through PQ is $y = 8 - x$. Thus, the equation of the perpendicular bisector is $y = x - 1$. This intersects the x-axis at $x = 1$, so the center of the circle is $1 + 0i$. The circle has to go through the points $4 + 4i$ and $5 + 3i$. Thus the radius is 5, using the Pythagorean theorem. Hence, the circle meets the x-axis at $M = -4 + 0i$ and $N = 6 + 0i$.

We need to translate the line Γ so that M goes to the origin. Thus, we need to translate by 4 and we need to apply the isometry $T_4\colon (x, y) \to (x + 4, y)$. Then, $P' = T_4(P) = (8, 4)$ and $Q' = T_4(Q) = (9, 3)$. Now, we need to invert in the unit circle and need to find the images of P' and Q'. We know what Φ does:

$$\Phi(P') = \Phi((8,4)) = \left(\frac{8}{80}, \frac{4}{80}\right) = \left(\frac{1}{10}, \frac{1}{20}\right)$$
$$\Phi(Q') = \Phi((9,3)) = \left(\frac{9}{90}, \frac{3}{90}\right) = \left(\frac{1}{10}, \frac{1}{30}\right)$$

Note that we now have these two images on a vertical (Euclidean) line. So the distance between the points $d_P(\Phi(P'), \Phi(Q')) = \log(1/20) - \log(1/30) = \log(3/2)$. Thus, the points P and Q are the same distance apart.

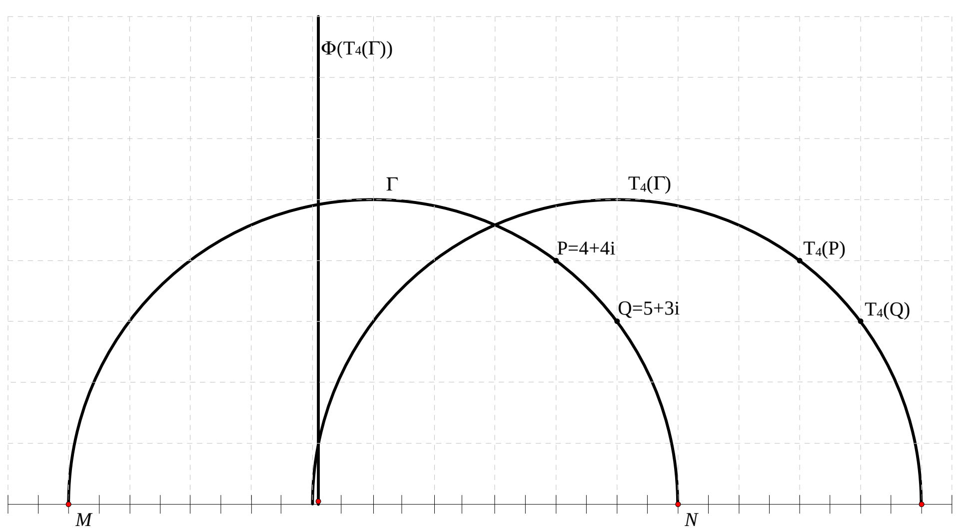


Figure 10.1: Isometries in $\mathscr{H}$

10.5 Fractional Linear Transformations

We want to be able to classify all of the isometries of the Poincaré half plane. It turns out that the group of direct isometries is easy to describe. We will describe them and then see why they are isometries.

A *fractional linear transformation* is a function of the form

$$T(z) = \frac{az+b}{cz+d}$$

where a, b, c, and d are complex numbers and $ad - bc \neq 0$. The domain of this function is the set of all complex numbers $\mathbb{C}$ together with the symbol, ∞, which will represent a point at infinity. Extend the definition of T to include the following

$$\begin{aligned} T(-d/c) &= \lim_{z \to -\frac{d}{c}} \frac{az+b}{cz+d} = \infty, \quad \text{if } c \neq 0, \\ T(\infty) &= \lim_{z \to \infty} \frac{az+b}{cz+d} = \frac{a}{c} \quad \text{if } c \neq 0, \\ T(\infty) &= \lim_{z \to \infty} \frac{az+b}{cz+d} = \infty \quad \text{if } c = 0. \end{aligned}$$

The fractional linear transformation, T, is usually represented by a 2×2 matrix

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and write $T = T_\gamma$. The matrix representation for T is not unique, since T is also represented by

$$k\gamma = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

for any scalar $k \neq 0$. We define two matrices to be *equivalent* if they represent the same fractional linear transformation. We will write $\gamma \equiv \gamma'$.

Lemma 10.5

$$T_{\gamma_1\gamma_2} = T_{\gamma_1}(T_{\gamma_2}(z)).$$

From this the following theorem follows.

