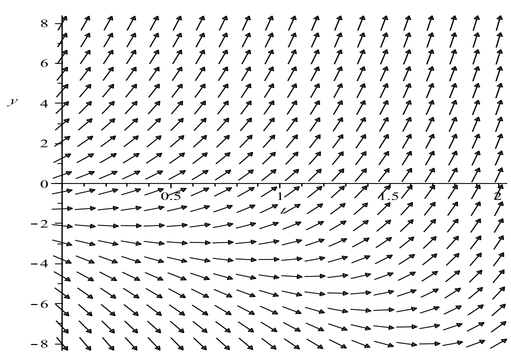
CHAPTER

2

First Order Differential Equations

2.1

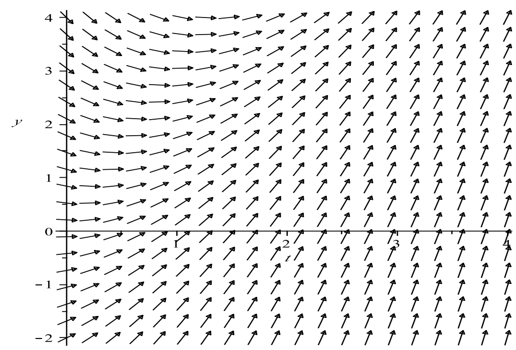
5.(a)



(b) If $y(0) > -3$, solutions eventually have positive slopes, and hence increase without bound. If $y(0) \le -3$, solutions have negative slopes and decrease without bound.

(c) The integrating factor is $\mu(t) = e^{-\int 2dt} = e^{-2t}$. The differential equation can be written as $e^{-2t}y' - 2e^{-2t}y = 3e^{-t}$, that is, $(e^{-2t}y)' = 3e^{-t}$. Integration of both sides of the equation results in the general solution $y(t) = -3e^t + c\,e^{2t}$. It follows that all solutions will increase exponentially if $c > 0$ and will decrease exponentially if $c \le 0$. Letting $c = 0$ and then $t = 0$, we see that the boundary of these behaviors is at $y(0) = -3$.

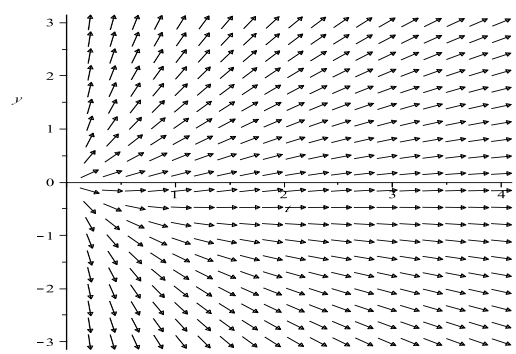
9.(a)



(b) All solutions eventually have positive slopes, and hence increase without bound.

(c) The integrating factor is $\mu(t) = e^{\int (1/2)\,dt} = e^{t/2}$. The differential equation can be written as $e^{t/2}y' + e^{t/2}y/2 = 3t\,e^{t/2}/2$, that is, $(e^{t/2}\,y/2)' = 3t\,e^{t/2}/2$. Integration of both sides of the equation results in the general solution $y(t) = 3t - 6 + c\,e^{-t/2}$. All solutions approach the specific solution $y_0(t) = 3t - 6$.

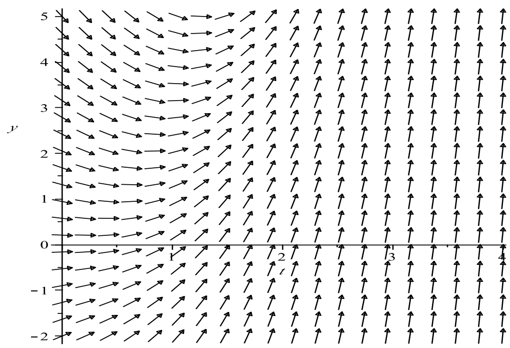
10.(a)



(b) For $y > 0$, the slopes are all positive, and hence the corresponding solutions increase without bound. For $y < 0$, almost all solutions have negative slopes, and hence solutions tend to decrease without bound.

(c) First divide both sides of the equation by t $(t > 0)$. From the resulting standard form, the integrating factor is $\mu(t) = e^{-\int (1/t)\,dt} = 1/t$. The differential equation can be written as $y'/t - y/t^2 = t\,e^{-t}$, that is, $(y/t)' = t\,e^{-t}$. Integration leads to the general solution $y(t) = -te^{-t} + c\,t$. For $c \neq 0$, solutions diverge, as implied by the direction field. For the case $c = 0$, the specific solution is $y(t) = -te^{-t}$, which evidently approaches zero as $t \to \infty$.

12.(a)



(b) All solutions eventually have positive slopes, and hence increase without bound.

