Chapter 12

Hyperbolic Trigonometry

Trigonometry is the study of the relationships among sides and angles of a triangle. In Euclidean geometry we use similar triangles to define the trigonometric functions—but the theory of similar triangles in not valid in hyperbolic geometry. Thus, we must them in terms of their power series expansion for any real number, as in Equations 10.1.

We define the hyperbolic trigonometric functions

$$\sinh x = \frac{e^x - e^{-x}}{2}$$
$$\cosh x = \frac{e^x + e^{-x}}{2}$$
$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all $x \in \mathbb{R}$, the power series expansions of the hyperbolic trigonometric functions are

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$
$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

and converge for all real x. In fact, using complex analysis and letting $i = \sqrt{-1}$, we can easily see that

$$\sinh x = -i\sin(ix) = i\sin\left(\frac{x}{i}\right)$$
$$\cosh x = \cos(ix) = \cos\left(\frac{x}{i}\right)$$

There are also the usual collections of hyperbolic trigonometry identities:

$$\cosh^2 x - \sinh^2 x = 1$$
$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$
$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$

A straightforward calculation using double angle formulas for the circular functions gives the following formulas:

$$\sin \Pi(x) = \operatorname{sech} x \tag{12.1}$$
$$\cos \Pi(x) = \tanh x \tag{12.2}$$
$$\tan \Pi(x) = \operatorname{csch} x \tag{12.3}$$

For example, to derive the first equation:

$$\begin{aligned}\sin \Pi(x) &= \sin(2 \arctan e^{-x}) \\ &= 2 \sin(\arctan e^{-x}) \cos(\arctan e^{-x}) \\ &= 2\left(\frac{e^{-x}}{1+e^{-2x}}\right) = 2\left(\frac{1}{e^{x}+e^{-x}}\right) \\ &= \frac{1}{\cosh x} = \operatorname{sech} x.\end{aligned}$$

This function $\Pi : (0, \frac{\pi}{2}) \to \mathbb{H}^2$ provides a connection between the hyperbolic and circular functions.

Given $\triangle ABC$, let $a = d_p(B, C)$, $b = d_p(A, C)$, and $c = d_p(A, B)$. First, we will derive some formulæof hyperbolic geometry. Let $B \in \mathbb{H}^2$. Let $x = d_p(O, B)$ be the Poincaré distance from O to B and let $t = d(O, B)$ be the Euclidean distance from O to B. From Lemma 10.11 we have

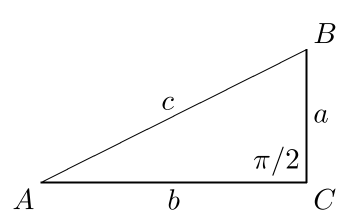
$$e^x = \frac{1+t}{1-t}.$$

We then get that

$$\begin{aligned}\sinh x &= \frac{2t}{1-t^2} \\ \cosh x &= \frac{1+t^2}{1-t^2} \\ \tanh x &= \frac{2t}{1+t^2}.\end{aligned}$$

Theorem 12.1 *Given any right triangle $\triangle ABC$ with $\angle C$ the right angle (having measure $\pi/2$, then*

$$\sin A = \frac{\sinh a}{\sinh c} \qquad \cos A = \frac{\tanh b}{\tanh c} \tag{12.4}$$
$$\cosh c = \cosh a \cosh b = \cot A \cot B \tag{12.5}$$
$$\cosh a = \frac{\cos A}{\sin B} \tag{12.6}$$



Before we prove these equations, compare them with the formulæ for a right triangle in Euclidean geometry.

1. Equation 12.5 is the hyperbolic analogue of the Pythagorean theorem.

$$\begin{aligned}\cosh c &= \cosh a \cosh b \\ 1 + \frac{c^2}{2!} + \frac{c^4}{4!} + \cdots &= \left(1 + \frac{a^2}{2!} + \frac{a^4}{4!} + \cdots\right) \cdot \left(1 + \frac{b^2}{2!} + \frac{b^4}{4!} + \cdots\right) \\ &= 1 + \left(\frac{a^2}{2!} + \frac{b^2}{2!}\right) + \cdots\end{aligned}$$

