Chapter 9

Poincaré's Disk Model for Hyperbolic Geometry

9.1 Saccheri's Work

Recall that Saccheri introduced a certain family of quadrilaterals. Look again at Section 7.3 to remind yourself of the properties of these quadrilaterals. Saccheri studied the three different possibilities for the summit angles of these quadrilaterals.

Hypothesis of the Acute Angle (HAA)	The summit angles are acute
Hypothesis of the Right Angle (HRA)	The summit angles are right angles
Hypothesis of the Obtuse Angle (HOA)	The summit angles are obtuse

Saccheri intended to show that the first and last could not happen, hence he would have found a proof for Euclid's Fifth Axiom. He was able to show that the Hypothesis of the Obtuse Angle led to a contradiction. This result is now know as the Saccheri-Legendre Theorem (Theorem 7.3). He was unable to arrive at a contradiction when he looked at the Hypothesis of the Acute Angle. He gave up rather than accept that there was another geometry available to study. It has been said that he wrote that the Hypothesis of the Acute Angle must be false "because God wants it that way."

9.2 The Poincaré Disk Model

When we adopt the *Hyperbolic Axiom*then there are certain ramifications:

1. The sum of the angles in a triangle is less than two right angles.
2. All similar triangles that are congruent, *i.e.* AAA is a congruence criterion.
3. There are no lines everywhere equidistant from one another.
4. If three angles of a quadrilateral are right angles, then the fourth angle is less than a right angle.
5. If a line intersects one of two parallel lines, it may not intersect the other.
6. Lines parallel to the same line need not be parallel to one another.
7. Two lines which intersect one another may both be parallel to the same line.

How can we visualize this? Surely it cannot be by just looking at the Euclidean plane in a slightly different way. We need a model with which we could study the hyperbolic plane. If it is to be a Euclidean object that we use to study the hyperbolic plane, $\mathscr{H}^2$, then we must have to make some major changes in our concept of point, line, and/or distance.

We need a model to *see* what $\mathscr{H}^2$ looks like. We know that it will not be easy, but we do not want some extremely difficult model to construct. We will work with a small subset of the plane, but give it a different way of measuring distance.

There are three traditional models for $\mathscr{H}^2$. They are known as the Klein model, the Poincaré Disk model, and the Poincaré Half-Plane model. We will start with the Disk model and move to the Half-Plane model later. There are geometric "isomorphisms" between these models, it is just that some properties are easier to see in one model than the other. The two Poincaré models tend to give us the opportunity to do computations more easily than the Klein model — though the Klein model is somewhat easier to describe.

In order to give a model for $\mathscr{H}^2$, we need to decide on a set of points, then determine what lines are and how to measure distance. For Poincaré's Disk Model we take the set of points that lie inside the unit circle, *i.e.*, the set

$$\mathscr{H}^2 = \{(x, y) \mid x^2 + y^2 < 1\}.$$

Note that points on the circle itself are **not** in the hyperbolic plane. However they do play an important part in determining our model. Euclidean points on the circle itself are called *ideal points*, *omega points*, *vanishing points*, or *points at infinity*.
[Note: Poincaré himself thought of this set as the set of all complex numbers with length less than 1

$$\mathscr{H}^2 = \{z \in \mathbb{C} \mid \|z\| < 1\}.$$

We will see why this is important when we study the Poincaré half plane model.]

A *unit* circle is any circle in the Euclidean plane is a circle with radius one.

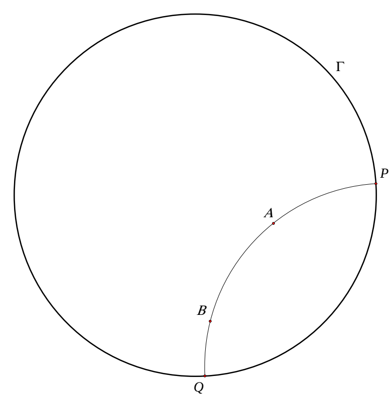

Figure 9.1: Poincaré line

Definition 9.1 *Given a unit circle Γ in the Euclidean plane, points of the hyperbolic plane are the points in the interior of Γ. Points on this unit circle are called omega points (Ω) of the hyperbolic plane.*

If we take Γ to be the unit circle centered at the origin, then we would think of the hyperbolic plane as $\mathscr{H}^2 = \{(x, y) \mid x^2 + y^2 < 1\}$ and the omega points are the points $\Omega = \{(x, y) \mid x^2 + y^2 = 1\}$. The points in the Euclidean plane satisfying $\{(x, y) \mid x^2 + y^2 > 1\}$ are called *ultraideal points*.