Theorem 10.1 *The set of fractional linear transformations forms a group under composition (matrix-multiplication).*

Proof: Theorem 10.5 shows us that this set is closed under our operation. The identity element is given by the identity matrix,

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The fractional linear transformation associated with this is

$$T_I(z) = \frac{z+0}{0z+1} = z.$$

The inverse of an element is

$$T_\gamma^{-1} = T_{\gamma^{-1}},$$

since

$$T_\gamma(T_{\gamma^{-1}}(z)) = T_I(z) = z.$$

We can also see that to find ${T_\gamma}^{-1}$ we set $w = T_\gamma(z)$ and solve for z.

$$\begin{aligned} w &= \frac{az+b}{cz+d} \\ (cz+d)w &= az+b \\ z &= \frac{dw-b}{-cw+a}. \end{aligned}$$

That is ${T_\gamma}^{-1}$ is represented by

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \equiv \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \gamma^{-1}.$$

Here we must use the condition that $ad - bc \neq 0$. ∎

In mathematical circles when we have such an interplay between two objects — matrices and fractional linear transformations — we write γz when $T_\gamma(z)$ is meant. Under this convention we may write

$$\gamma z = \begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \frac{az+b}{cz+d}.$$

Note that the second "=" is not *equals* in the usual sense, but is instead an assignment or a definition.

This follows the result of Theorem 10.5 in that

$$(\gamma_1\gamma_2)z = \gamma_1(\gamma_2 z),$$

however in general $k(\gamma z) \neq (k\gamma)z$. Note that

$$k(\gamma z) = \frac{k(az+b)}{cz+d},$$

while

$$(k\gamma z) = \gamma z = \frac{az+b}{cz+d}.$$

Recall the following definitions for any ring R:

$$\begin{aligned} M_{2\times 2}(R) &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a,b,c,d \in R \right\} \\ \mathrm{GL}_2(R) &= \{\gamma \in M_{2\times 2}(R) \mid \det(\gamma) \neq 0\} \\ \mathrm{SL}_2(R) &= \{\gamma \in \mathrm{GL}_2(R) \mid \det(\gamma) = 1\}. \end{aligned}$$

For the purposes of what we will be doing, we prefer the ring to be the field of complex numbers, $\mathbb{C}$, the field of real numbers, $\mathbb{R}$, the field of rational numbers, $\mathbb{Q}$, or the ring of integers $\mathbb{Z}$. $\mathrm{GL}_2(R)$ is called the *general linear group* over R, and $\mathrm{SL}_2(R)$ is called the *special linear group* over R.

There is another group, which is not as well known. This is the *projective special linear group* denoted by $\mathrm{PSL}_2(R)$. $\mathrm{PSL}_2(R)$ is obtained from $\mathrm{GL}_2(R)$ by identifying γ with $k\gamma$ for any $k \neq 0$. The group $\mathrm{PSL}_2(\mathbb{C})$ is isomorphic to the group of fractional linear transformations.

Remember that we wanted to classify the group of direct isometries on the upper half plane. We want to show that any 2×2 matrix with real coefficients and determinant 1 represents a fractional linear transformation which is an isometry of the Poincaré upper half plane.

Lemma 10.6 *The horizontal translation by a*

$$T_a(x, y) = (x + a, y),$$

can be thought of as a fractional linear transformation, represented by an element of $\mathrm{SL}_2(\mathbb{R})$.

PROOF: If $a \in \mathbb{R}$, then

$$T_a(x, y) = T_a(z) = z + a, \quad z \in \mathbb{C},$$

and this is represented by

$$\tau_a = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}.$$

This is what we needed. ∎

Lemma 10.7 *The map*

$$\varphi(x, y) = \left(\frac{-x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right),$$

which is inversion in the unit circle followed by reflection through $x = 0$, can be thought of as a fractional linear transformation which is represented by an element of $\mathrm{SL}_2(\mathbb{R})$.

PROOF: As a function of complex numbers, the map φ is

$$\varphi(z) = \varphi(x + iy) = \frac{-x + iy}{x^2 + y^2} = \frac{-(x - iy)}{(x + iy)(x - iy)} = -\frac{1}{z}.$$

This map is generated by

$$\sigma = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

∎

Theorem 10.2 *The group* $\mathrm{SL}_2(\mathbb{R})$ *is generated by σ and the maps τ_a for $a \in \mathbb{R}$.*

PROOF: Note that

$$\begin{aligned} \sigma\tau_r &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & r \end{bmatrix} \end{aligned}$$

so

$$\sigma\tau_s\sigma\tau_r = \begin{bmatrix} 0 & -1 \\ 1 & s \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & r \end{bmatrix} = \begin{bmatrix} -1 & -r \\ s & rs-1 \end{bmatrix}$$

and

$$\sigma\tau_t\sigma\tau_s\sigma\tau_r = \begin{bmatrix} 0 & -1 \\ 1 & t \end{bmatrix}\begin{bmatrix} -1 & -r \\ s & rs-1 \end{bmatrix} = \begin{bmatrix} -s & 1-rs \\ st-1 & rst-r-t \end{bmatrix}$$