Thus, if $\triangle ABC$ is sufficiently small so that higher powers of a, b, and c are negligible, then

$$\begin{aligned}1 + \frac{c^2}{2} &\approx 1 + \frac{1}{2}(a^2 + b^2) \\ c^2 &\approx a^2 + b^2\end{aligned}$$

2. Equation 12.4 says that for $\triangle ABC$ sufficiently small $\sin A \approx \frac{a}{c}$ and $\cos A \approx \frac{b}{c}$. How close are these approximations? Consider right triangles with $\angle A$ fixed and let c approach 0. Since $a < c$, $a \to 0$.

$$\frac{1}{\sinh x} \approx \frac{1}{c + \frac{c^3}{3!}} = \frac{1}{c}\frac{1}{1+u} = \frac{1}{c}(1 - u + u^2 - u^3 + \ldots)$$

where $\lim\limits_{c \to 0} = 0$. Thus,

$$\begin{aligned}\frac{\sinh a}{\sinh c} &= \frac{a}{c}(1 + \frac{a^2}{3!} + \frac{a^4}{5!} + \cdots)(1 - u + u^2 - u^3 + \cdots) \\ &\approx \frac{a}{c}\end{aligned}$$

Thus,

$$\lim_{c \to 0} \frac{a}{c} = \lim_{c \to 0} \frac{\sinh a}{\sinh c} = \sin A.$$

3. Equation 12.6 and the second equality in Equation 12.5 have no Euclidean parallels for there the angles do not determine the lengths of the sides.

First recall that a point $P = ie^p$ in $\mathscr{H}$ is a distance p from i. Also, remember that

$$\phi = \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$$

sends $\mathscr{H}$ to $\mathscr{D}$. In this remember that $\phi(i) = 0$, and note that

$$\phi(ie^p) = \frac{ie^p - i}{e^p + 1} = i \tanh(p/2).$$

Thus, a point that is a hyperbolic distance p away from zero in $\mathscr{D}$ is a Euclidean distance $\tanh(p/2)$ away from zero.

Thus, we may choose our right triangle $\triangle ABC$ with right angle C and sides of length $|AC| = b$ and $|BC| = a$ to be the triangle in the Poincaré disk model $\mathscr{D}$ with vertices $C = 0$, $A = \tanh(b/2)$ and $B = i\tanh(a/2)$.

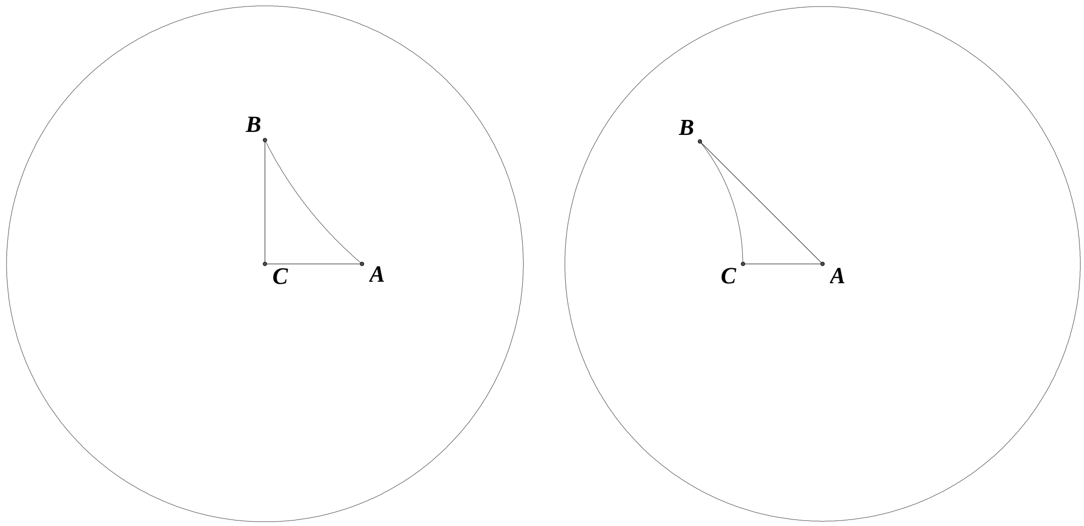


Figure 12.1: Right triangles in $\mathscr{D}$

However, for computation purposes, it turns out that we will be better off by making $A = 0$ as in the right hand figure above. To do this we need to find a direct isometry γ of $\mathscr{D}$ which sends A to 0 and the line AC to itself. So, we want $\gamma(A) = 0$, $\gamma(1) = 1$, and $\gamma(-1) = -1$. Then

$$(z, A; 1, -1) = (\gamma z, 0; 1, -1)$$

from which it follows that

$$\gamma = \begin{bmatrix} -1 & A \\ A & -1 \end{bmatrix}$$

Now, apply γ to B.