We now have what our points will be. We see that we are going to have to modify our concept of line in order to have the Hyperbolic Axiom to hold.

Definition 9.2 *Given a unit circle Γ in the Euclidean plane, lines of the hyperbolic plane are arcs of circles drawn orthogonal*[1] *to Γ and located in the interior of Γ.*

[1]Circles are orthogonal to one another when their radii at the points of intersection are perpendicular.

9.2.1 Construction of Lines

This sounds nice, but how do you draw them?

1. Start with a circle Γ centered at O and consider the ray $\overrightarrow{OA}$, where A lies on the circle, Γ.

2. Construct the line perpendicular to $\overrightarrow{OA}$ at A.

3. Choose a point P on this perpendicular line for the center of the second circle and make PA the radius of a circle centered at P.

4. Let B denote the second point of intersection with circle Γ. Then the arc AB represents a line in this model.

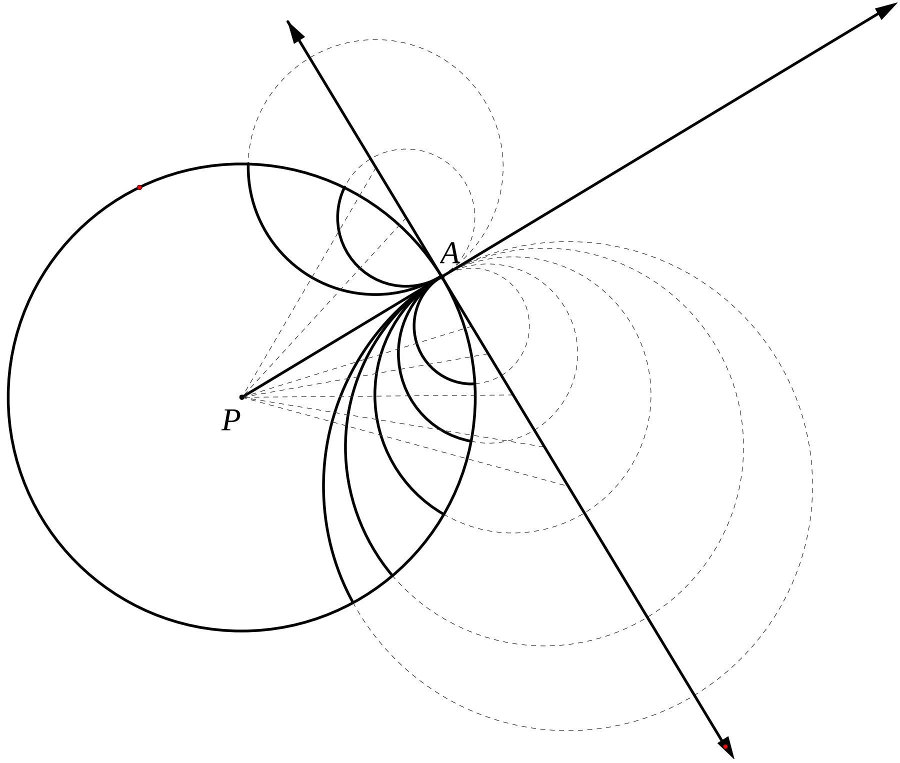

Figure 9.2: Poincaré lines through A

Now, how do you construct these lines in more general circumstances? There are three cases we need to consider.
Case I:$A, B \in \Gamma$
Case II: $A \in \Gamma$ and B lies inside Γ
Case III: A and B both lie inside Γ.
Case I: Construct rays $\overrightarrow{PA}$ and $\overrightarrow{PB}$ where P is the center of the circle Γ. Construct the lines perpendicular to $\overrightarrow{PA}$ and $\overrightarrow{PB}$ at A and B respectively. Let Q be the point of intersection of those two lines. The circle Ω centered at Q with radius QA intersects Γ at A and B. The line between A and B is the arc of Ω that lies inside Γ.

Note that this arc is clearly orthogonal to Γ by its construction.