Now, we know what the composition of these transformations will look like. To say that these generate SL)$_2(\mathbb{R})$ means is that for an arbitrary

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL})_2(\mathbb{R})$$

we need to find $r, s, t \in \mathbb{R}$ so that

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -s & 1-rs \\ st-1 & rst-r-t \end{bmatrix}.$$

This cannot be too difficult for $a \neq 0$. Set $s = -a$, solve $b = 1 - rs = 1 + ra$ for r and solve $c = st - 1 = -at - 1$ for t. This gives

$$r = \frac{b-1}{a} \quad \text{and} \quad t = \frac{-1-c}{a}.$$

Since $\det(\gamma) = 1$, this forces $d = rst - r - t$. Thus, for $a \neq 0$, then γ can be written as a product involving only σ and translations. If $a = 0$, then $c \neq 0$, since $ad - bc = 1$, and

$$\sigma\gamma = \begin{bmatrix} -c & -d \\ a & b \end{bmatrix},$$

which can be written as a suitable product. Thus $\mathrm{SL}_2(\mathbb{R})$ is generated by the translations and σ. ∎

Lemma 10.8 *The group $\mathrm{SL}_2(\mathbb{R})$, when thought of as a group of fractional linear transformations, is a subgroup of the isometries of the Poincaré upper half plane.*

Lemma 10.9 *If $\gamma \in \mathrm{GL}_2(\mathbb{R})$ and $\det\gamma > 0$, then γ is an isometry of the Poincaré upper half plane.*

Theorem 10.3 *The image of a circle or line in $\mathbb{C}$ under the action of a fractional linear transformation $\gamma \in \mathrm{SL}_2(\mathbb{C})$ is again a circle or a line.*

10.6 Cross Ratio

Henri Poincaré was studying this *cross ratio* when he discovered this particular representation of the hyperbolic plane.

Let a, b, c, d be elements of the extended complex numbers, $\mathbb{C}\bigcup\{\infty\}$, at least three of which are distinct. The *cross ratio* of a, b, c, and d is defined to be

$$[a, b; c, d] = \frac{\dfrac{a-c}{a-d}}{\dfrac{b-c}{b-d}} = \frac{(a-c)(b-d)}{(a-d)(b-c)}.$$

The algebra for the element ∞ and division by zero is the same as it is for fractional linear transformations.

$$\begin{aligned}
\text{If } a = d,\ \ [a, b; c, d] &= \infty \\
[\infty, b; c, d] &= \frac{b-d}{b-c} = [a, b; c, \infty] \\
[a, \infty; c, d] &= \frac{a-c}{a-d} = [a, b; \infty, d]
\end{aligned}$$

If we fix three distinct elements a, b, and $c \in \mathbb{C}\bigcup\{\infty\}$, and consider the fourth element as a variable z, then we get a fractional linear transformation:

$$T(z) = (z, a; b, c) = \frac{\dfrac{z-b}{z-c}}{\dfrac{a-b}{a-c}}.$$

This is the unique fractional linear transformation T with the property that

$$T(a) = 1, \quad T(b) = 0, \quad \text{and} \quad T(c) = \infty,$$

that is, it sends a to 1, b to 0 and c to ∞.

We need to look at several examples to see why we want to use the cross ratio.

Example 10.1 Find the fractional linear transformation which sends 1 to 1, $-i$ to 0 and -1 to ∞.

From above we need to take: $a = 1$, $b = -i$, and $c = -1$. Thus, set

$$\begin{aligned}
w &= (z, 1; -i, -1) \\
&= \frac{z+i}{z+1} \Big/ \frac{1+i}{1+1} \\
&= \frac{2z+2i}{(1+i)(z+1)}
\end{aligned}$$

In matrix notation,

$$w = \begin{bmatrix} 2 & 2i \\ 1+i & 1+i \end{bmatrix} z.$$

Example 10.2 Find the fractional linear transformation which fixes i, sends ∞ to 3, and 0 to $-1/3$.