$$\gamma(B) = \frac{-B + A}{AB - 1} = \frac{-i\tanh(a/2) + \tanh(b/2)}{i\tanh(a/2)\tanh(b/2) - 1}.$$

Thus, for our proof we may use the triangle $\triangle ABC$ in $\mathscr{D}$ with

$$\begin{aligned} A &= 0 \\ B &= \frac{-i\tanh(a/2) + \tanh(b/2)}{i\tanh(a/2)\tanh(b/2) - 1} \\ C &= -\tanh(b/2). \end{aligned}$$

Proof: To prove the Hyperbolic Pythagorean Theorem, we may assume that $A = O$ is the origin. Since B is a distance $\tanh(c/2)$ away from zero, we get

$$\begin{aligned} \tanh(c/2) &= \left| \frac{-B + A}{AB - 1} \right| \\ \tanh^2(c/2) &= \frac{\tanh^2(a/2) + \tanh^2(b/2)}{\tanh^2(a/2)\tanh^2(b/2) + 1} \end{aligned}$$

From our hyperbolic trigonometry identities, you can easily show that $\operatorname{sech}^2 x = 1 - \tanh^2 x$, so

$$\begin{aligned}
\operatorname{sech}^2(c/2) &= 1 - \tanh^2(c/2) \\
&= \frac{\tanh^2(a/2)\tanh^2(b/2) - \tanh^2(a/2) - \tanh^2(b/2) + 1}{\tanh^2(a/2)\tanh^2(b/2) + 1} \\
&= \frac{(\tanh^2(a/2) - 1)(\tanh^2(b/2) - 1)}{\tanh^2(a/2)\tanh^2(b/2) + 1} \\
\cosh^2(c/2) &= \frac{(\tanh^2(a/2)\tanh^2(b/2) + 1)}{\operatorname{sech}^2(a/2)\operatorname{sech}^2(b/2)} \\
&= (\tanh^2(a/2)\tanh^2(b/2) + 1)\cosh^2(a/2)\cosh^2(b/2) \\
&= \sinh^2(a/2)\sinh^2(b/2) + \cosh^2(a/2)\cosh^2(b/2) \\
&= 2\cosh^2(a/2)\cosh^2(b/2) - \cosh^2(a/2) - \cosh^2(b/2) + 1.
\end{aligned}$$

Again, using the hyperbolic trigonometry identity $\cosh 2x = 2\cosh^2 x - 1$, we have

$$\begin{aligned}
\cosh c &= 2\cosh^2(c/2) - 1 \\
&= 4\cosh^2(a/2)\cosh^2(b/2) - 2\cosh^2(a/2) - 2\cosh^(b/2) + 1 \\
&= (2\cosh^2(a/2) - 1)(2\cosh^2(b/2) - 1) \\
&= \cosh a \cosh b
\end{aligned}$$

Since A sits at the origin and angles are measured with a Euclidean protractor, we can find the angle at A using regular trigonometry. We just need to know what the lengths of the sides are

First, we need to find the coordinates for B. Rationalize the denominator for B.

$$B = \frac{(\tanh^2(a/2) + 1)\tanh(b/2) + i\tanh(a/2)(\tanh^2(b/2) - 1)}{\tanh^2(a/2)\tanh^2(b/2) + 1}$$

Note that

$$\begin{aligned}
\tanh^2(b/2) - 1 &= -\operatorname{sech}^2(b/2) \\
\tanh^2(a/2) + 1 &= \frac{\sinh^2(a/2) + \cosh^2(a/2)}{\cosh^2(a/2)} = \frac{\cosh a}{\cosh^2(a/2)}
\end{aligned}$$