Case II: Construct rays $\overrightarrow{PA}$ and $\overrightarrow{PB}$ where P is the center of the circle Γ. Construct the line perpendicular to $\overrightarrow{PA}$ at A. Draw segment AB and construct its perpendicular bisector. Let Q be the point of intersection of this line and the tangent line to Γ at A. The circle Ω centered at Q with radius QA contains A and B. The line containing A and B is the arc of Ω that lies inside Γ.

This arc, as constructed is orthogonal to Γ at A. We want to see that it is orthogonal at the other point of intersection with the circle. Let that point of intersection be X. Then, $X \in \Gamma$ means that $PA \cong PX$. Since X lies on our second circle it follows that $QX \cong QA$. Since $PQ \cong PQ$, we have that $\triangle PAQ \cong \triangle PXQ$, which means that $\angle PXQ$ is a right angle, as we wanted to show.

Case III: Construct the ray $\overrightarrow{PA}$ and then construct the line perpendicular to $\overrightarrow{PA}$ at A. This intersects Γ in points X and Y. Construct the tangents to Γ at X and at Y. These tangent lines intersect at a point C. The circle Ω centered at Q is the circle passing through A, B, and C. The line containing A and B is the arc of Ω that lies inside Γ.

From our construction, we have that $\triangle PXC \sim \triangle PAX$ and it follows that $|PA||PC| = |PX|^2 = r^2$. Now, Q lies on the perpendicular bisectors of AC and AB as Ω is the circumcircle for $\triangle ABC$. There is a point T on the circle Ω so that the tangent line to Ω at T passes through P.

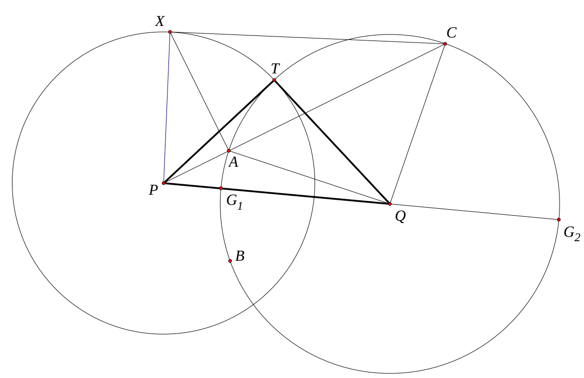

Figure 9.3: Poincaré line in Case III

Construct the line through P and Q which intersects the circle in two points G_1 and G_2 so that G_1 lies between P and Q. Now,

$$\begin{aligned}|PT|^2 &= |PQ|^2 - |QT|^2 \\ &= (|PQ| - |QT|)(|PQ| + |QT|) \\ &= (|PQ| - |QG_1|)(|PQ| + |QG_2|) \\ &= |PG_1||PG_2| \text{ which by Theorem 5.3,} \\ &= |PA||PC| = r^2\end{aligned}$$

Therefore, T lies on the circle Γ and Γ and Ω are orthogonal at that point. A similar argument shows that they are orthogonal at the other point of intersection.

9.2.2 Distance

Now, this area inside the unit circle must represent the infinite hyperbolic plane. This means that our standard distance formula will not work. We introduce a distance metric by

$$d\rho = \frac{2dr}{1 - r^2}$$

where ρ represents the hyperbolic distance and r is the Euclidean distance from the center of the circle. Note that $d\rho \to \infty$ as $r \to 1$. This means that lines are going to have infinite extent.

The relationship between the Euclidean distance of a point from the center of the circle and the hyperbolic distance is:

$$\rho = \int_0^r \frac{2du}{1-u^2} = \log\left(\frac{1+r}{1-r}\right) = 2\tanh^{-1} r,$$

or $r = \tanh\frac{\rho}{2}$.

Now, for those of you who don't remember ever having seen this function $\tanh(x)$, we give a little review. The hyperbolic trigonometric functions $\cosh(x)$ and $\sinh(x)$ are defined by:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$
$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

and

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x}-1}{e^{2x}+1}.$$

We will study these in more depth later.

Now, we can use this to define the distance between two points on a Poincaré line. Given two hyperbolic points A and B, let the Poincaré line intersect the circle in the omega points P and Q. Define

$$(AB, PQ) = \frac{AP/AQ}{BP/BQ} = \frac{AP \cdot BQ}{AQ \cdot BP},$$

to be the *cross ratio* of A and B with respect to P and Q, where AP denotes the the Euclidean arclength. Define the hyperbolic distance from A to B to be

$$d(A, B) = \log |AB, PQ|.$$

We will prove the following later.