This doesn't seem to fit our model. However, we can combine two of our transformations to get this one. First think of sending

$$\begin{aligned} i &\to 1 & & & i &\to 1 \\ \infty &\to 0 & &\text{and} & 3 &\to 0 \\ 0 &\to \infty & & & -\frac{1}{3} &\to \infty \end{aligned}$$

If we go by the first transformation followed by the ***inverse*** of the second we will send:

$$\begin{aligned} i \to 1 \to &\quad i \\ \infty \to 0 \to &\quad 3 \\ 0 \to \infty \to &\ -\frac{1}{3} \end{aligned}$$

Let

$$\gamma_1 z = (z, i; \infty, 0)$$

and

$$\gamma_2 w = (w, i; 3, -1/3).$$

So, $\gamma_1(i) = 1$, $\gamma_1(\infty) = 0$, $\gamma_1(0) = \infty$, $\gamma_2(i) = 1$, $\gamma_2(3) = 0$, and $\gamma_2(-1/3) = \infty$. Therefore, $\gamma_2^{-1}(1) = i$, $\gamma_2^{-1}(0) = 3$, and $\gamma_2^{-1}(\infty) = -1/3$. Now, compose these functions:

$$\gamma = \gamma_2^{-1}\gamma_1.$$

Let's check what γ does: $\gamma(i) = i$, $\gamma(\infty) = 3$ and $\gamma(0) = -1/3$, as desired.

Now, set $w = \gamma(z)$ and

$$\begin{aligned} w &= \gamma_2^{-1}\gamma_1(z) \\ \gamma_2(w) &= \gamma_1(z) \\ (w, i; 3, -1/3) &= (z, i; \infty, 0). \end{aligned}$$

Now, we need to solve for z:

$$\begin{aligned} \frac{w-3}{w+1/3} \Big/ \frac{i-3}{i+1/3} &= \frac{z-\infty}{z-0} \Big/ \frac{i-\infty}{i-0} \\ \frac{(3i+1)w - 3(3i+1)}{3(i-3)w + i - 3} &= \frac{i}{z} \\ z &= \frac{(-3(3i+1)w - (3i+1)}{(3i+1)w - 3(3i+1)} \\ &= \frac{3w+1}{-w+3} \\ &= \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix} \end{aligned}$$

Then, using our identification, we will get that

$$w = \frac{1}{10}\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \equiv \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} z$$

10.7 Translations

Now, the Poincaré upper half plane is a model for the hyperbolic plane. We need to check this, but first let's look at what transformational properties we can discover with the machinery that we have built.

TRANSFORMATIONAL PROPERTY 1: Given any two points P and Q, there exists an isometry f such that $f(P) = Q$.

Let $P = a+bi$ and $Q = c+di$. We have many choices, but we will start with an isometry that also fixes the point at ∞. In some sense, this is a *nice* isometry, since it does not map any regular point to infinity nor infinity to any regular point. Now, since $f(\infty) = \infty$ and $f(P) = Q$, f must send the line through P and ∞ to the line through Q and ∞. This means that the vertical line at $x = a$ is sent to the vertical line at $x = c$. Thus, $f(a) = c$. This now means that we have to have

$$\begin{aligned}(w, c+di; c, \infty) &= (z, a+bi; a, \infty)\\ \frac{w-c}{di} &= \frac{z-a}{bi}\\ w &= \frac{d(z-a)}{b} + c\\ &= \begin{bmatrix} d & bc-ad \\ 0 & b \end{bmatrix} z.\end{aligned}$$

Since $b > 0$ and $d > 0$, then the determinant of this matrix is positive. That and the fact that all of the entries are real means that it is an element of $\mathrm{PSL}_2(\mathbb{R})$ and is an isometry of the Poincaré upper half plane.

We claim that this map that we have chosen is a translation. Now, a translations has no fixed points. How do we show that this map has no fixed points? A fixed point would be a point z_0 so that $f(z_0) = z_0$. If this is the case, then solve for z below:

$$\begin{aligned}\frac{d(z_0-a)}{b} + c &= z_0\\ z_0 &= \frac{ad-bc}{d-b}\end{aligned}$$

But, note that a, b, c, and d are all real numbers. Thus, if $b \neq d$ then z_0 is a real number and is **not** in the upper half plane. Thus, this map has no fixed points in $\mathscr{H}^2$ and is a translation. If $b = d$, then $z_0 = \infty$, and again there are no solutions in the upper half plane, so the map is a translation.

In the Poincaré upper half plane, we can classify our translations by how many fixed points there are on the line at infinity (that is, in $\mathbb{R} \bigcup \infty$.) Let

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then $\gamma(z) = z$ if

$$cz^2 + (d-a)z - b = 0.$$

Now, if $c \neq 0$, then this is a quadratic equation with discriminant

$$\Delta = (d-a)^2 - 4bc.$$

Thus, there is a fixed point in $\mathscr{H}^2$ if $\Delta < 0$, and no fixed points if $\Delta \geq 0$. If $\Delta = 0$ then there is exactly one fixed point on the line at infinity. In this case the translation is called a *parabolic translation.* If $\Delta > 0$ the translation is called a *hyperbolic translation.*

10.8 Rotations

What are the rotations in the Poincaré upper half plane? What fractional linear transformations represent rotations?