Thus,

$$\begin{aligned}
B &= \frac{(\cosh a \tanh(b/2)\cosh^2(b/2) - i\tanh(a/2)\cosh^2(a/2)}{\sinh^2(a/2)\sinh^2(b/2) + \cosh^2(a/2)\cosh^2(b/2)} \\
&= \frac{\cosh a \sinh b - i\sinh a}{2(\sinh^2(a/2)\sinh^2(b/2) + \cosh^2(a/2)\cosh^2(b/2)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\cos A &= \frac{B_x}{|B|} \\
&= \frac{\dfrac{\cosh a \sinh b}{2(\sinh^2(a/2)\sinh^2(b/2)+\cosh^2(a/2)\cosh^2(b/2))}}{\tanh^2(c/2)} \\
&= \frac{\cosh a \sinh b}{2(\sinh^2(a/2)\sinh^2(b/2)+\cosh^2(a/2)\cosh^2(b/2))}\sqrt{\frac{\tanh^2(a/2)\tanh^2(b/2)+1}{\tanh^2(a/2)+\tanh^2(b/2)}} \\
&= \frac{\cosh a \sinh b}{2(\sinh^2(a/2)\sinh^2(b/2)+\cosh^2(a/2)\cosh^2(b/2))}\sqrt{\frac{\sinh^2(a/2)\cosh^2(b/2)+\cosh^2(a/2)\sinh^2(b/2)}{\sinh^2(a/2)\sinh^2(b/2)+\cosh^2(a/2)\cosh^2(b/2)}} \\
&= \frac{\cosh a \sinh b}{\sqrt{(\sinh^2(a/2)\sinh^2(b/2)+\cosh^2(a/2)\cosh^2(b/2))(\sinh^2(a/2)\cosh^2(b/2)+\cosh^2(a/2)\sinh^2(b/2))}} \\
&= \frac{2\cosh a \sinh b}{\sqrt{[(\cosh a-1)(\cosh b-1)+(\cosh a+1)(\cosh b+1)][(\cosh a-1)(\cosh b+1)+(\cosh b-1)(\cosh a+1)]}} \\
&= \frac{\cosh a \sinh b}{\sqrt{(\cosh a\cosh b+1)(\cosh a\cosh b-1)}} \\
&= \frac{\cosh a \sinh b}{\sqrt{\cosh^2 a\cosh^2 b-1}}
\end{aligned}$$

Likewise,

$$\sin A = \frac{\sinh a}{\sqrt{\cosh^2 a\cosh^2 b-1}}$$

Now,

$$\begin{aligned}
\cosh^2 a\cosh^2 b - 1 &= \cosh^2 c - 1 \\
&= \sinh^2 c
\end{aligned}$$

since $\cosh a\cosh b = \cosh c$ by the Hyperbolic Pythagorean Theorem. Thus,

$$\begin{aligned}
\cos A &= \frac{\cosh a \sinh b}{\sinh c} \\
&= \frac{\dfrac{\cosh a \sinh b}{\cosh b}}{\dfrac{\sinh c}{\cosh b}} \\
&= \frac{\tanh b \cosh a \cosh b}{\sinh c} \\
&= \frac{\tanh b \cosh c}{\sinh c} \\
\cos A &= \frac{\tanh b}{\tanh c} \\
\sin A &= \frac{\sinh a}{\sinh c}.
\end{aligned}$$

∎

Theorem 12.2 *For any triangle $\triangle ABC$ in the hyperbolic plane*

$$\cosh c = \cosh a\, \cosh b - \sinh a\, \sinh b \cos C \tag{12.7}$$

Hyperbolic Law of Cosines

$$\frac{\sinh a}{\sin A} = \frac{\sinh b}{\sin B} = \frac{\sinh c}{\sin C} \tag{12.8}$$

Hyperbolic Law of Sines

Theorem 12.3 (Hyperbolic Law of Cosines for Angles) *Let $\triangle ABC$ be a triangle in $\mathbb{H}^2$ with sides a, b, and c opposite the angles A, B, and C. Then*

$$\cos C = -\cos A \cos B + \sin A \sin B \cosh c.$$

Note that this theorem is the tool that allows us to solve for the sides of a triangle given the three angles (AAA).

Since the area of a triangle is determined by its angles, and since the sides of a triangle determine the angles, there must be a formula for the area of a triangle in terms of its sides: a Heron's Formula for Hyperbolic Geometry.