Theorem 9.1 *If a point A in the interior of Γ is located at a Euclidean distance $r < 1$ from the center O, its hyperbolic distance from the center is given by*

$$d(A, O) = \log \frac{1+r}{1-r}.$$

Lemma 9.1 *The hyperbolic distance from any point in the interior of Γ to the circle itself is infinite.*

9.2.3 Parallel Lines

It is easy to see that the Hyperbolic Axiom works in this model. Given a line $\overleftrightarrow{AB}$ and a point $D \notin \overleftrightarrow{AB}$, then we can draw at least two lines through D that do not intersect $\overleftrightarrow{AB}$.

Call these two lines through D lines ℓ_1 and ℓ_2. Notice now how two of our previous results do not hold, as we remarked earlier. We have that $\overleftrightarrow{AB}$ and ℓ_1 and $\overleftrightarrow{AB}$ and ℓ_2 are parallel, ***but*** ℓ_1 and ℓ_2 are not parallel. Note also that ℓ_2 intersects one of a pair of parallel lines (ℓ_1), but does not intersect the other parallel line ($\overleftrightarrow{AB}$).

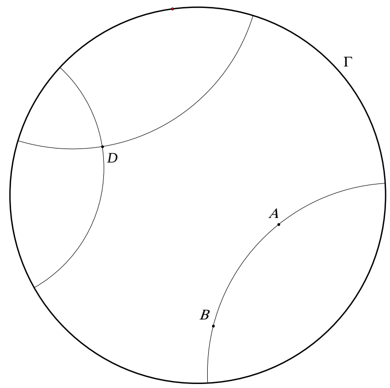

Figure 9.4: Multiple parallels through A

As we now know, the hyperbolic plane has two types of parallel lines. The definition that we will give here will depend explicitly on the model that we have chosen. Consider the hyperbolic line $\overleftrightarrow{AB}$ which intersects the circle Σ in the ideal points Λ and Ω. Take a point $D \notin \overleftrightarrow{AB}$. Construct the line through Λ and D. Since this line does not intersect the line $\overleftrightarrow{AB}$ inside the circle, these two hyperbolic lines are parallel. However, they seem to be approaching one another as we go "to infinity". Since there are two "ends" of the Poincaré line $\overset{\frown}{AB}$, there are two of these lines. The line $\overset{\frown}{AB}$ and $\overset{\frown}{D\Lambda}$ are *horoparallel*. The defining property is as follows.

Definition 9.3 *Let $P \in \overset{\frown}{AB}$. Consider the collection of lines $\overset{\frown}{DP}$ as P goes to Ω or Λ. The first line through D in this collection that does not intersect $\overset{\frown}{AB}$ in $\mathscr{H}^2$ is the horoparallel line to $\overset{\frown}{AB}$ in that direction.*

Drop a perpendicular from D to $\overset{\frown}{AB}$ and label this point of intersection M. Angles $\angle \Lambda DM$ and $\angle \Omega DM$ are called *angles of parallelism.*

Theorem 9.2 *The angles of parallelism associated with a given line and point are congruent.*

Proof: Assume not, *i.e.*, assume $\angle \Lambda DM \neq \angle \Omega DM$. Then one angle is greater than the other. Without loss of generality, we may assume that $\angle \Lambda DM < \angle \Omega DM$. Then there is a point E in the interior of $\angle \Omega DM$ such that $\angle \Lambda DM = \angle EDM$. The line $\overset{\frown}{ED}$ must intersect $\overset{\frown}{AB}$ since $\overset{\frown}{D\Omega}$ is the limiting parallel line to $\overset{\frown}{AB}$ in that direction. Let the point of intersection be F. Choose G on $\overset{\frown}{AB}$ on the opposite side of $\overset{\frown}{DM}$ from F so that $FM = GM$. Then $\triangle GMD \cong \triangle FMD$. This implies that $\angle GDM = \angle FDM = \angle \Lambda DM$. This means that $\overset{\frown}{D\Omega}$ intersects $\overset{\frown}{AB}$ at G. This contradicts the condition that $\overset{\frown}{D\Omega}$ is limiting parallel to $\overset{\frown}{AB}$. Thus, the angles of parallelism are congruent. ∎