A rotation will fix only one point. Let $P = a + bi$. We want to find the rotation that fixes P and rotates counterclockwise through an angle of θ.

First, find the (Euclidean) line through P which makes an angle θ with the vertical line through P. Find the perpendicular to this line, and find where it intersects the x-axis. The circle centered at this intersection and through P is the image of the vertical line under the rotation. Let this circle intersect the x-axis at points M and N. Then the rotation is given by

$$(w, P; N, M) = (z, P; a, \infty).$$

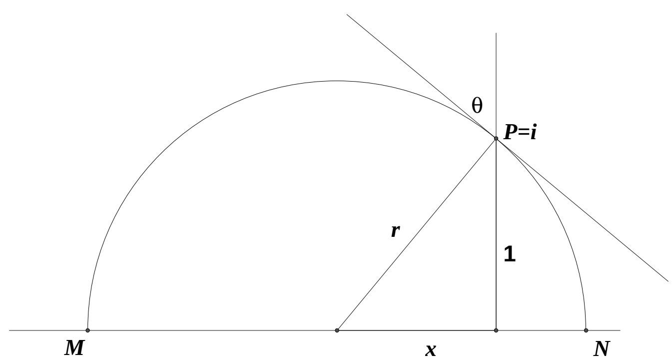
We want to find an easy point to rotate, then we can do this in general. It turns out that the simplest case is to rotate about $P = i$.

Here let the center of the half circle be at $-x$, and let the (Euclidean) radius of the circle be r. Then $x = r\cos\theta$, $r\sin\theta = 1$, $M = -r - x$, and $N = r - x$. So we have to solve

$$(w, i; r - x, -r - x) = (z, i; 0, \infty).$$

After quite a bit of algebraic manipulation, we get

$$w = \rho_\theta z = \begin{bmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} z$$



For an arbitrary point $P = a + bi$ we need to apply a translation that sends P to i and then apply the rotation, and then translate back. The translation from $P = a + bi$ to $0 + i$ is

$$\gamma = \begin{bmatrix} 1 & -a \\ 0 & b \end{bmatrix}.$$

The inverse translation is

$$\gamma^{-1} = \begin{bmatrix} b & a \\ 0 & 1 \end{bmatrix}.$$

Thus, the rotation about P is

$$\begin{aligned} \gamma^{-1}\rho_\theta\gamma &= \begin{bmatrix} b & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} \begin{bmatrix} 1 & -a \\ 0 & b \end{bmatrix} \\ &= \begin{bmatrix} b\cos\frac{\theta}{2} - a\sin\frac{\theta}{2} & (a^2 + b^2)\sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & a\sin\frac{\theta}{2} + b\cos\frac{\theta}{2} \end{bmatrix} \end{aligned}$$

10.9 Reflections

We have not yet described all of the isometries of the Poincaré upper half plane. We did see that the reflection through the imaginary axis is given by

$$R_0(x, y) = (-x, y),$$

which is expressed in complex coordinates as

$$R_0(z) = -\overline{z}.$$

Note that in terms of a matrix representation, we can represent $R_0(z)$ by

$$R_0(z) = \mu\overline{z} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \overline{z}$$

Now, to reflect through the line ℓ in $\mathscr{H}^2$, first use the appropriate isometry, γ_1 to move the line ℓ to the imaginary axis, then reflect and move the imaginary axis back to ℓ:

$$\gamma_1^{-1}\mu\overline{\gamma_1 z} = \gamma_1^{-1}\mu\gamma_1\overline{z}.$$

Note that $\mu^2 = 1$ and that $\mu\gamma\mu \in \mathrm{SL}_2(\mathbb{R})$ for all $\gamma \in \mathrm{SL}_2(\mathbb{R})$, since $\det\mu = -1$. Therefore,

$$\gamma_1^{-1}\mu\gamma_1\overline{z} = \gamma_1^{-1}(\mu\gamma_1\mu)\mu\overline{z} = \gamma_2\mu\overline{z} = \gamma_2(-\overline{z}),$$

where $\gamma_2 \in \mathrm{SL}_2(\mathbb{R})$. Thus, every reflection can be written in the form $\gamma(-\overline{z})$ for some $\gamma \in \mathrm{SL}_2(\mathbb{R})$.

Theorem 10.4 *Every isometry f of $\mathscr{H}$ which is not direct can be written in the form*

$$f(z) = \gamma(-\overline{z})$$

for some $\gamma \in \mathrm{SL}_2(\mathbb{R})$. Furthermore, if

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then $f(z)$ is a reflection if and only if $a = d$.