Theorem 12.4 (Heron's Formula for Hyperbolic Geometry) *Let $\triangle ABC$ be a triangle in $\mathscr{H}^2$ with sides a, b, and c opposite the angles A, B, and C. Let*

$$s = \frac{a+b+c}{2}$$

be the semiperimeter. Let $K = |\triangle ABC| =$ area of $\triangle ABC$. Then

$$1 - \cos(K) = \frac{4 \sinh s \sinh(s-a) \sinh(s-b) \sinh(s-c)}{(1+\cosh a)(1+\cosh b)(1+\cosh c)}.$$

Let us take a look at a specific example. Consider equilateral triangles. In Euclidean geometry all are similar, since they all must have angles measuring 60°. If this were true in hyperbolic geometry, they would have to be congruent by AAA. What then are the angles in an equilateral triangle of differing sides? Look at the following table and see if you can tell what is happening.

Sides	Radians	Degrees
10	0.0135	0.77
5	0.1633	9.35
2.5	0.5359	30.71
1.5	0.7930	45.43
1.0	0.9188	52.64
0.5	1.0122	57.99
0.1	1.0458	59.92

Table 12.1: Hyperbolic Equilateral Triangles

Other interesting examples would be how the angles in a right isosceles triangle vary with the sides and what triangle is analogous to the standard 30-60-90 triangle in Euclidean geometry.

12.1 Circumference and Area of a Circle

Theorem 12.5 *The circumference C of a circle of radius r is*

$$C = 2\pi \sinh r.$$

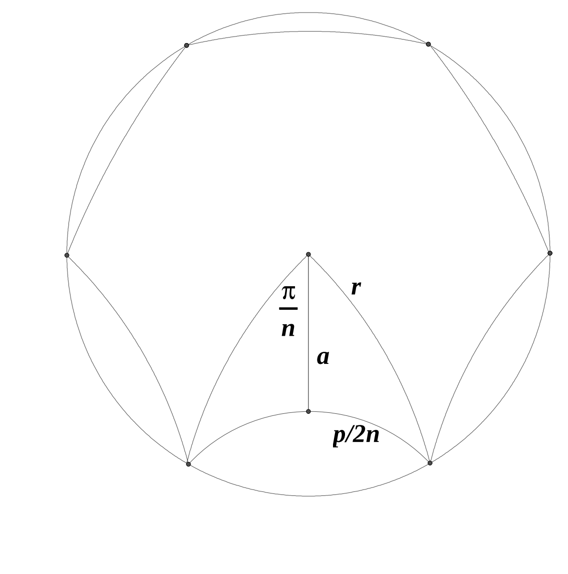


Figure 12.2: Hyperbolic Circle

In Euclidean geometry $C = \lim_{n\to\infty} p_n$ where p_n is the perimeter of the regular n-gon inscribed in the circle.

$$\begin{aligned} p_n = 2nr \sin\frac{\pi}{n} &= 2nr\left[\frac{\pi}{n} - \frac{1}{3!}\left(\frac{\pi}{n}\right)^3 + \frac{1}{5!}\left(\frac{\pi}{n}\right)^5 - \cdots\right] \\ &= 2\pi r - 2\pi r\left[\frac{1}{3!}\left(\frac{\pi}{n}\right)^2 + \frac{1}{5!}\left(\frac{\pi}{n}\right)^4 - \cdots\right] \end{aligned}$$

Thus, $C = \lim_{n\to\infty} p_n = 2\pi r$.

In hyperbolic geometry we can still compute the perimeter and compute the limit, but we will use Theorem 12.1 to compute the perimeter.