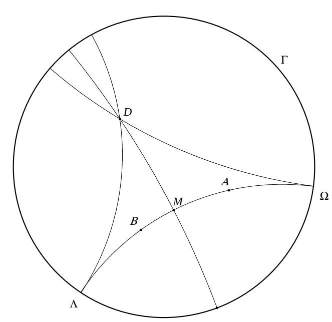

Figure 9.5: Limiting Parallel Poincaré Lines

Theorem 9.3 *The angles of parallelism associated with a given line and point are acute.*

PROOF: Assume not, *i.e.*, assume that $\angle MD\Omega > 90°$. Then there is a point E interior to $\angle MD\Omega$ so that $\angle MDE = 90°$. Then, since $\overset{\frown}{DE}$ and $\overset{\frown}{AB}$ are perpendicular to the same line, they are parallel. Thus, $\overset{\frown}{DE}$ does not intersect $\overset{\frown}{AB}$ which contradicts the condition that $D\Omega$ is the limiting parallel line.

If the angle of parallelism is $90°$ then we can show that we have Euclidean geometry. Thus, in $\mathscr{H}^2$ the angle of parallelism is acute. ∎

Theorem 9.4 (Lobachevskii's Theorem) *Given a point P at a hyperbolic distance ρ from a hyperbolic line $\overset{\frown}{AB}$ (i.e., $d(P, M) = \rho$), the angle of parallelism, θ, associated with the line and the point satisfies*

$$e^{-\rho} = \tan\left(\frac{\theta}{2}\right).$$

Note then that

$$\lim_{\rho\to 0}\theta = \frac{\pi}{2} \text{ and } \lim_{\rho\to\infty}\theta = 0.$$

PROOF: The proof of this is interesting in that we play one geometry against the other in order to arrive at our conclusion.

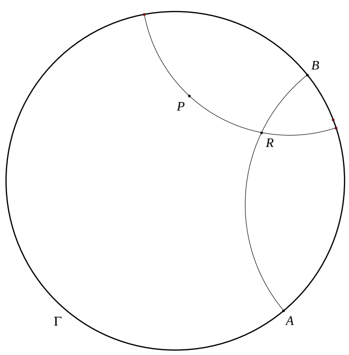

Figure 9.6: Lobachevskii's Theorem

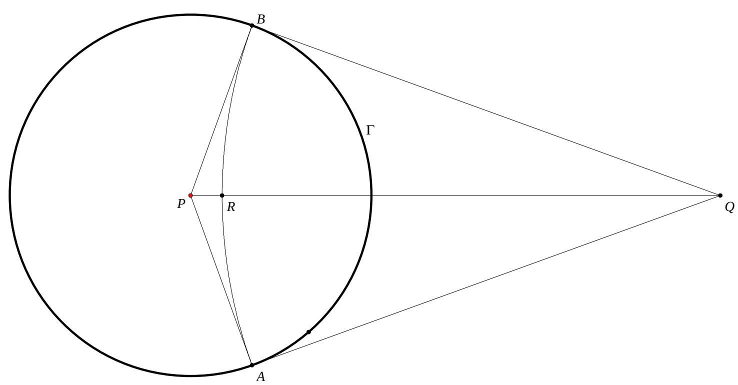

Figure 9.7: After the first translation

We are given a line $\overset{\frown}{AB}$ and a point P not on the line. Construct the line through P which is perpendicular to $\overset{\frown}{AB}$. Call the point of intersection R as in Figure 9.6. Then we have that $\rho = d(P, R)$. We can translate P to the center of the unit circle and translate our line to a line so that our line perpendicular to AB is a radius of Γ as we have done in Figure 9.7. Construct the radii from P to the ideal points A and B and construct the lines tangent to Γ at these points. These tangent lines intersect at a point Q which lies on $\overrightarrow{PR}$. Now, since we have moved our problem to the center of the circle, we can use our previous result to see that if r is the Euclidean distance from P to R, then we have

$$\rho = \log\frac{1+r}{1-r},$$

or rewriting this we have

$$e^{\rho} = \frac{1+r}{1-r} \text{ or } e^{-\rho} = \frac{1-r}{1+r}.$$

Now, we are talking about Euclidean distances (with r) and using our Euclidean right triangles with radius 1 we have that:

$$r = QP - QR = QP - QA = \sec \angle QPA - \tan \angle QPA = \sec\theta - \tan\theta = \frac{1-\sin\theta}{\cos\theta}.$$

