Exercise Set #7

1. What is the distance between the points $4 + i/2$ and $4 + 5i$ in the Poincaré upper half plane, H?

2. What is the distance between the points $-2 + 2i$ and $-2 + 7i$ in the Poincaré upper half plane, H?

3. Prove that the dilation $\delta_\lambda(x, y) = (\lambda x, \lambda y)$ preserves the Poincaré arclength element.

4. Let $P = 4 + 4i$ and $Q = 5 + 3i$. Find $M$ and $N$, the endpoints of the Poincaré line through $P$ and $Q$.

5. Let $P = 12i$ and $Q = 7 + 5i$. Find $M$ and $N$, the endpoints of the Poincaré line through $P$ and $Q$.

6. In the upper half plane model, H, carefully draw the asymptotic triangle with vertices $i, 1 + i,$ and $1$. Is the map

$$\gamma = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

an isometry of H? In the same diagram, carefully draw the image of the asymptotic triangle under the action of $\gamma$.

7. In the upper half plane model, H, carefully draw the asymptotic triangle with vertices $i$, $-1 + i,$ and $1 + i$. In the same diagram, carefully draw the image of this triangle under the isometry

$$\gamma = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

8. Let $P = \frac{8 + i}{13}, Q = \frac{13 + i}{20}$, and $\gamma = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$. What are $\gamma P$ and $\gamma Q$? Sketch $P$, $Q$ and their images. Is $\gamma$ an isometry? Why? Use all of this information to find the distance between $P$ and $Q$ in H.

9. Let $P = 2 + 4i$ and $Q = \frac{6 + 4i}{3}$ be two points in the upper half plane, H. Let

$$\gamma = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}.$$

What are $\gamma P$ and $\gamma Q$? What is the Poincaré distance from $P$ to $Q$ in H.

10. Suppose that $T$ is a fractional linear transformation such that $T(1) = 1$, $T(0) = 0$, and $T(\infty) = \infty$. Prove that $T$ is the identity map. That is, show that $T(z) = z$ for all $z$. 