10.10 Distance and Lengths

We want a formula for the distance between two points or the length of any line segment. We know this for two points on the same vertical line: if $P = a + bi$ and $Q = a + ci$, then

$$\begin{aligned} |PQ| &= \left| \int_b^c \frac{dy}{y} \right| \\ &= |\log(c/b)| \end{aligned}$$

Now, what will happen if P and Q don't lie on a vertical line segment. Then there is a half circle with center on the x-axis which goes through both P and Q. Let this half circle have endpoints M and N. Since isometries preserve distance, we will look at the image of

σ which sends P to i and PQ to a vertical line. This is the transformation that sends P to i, M to 0 and N to ∞. Since the image of Q will lie on this line, Q is sent to some point $0 + ci$ for some c. Then

$$|PQ| = |\log(c/1)| = |\log(c)|.$$

Note that

$$(\sigma z, i; 0, \infty) = (z, P; M, N)$$

and in particular, since $\sigma(Q) = ci$ and $(\sigma z, i; 0, \infty) = \dfrac{\sigma z}{i}$, we get

$$c = (Q, P; M, N),$$

so

$$|PQ| = |\log(Q, P; M, N)|.$$

10.11 The Area of Triangles

Let's compute the area of a doubly asymptotic triangle. We want to compute the area of the doubly asymptotic triangle with vertices at $P = e^{i(\pi-\theta)}$ in $\mathscr{H}$, and vertices at infinity of 1 and ∞. The angle at P for this doubly asymptotic triangle has measure θ. Consider Figure 10.2.

The area element for the Poincaré upper half plane model is derived by taking a small (Euclidean) rectangle with sides oriented horizontally and vertically. The sides approximate hyperbolic segments, since the rectangle is very small. The area would then be a product of the height and width (measured with the hyperbolic arclength element). The vertical sides of the rectangle have Euclidean length Δy, and since y is essentially unchanged, the hyperbolic length is $\dfrac{\Delta y}{y}$. The horizontal sides have Euclidean length Δx and hence hyperbolic length $\dfrac{\Delta x}{y}$. This means that the area element is given by $\dfrac{dxdy}{y^2}$.

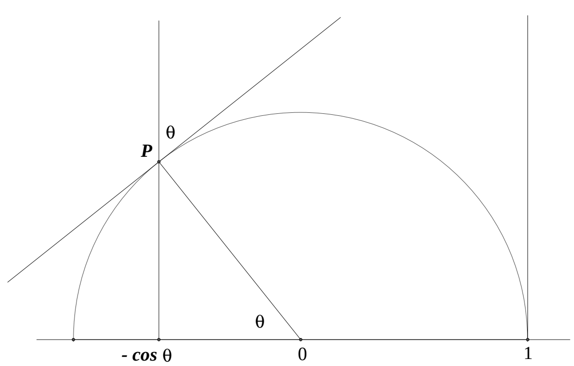


Figure 10.2: Doubly Asymptotic Triangle

Lemma 10.10 *The area of a doubly asymptotic triangle $P\Omega\Theta$ with points Ω and Θ at infinity and with angle $\Omega P \Theta = P$ has area*

$$|\triangle P\Omega\Theta| = \pi - P,$$

where P is measured in radians.

PROOF: Let the angle at P have measure θ. Then $\triangle P\Omega\Theta$ is similar to the triangle in Figure 10.2 and is hence congruent to it. Thus, they have the same area. The area of the triangle in

Figure 10.2 is given by

$$\begin{aligned} A(\theta) &= \int_{-\cos\theta}^{1} \int_{\sqrt{1-x^2}}^{\infty} \frac{1}{y^2}\, dx dy \\ &= \int_{-\cos\theta}^{1} \frac{dx}{\sqrt{1-x^2}} \\ &= \arccos(-x)|_{-\cos\theta}^{1} = \pi - \theta \end{aligned}$$

∎

Corollary 4 *The area of a trebly asymptotic triangle is π.*

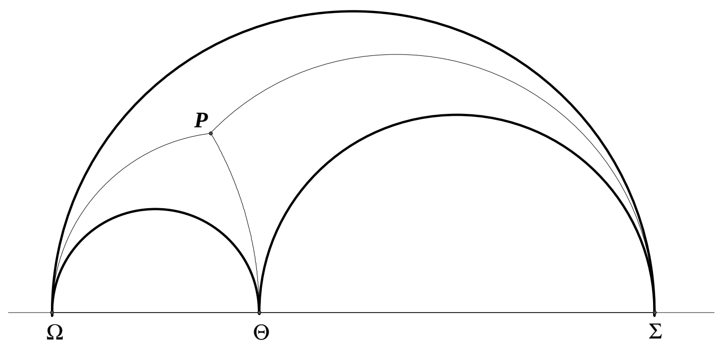


Figure 10.3: Trebly Asymptotic Triangle

Proof: : Let $\triangle\Omega\Theta\Sigma$ be a trebly asymptotic triangle, and let P be a point in the interior. Then

$$\begin{aligned} |\triangle\Omega\Theta\Sigma| &= |\triangle P\Omega\Sigma| + |\triangle P\Theta\Sigma| + |\triangle P\Omega\Theta| \\ &= (\pi - \angle\Omega P\Sigma) + (\pi - \angle\Theta P\Sigma) + (\pi - \angle\Omega P\Theta) \\ &= 3\pi - 2\pi = \pi \end{aligned}$$