Now, algebra leads us to:

$$\begin{aligned}
e^{-\rho} &= \frac{1-r}{1+r} \\
&= \frac{\cos\theta + \sin\theta - 1}{\cos\theta - \sin\theta + 1} \\
&= \frac{\cos\theta + \sin\theta - 1}{\cos\theta - \sin\theta + 1}\frac{\cos\theta + \sin\theta + 1}{\cos\theta + \sin\theta + 1} \\
&= \frac{\cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta - 1}{\cos^2\theta + 2\cos\theta - \sin^2\theta + 1} \\
&= \frac{2\sin\theta\cos\theta}{2\cos^2\theta + 2\cos\theta} = \frac{\sin\theta}{1+\cos\theta} \\
&= \frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{\left(2\cos^2\left(\frac{\theta}{2}\right) - 1\right) + 1} \\
&= \tan\left(\frac{\theta}{2}\right)
\end{aligned}$$

∎

9.2.4 Hyperbolic Circles

Now, if we have a center of a circle that is not at the center P of the unit circle Σ, we know that the hyperbolic distance in one direction looks skewed with respect to the Euclidean distance. That would lead us to expect that a circle in this model might take on an elliptic or oval shape. We will prove later that this is not the case. In fact, hyperbolic circles embedded in Euclidean space retain their circular appearance — their centers are offset!

Theorem 9.5 *Given a hyperbolic circle with radius R, the circumference C of the circle is given by $C = 2\pi\sinh(R)$.*

9.2.5 Common Figures in the Disk Model

What do some of the common figures, with which we have become accustomed, look like in the Poincaré Disk Model?

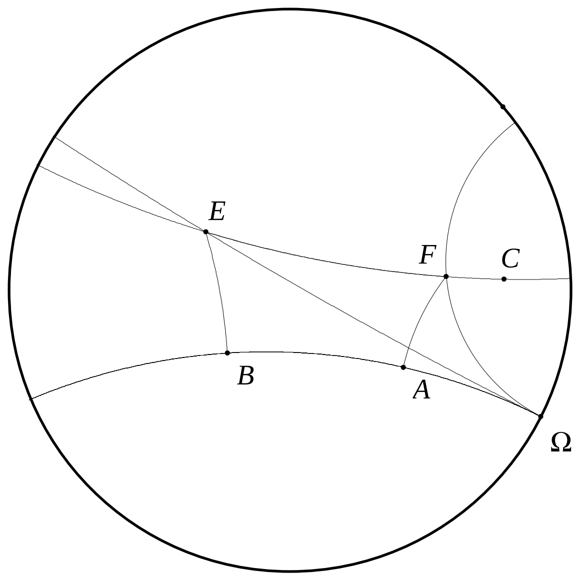

Figure 9.8: Saccheri quadrilateral in the Poincaré Disk

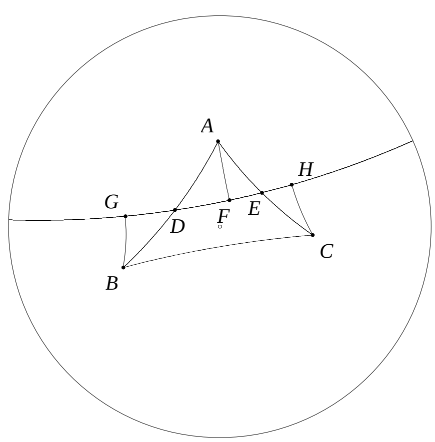

Figure 9.9: Acute Triangle

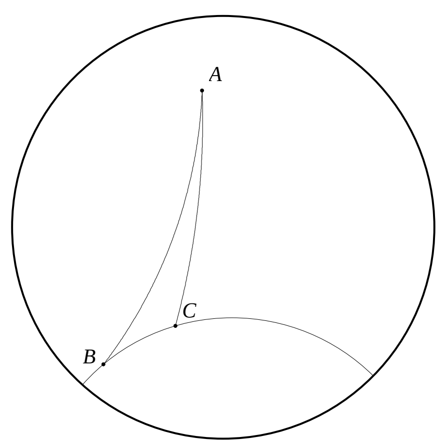

Figure 9.10: Obtuse Triangle