The proof is nothing but the following computation.

$$
\begin{aligned}
\sin\frac{\pi}{2n} &= \frac{\sinh\frac{p}{2n}}{\sinh r}\\
\sinh\frac{p}{2n} &= \sinh r\sin\frac{\pi}{n}\\
\sum_{j=0}^{\infty}\frac{\left(\frac{p}{2n}\right)^{2j+1}}{(2j+1)!} &= \sinh r\left(\sum_{j=0}^{\infty}(-1)^j\frac{\left(\frac{\pi}{n}\right)^{2j+1}}{(2j+1)!}\right)\\
\frac{p}{2n}\left(\sum_{j=0}^{\infty}\frac{\left(\frac{p}{2n}\right)^{2j}}{(2j+1)!}\right) &= \frac{\pi}{n}\sinh r\left(\sum_{j=0}^{\infty}(-1)^j\frac{\left(\frac{\pi}{n}\right)^{2j}}{(2j+1)!}\right)\\
p\left(1+\frac{1}{3!}\left(\frac{p}{2n}\right)^2+\cdots\right) &= 2\pi\sinh r\left(1-\frac{1}{3!}\left(\frac{\pi}{n}\right)^2+\cdots\right)\\
C &= \lim_{n\to\infty}p = 2\pi\sinh r.
\end{aligned}
$$

Let K be the area of $\triangle ABC$, so $K = \pi - \angle A - \angle B - \angle C$. Let $\triangle ABC$ have a right angle at C, then $K = \frac{\pi}{2} - A - B$.

Theorem 12.6 $\tan K/2 = \tanh a/2\ \tanh b/2$.

Once again the proof is a computation.

$$
\begin{aligned}
\tanh^2\frac{a}{2}\ \tanh^2\frac{b}{2} &= \frac{\cosh a - 1}{\cosh a + 1}\cdot\frac{\cosh b - 1}{\cosh b + 1}\\
&= \frac{\cos A - \sin B}{\cos A + \sin B}\cdot\frac{\cos B - \sin A}{\cos B + \sin A}\\
&= \frac{1 - \sin(A+B)\cos(A-B)}{1 + \sin(A+B)\cos(A-B)}\\
&= \frac{1-\cos K}{1+\cos K} = \tan^2\frac{K}{2}.
\end{aligned}
$$

Using this and our limiting approach we can now compute the area of a circle.

Theorem 12.7 *The area, A, of a circle of radius r is*

$$A = 4\pi\sinh^2\frac{r}{2}.$$

PROOF: We do this just as before. If K_n is the area of the inscribed regular n-gon, then $A = \lim_{n\to\infty} K_n$. In the right triangle in Figure 12.2 let K, a, and p denote K_n, a_n and p_n.

The area of the right triangle is $K_n/2n$.

$$\begin{aligned}
\tan\frac{K}{4n} &= \tanh\frac{a}{2}\tanh\frac{p}{4n} \\
4n\tan\frac{K}{4n} &= (\tanh\frac{a}{2})\,4n\tanh\frac{p}{4n} \\
\text{Now,} & \\
4n\tan\frac{K}{4n} &= 4n\left[\frac{K}{4n} + \frac{1}{3}\left(\frac{K}{4n}\right)^3 + \frac{2}{15}\left(\frac{K}{4n}\right)^5 + \cdots\right] \\
&= K[1 + \frac{1}{3}\left(\frac{K}{4n}\right)^2 + \frac{2}{15}\left(\frac{K}{4n}\right)^4 + \cdots] \\
\text{and} & \\
4n\tanh\frac{p}{4n} &= 4n\left[\frac{p}{4n} - \frac{1}{3}\left(\frac{p}{4n}\right)^3 + \frac{2}{15}\left(\frac{p}{4n}\right)^5 + \cdots\right] \\
&= p[1 + \frac{1}{3}\left(\frac{p}{4n}\right)^2 - \frac{2}{15}\left(\frac{p}{4n}\right)^4 + \cdots]
\end{aligned}$$

Thus, we find that

$$\begin{gathered}
\lim_{n\to\infty} 4n\tan\frac{K}{4n} = \lim_{n\to\infty} K = A \\
\lim_{n\to\infty} 4n\tanh\frac{p}{4n} = \lim_{n\to\infty} p = C \\
\lim_{n\to\infty} a = r
\end{gathered}$$

Putting all of this together we have that

$$\begin{aligned}
A &= C\tanh\frac{r}{2} \\
&= 2\pi\sinh r\tanh\frac{r}{2} \\
&= 2\pi\sinh r\frac{\cosh r - 1}{\sinh r} \\
&= 2\pi(\cosh r - 1) \\
&= 4\pi\sinh^2\frac{r}{2}.
\end{aligned}$$

Thus, we have computed the area of a circle. ∎