∎

Corollary 5 *Let $\triangle ABC$ be a triangle in $\mathscr{H}$ with angle measures A, B, and C. Then the area of $\triangle ABC$ is*

$$|\triangle ABC| = \pi - (A + B + C),$$

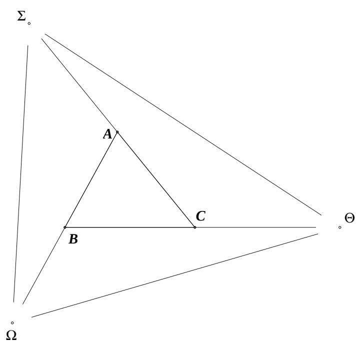
where the angles are measured in radians.

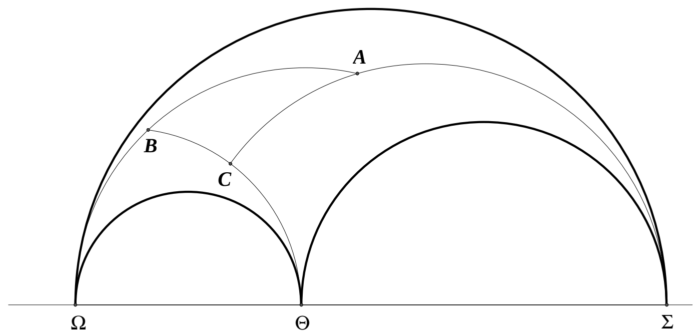
In the figure below, the figure on the left is just an abstract picture from the hyperbolic plane. The figure on the right comes from the Poincaré model, $\mathscr{H}$.

Proof: Construct the triangle $\triangle ABC$ and continue the sides as rays AB, BC, and CA. Let these approach the ideal points Ω, Θ, and Σ, respectively. Now, construct the common parallels $\Omega\Theta$, $\Theta\Sigma$, and $\Sigma\Omega$. These form a trebly asymptotic triangle whose area is π. Thus,

$$\begin{aligned} |\triangle ABC| &= \pi - |\triangle A\Sigma\Omega| - |\triangle B\Omega\Theta| - |\triangle C\Theta\Sigma| \\ &= \pi - (\pi - (\pi - A)) - (\pi - (\pi - B)) - (\pi - (\pi - C)) \\ &= \pi - (A + B + C). \end{aligned}$$

∎





10.12 Connection to the Poincaré Disk Model

Consider the fractional linear transformation in matrix form

$$\phi = \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$$

or

$$w = \frac{z-i}{1-iz}.$$

This map sends 0 to $-i$, 1 to 1, and ∞ to i. This map sends the upper half plane to the interior of the unit disk. The image of $\mathscr{H}$ under this map is the Poincaré disk model, $\mathscr{D}$.

Under this map lines and circles perpendicular to the real line are sent to circles which are perpendicular to the boundary of $\mathscr{D}$. Thus, hyperbolic lines in the Poincaré disk model are the portions of Euclidean circles in $\mathscr{D}$ which are perpendicular to the boundary of $\mathscr{D}$.

There are several ways to deal with points in this model. We can express points in terms of polar coordinates:

$$\mathscr{D} = \{re^{i\theta} \mid 0 \leq r < 1\}.$$

We can show that the arclength segment is

$$ds = \frac{2\sqrt{dr^2 + r^2 d\theta^2}}{1-r^2}.$$

The group of proper isometries in $\mathscr{D}$ has a description similar to the description on $\mathscr{H}$. It is the group

$$\Gamma = \left\{ \gamma \in \mathrm{SL}_2(\mathbb{C}) \mid \gamma = \begin{bmatrix} a & b \\ \overline{b} & \overline{a} \end{bmatrix} \right\}$$

All improper isometries of $\mathscr{D}$ can be written in the form $\gamma(-\overline{z})$ where $\gamma \in \Gamma$.

