MATH 6118

Neutral Geometry

I was going to try powdered water, but I didn't know what to add.

Undefined Terms

- Point
- Line
- Distance
- Half-plane
- Angle
- Measure
- Area

Axioms

1. (Existence Axiom) The collection of all points forms a nonempty set. There is more than one point in that set.
2. (Incidence Axiom) Every line is a set of points. For every pair of distinct points A and B there is exactly one line ℓ containing A and B.

Definition: Two lines ℓ and m are <u>parallel</u> if they do not intersect.

<u>Theorem</u>: If ℓ and m are two distinct, non-parallel lines, then there is exactly one point P that lies on both ℓ and m.

Proof:

Distance

Axiom 3: (Ruler Axiom) For every pair of points P and Q there is a real number d(P,Q). For each line ℓ there is a 1-1 correspondence from ℓ to R so that P and Q correspond to x,y (real numbers) then d(P,Q)=|x – y|.

Definitions

Definition: C is between A and B, A*B*C, if C lies on the line AB and

d(A,C) + d(C,B) = d(A,B)

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$AB = \{A,B\} \cup \{P \mid A*B*C\}$

$\overrightarrow{AB} = AB \cup \{P \mid A*B*P\}$

Definitions

Definition: The <u>length</u> of segment AB is d(A,B). Two segments are congruent if they have the same length.

Theorem: If P and Q are any points, then

$$d(P,Q) = d(Q,P)$$
$$d(P,Q) \geq 0$$
$$d(P,Q) = 0 \Leftrightarrow P = Q$$

Plane Separation

Axiom: For every line ℓ the points that do not lie on ℓ form two disjoint nonempty sets (H_1 and H_2) so that:

i) H_1 and H_2 are convex;

ii) If P is in H_1 and Q is in H_2 then PQ intersects ℓ.

Definition

For a line ℓ and external points A,B

A and B are on the same side if AB does not intersect ℓ. A and B are on opposite sides of ℓ if AB does intersect ℓ.

An angle is the union of two nonopposite rays sharing the same endpoint.

The interior of the angle is the intersection of two half planes.

Definition

If A, B, C are non-collinear points, the triangle ABC is the union of the three segments AB, BC, and AC.

Pasch's Theorem

Let ΔABC be a triangle and ℓ a line so that none of A,B, and C are on ℓ. If ℓ intersects AB, then ℓ intersects either AC or BC.

Proof:

Angle Measure

Axiom: (Protractor Axiom) For every angle BAC there is a number m(BAC) so that:
i) $0 \leq m(BAC) \leq 180$,
ii) m(BAC) = 0 iff AB = AC,
iii) you can construct an angle of measure r on either side of a line;
iv) if AD is between AB and AC
m(BAD) + m(DAC) = m(BAC)

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Betweenness

Coordinate function: a 1-1 correspondence f: ℓ → R so that d(P,Q) = |f(P)-f(Q)|.

Theorem: If A,B,C lie on ℓ, then C is between A and B iff f(A) < f(C) < f(B) or f(A) > f(C) > f(B).

Lemma: If A,B,C on ℓ the exactly one of them lies between the other two.

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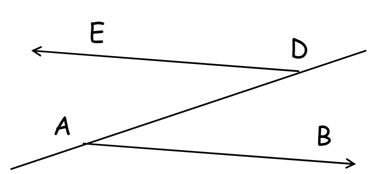
Betweenness

Theorem: Let ℓ be a line and A,D on ℓ. If B and E on opposite sides of ℓ then rays AB and DE do not intersect.



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Betweenness

Theorem: Each angle has a unique bisector.

Crossbar Theorem: Given ΔABC, let D be in the interior of angle BAC. Then ray AD must intersect BC.

Triangle Congruency

Definition: Two triangles are congruent if there is a 1-1 correspondence between the vertices so that the corresponding sides are congruent and corresponding angles are congruent.

SAS Axiom: Given ΔABC and ΔDEF so that AB ≈ DE, BC ≈ EF, and ABC ≈ DEF, then ΔABC ≈ ΔDEF.

Neutral Geometry Results

Theorem: (ASA) Given ΔABC and ΔDEF so that CAB ≈ FDE, AB ≈ DE, and ABC ≈ DEF, then ΔABC ≈ ΔDEF.

Theorem: In ΔABC if ABC ≈ ACB, then AB ≈ DE.

Neutral Geometry Results

Theorem: (Existence of Perpendiculars) Given line ℓ and P not on ℓ, there exists a line m through P that is perpendicular to ℓ.

Alternate Interior Angles

Theorem: (Alternate Interior Angles Theorem) If two lines are cut by a transversal so that there is a pair of congruent alternate interior angles, then the two lines are parallel.

Existence of Parallel Lines

Theorem: If m and n are distinct lines both perpendicular to ℓ, then m and n are parallel.

Uniqueness of Perpendiculars

Theorem: If P is not on ℓ, then the perpendicular dropped from P to ℓ is unique.

Exterior Angle Theorem

Theorem: An exterior angle of a triangle is greater than either remote interior angle.

Angle Angle Side Criterion

Theorem: Given ΔABC and ΔDEF so that AC ≈ DF, BAC ≈ EDF, and ABC ≈ DEF, then ΔABC ≈ ΔDEF.

Hypotenuse Leg Criterion

Theorem: Two right triangles are congruent if the hypotenuse and leg of one are congruent respectively to the hypotenuse and leg of the other.

Side-Side-Side Criterion

Theorem: Given ΔABC and ΔDEF so that $AC \approx DF$, $AB \approx DE$, and $BC \approx EF$, then $\Delta ABC \approx \Delta DEF$.

Saccheri-Legendre Theorem

Lemma: The sum of the measures of any two angles of a triangle is less than 180.

Lemma: If A, B, and C are non-collinear, then $|AC| < |AB| + |BC|$

Saccheri-Legendre Theorem

Theorem: (Saccheri-Legendre) The sum of the angles I any triangle is less than or equal to 180.

Lemma: If A, B, and C are non-collinear, then $|AC| < |AB| + |BC|$

Defect of a Triangle

Definition: The defect of a triangle ΔABC is the number:

defect(ΔABC) = 180 – (m(A)+m(B)+m(C))

Theorem: (Additivity of Defect) Let ΔABC be any triangle and D lie on AB, then:

def(ΔABC) = def(ΔACD) + def(ΔBCD)

Quality of Defect

Theorem:

a) If there exists a triangle of defect 0, then a rectangle exists.
b) I a rectangle exists, then every triangle has defect 0.

Quality of Defect

Path of Proof

ΔABC has defect 0 → there is a right triangle with defect 0 → we can construct a rectangle → we can construct arbitrarily large rectangles → all right triangles have defect 0 → all triangles have defect 0

Positive Defect

Corollary: If there is a triangle with positive defect then all triangles have positive defect.