Lemma 10.11 *If $d_p(O,B) = x$, then*

$$d(O,B) = \frac{e^x - 1}{e^x + 1}.$$

PROOF: If Ω and Λ are the ends of the diameter through OB then

$$\begin{aligned} x &= \log(O, B; \Omega, \Lambda) \\ e^x &= \frac{O\Omega \cdot B\Lambda}{O\Lambda \cdot B\Omega} \\ &= \frac{B\Lambda}{B\Omega} = \frac{1+OB}{1-OB} \\ OB &= \frac{e^x+1}{e^x-1} \end{aligned}$$

which is what was to be proven. ∎

10.13 Angle of Parallelism

Let $\Pi(d)$ denote the radian measure of the angle of parallelism corresponding to the hyperbolic distance d. We can define the standard trigonometric functions, not as before—using right triangles—but in a standard way. Define

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \tag{10.1}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \tag{10.2}$$

$$\tan x = \frac{\sin x}{\cos x}. \tag{10.3}$$

In this way we have avoided the problem of the lack of similarity in triangles, the premise upon which all of real Euclidean trigonometry is based. What we have done is to define these functions *analytically*, in terms of a power series expansion. These functions are defined for all real numbers x and satisfy the usual properties of the trigonometric functions.

Theorem 10.5 (Bolyai-Lobachevskii Theorem) *In the Poincaré model of hyperbolic geometry the angle of parallelism satisfies the equation*

$$e^{-\rho} = \tan\left(\frac{\Pi(\rho)}{2}\right).$$

PROOF: By the definition of the angle of parallelism, $d = d_p(P, Q)$ for some point P to its foot Q in some p-line ℓ. Now, $\Pi(d)$ is half of the radian measure of the fan angle at P, or is the radian measure of $\angle QP\Omega$, where $P\Omega$ is the limiting parallel ray to ℓ through P.

We may choose ℓ to be a diameter of the unit disk and $Q = O$, the center of the disk, so that P lies on a diameter of the disk perpendicular to ℓ.

The limiting parallel ray through P is the arc of a circle δ so that

(i) δ is orthogonal to Γ,

1. ℓ is tangent to δ at Ω.

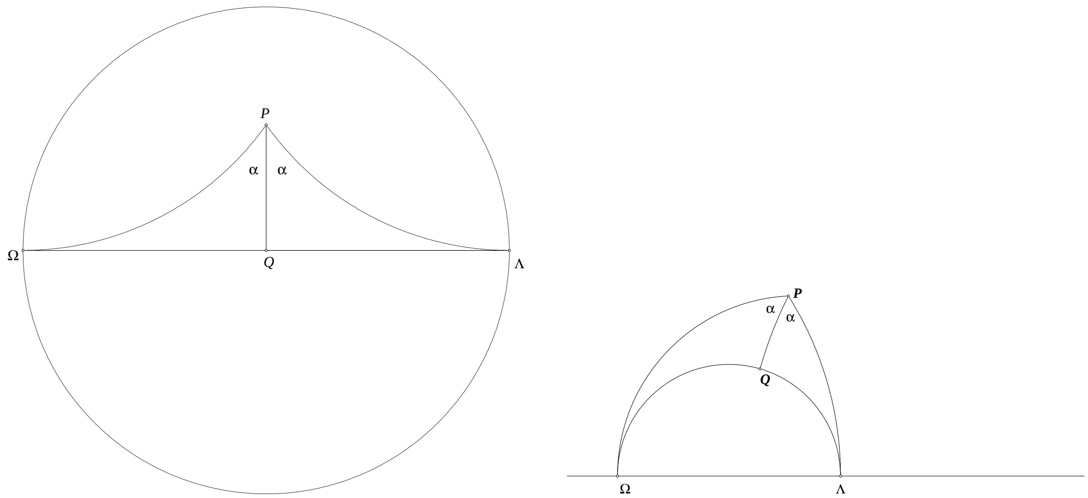


Figure 10.4: Angle of Parallelism: left in $\mathscr{D}$, right in $\mathscr{H}$

The tangent line to δ at P must meet ℓ at a point R inside the disk. Now $\angle QP\Omega = \angle Q\Omega P = \beta$ radians. Let us denote $\Pi(d) = \alpha$. Then in $\triangle PQ\Omega$, $\alpha + 2\beta = \frac{\pi}{2}$ or

$$\beta = \frac{\pi}{4} - \frac{\alpha}{2}.$$

Now, $d(P,Q) = r\tan\beta = r\tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)$. Applying Lemma 10.11 we have

$$e^d = \frac{r + d(P,Q)}{r - d(P,Q)} = \frac{1 + \tan\beta}{1 - \tan\beta}.$$

Using the identity $\tan(\frac{\pi}{4} - \frac{\alpha}{2}) = \dfrac{1 - \tan\alpha/2}{1 + \tan\alpha/2}$ it follows that

$$e^d = \frac{1}{\tan\alpha/2}.$$

Simplifying this it becomes

$$e^{-d} = \tan\left(\frac{\Pi(d)}{2}\right).$$

Also, we can write this as $\Pi(d) = 2\arctan(e^{-d})$. ∎