(c) The integrating factor is $\mu(t) = e^{t/2}$. The differential equation can be written as $e^{t/2}y' + e^{t/2}y/2 = 3t^2/2$, that is, $(e^{t/2}\,y/2)' = 3t^2/2$. Integration of both sides of the equation results in the general solution $y(t) = 3t^2 - 12t + 24 + c\,e^{-t/2}$. It follows that all solutions converge to the specific solution $3t^2 - 12t + 24$.

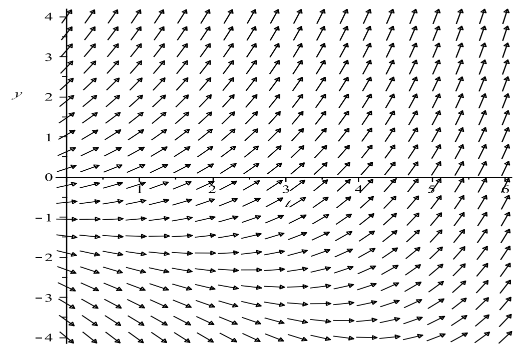
14. The integrating factor is $\mu(t) = e^{2t}$. After multiplying both sides by $\mu(t)$, the equation can be written as $(e^{2t}\,y)' = t$. Integrating both sides of the equation results in the general solution $y(t) = t^2e^{-2t}/2 + c\,e^{-2t}$. Invoking the specified condition, we require that $e^{-2}/2 + c\,e^{-2} = 0$. Hence $c = -1/2$, and the solution to the initial value problem is $y(t) = (t^2 - 1)e^{-2t}/2$.

16. The integrating factor is $\mu(t) = e^{\int (2/t)\,dt} = t^2$. Multiplying both sides by $\mu(t)$, the equation can be written as $(t^2\,y)' = \cos t$. Integrating both sides of the equation results in the general solution $y(t) = \sin t/t^2 + c\,t^{-2}$. Substituting $t = \pi$ and setting the value equal to zero gives $c = 0$. Hence the specific solution is $y(t) = \sin t/t^2$.

17. The integrating factor is $\mu(t) = e^{-2t}$, and the differential equation can be written as $(e^{-2t}\,y)' = 1$. Integrating, we obtain $e^{-2t}\,y(t) = t + c$. Invoking the specified initial condition results in the solution $y(t) = (t+2)e^{2t}$.

19. After writing the equation in standard form, we find that the integrating factor is $\mu(t) = e^{\int (4/t)\,dt} = t^4$. Multiplying both sides by $\mu(t)$, the equation can be written as $(t^4\,y)' = t\,e^{-t}$. Integrating both sides results in $t^4y(t) = -(t+1)e^{-t} + c$. Letting $t = -1$ and setting the value equal to zero gives $c = 0$. Hence the specific solution of the initial value problem is $y(t) = -(t^{-3} + t^{-4})e^{-t}$.

22.(a)

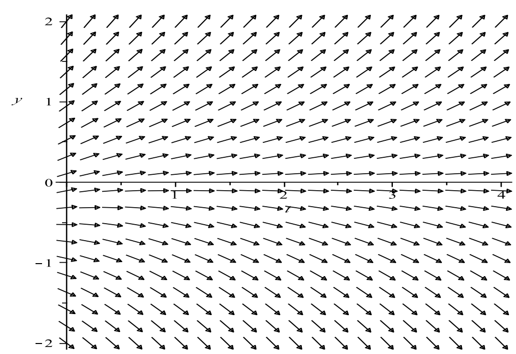


The solutions eventually increase or decrease, depending on the initial value a. The critical value seems to be $a_0 = -2$.

(b) The integrating factor is $\mu(t) = e^{-t/2}$, and the general solution of the differential equation is $y(t) = -3e^{t/3} + c\, e^{t/2}$. Invoking the initial condition $y(0) = a$, the solution may also be expressed as $y(t) = -3e^{t/3} + (a+3)\, e^{t/2}$. The critical value is $a_0 = -3$.

(c) For $a_0 = -3$, the solution is $y(t) = -3e^{t/3}$, which diverges to $-\infty$ as $t \to \infty$.

23.(a)

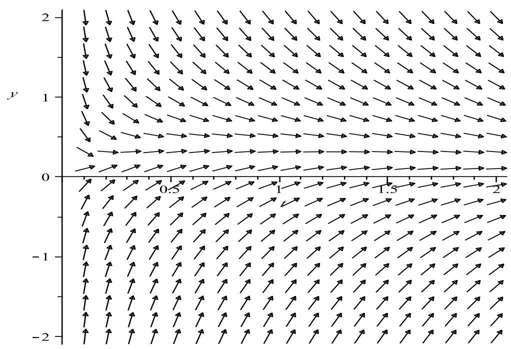


Solutions appear to grow infinitely large in absolute value, with signs depending on the initial value $y(0) = a_0$. The direction field appears horizontal for $a_0 \approx -1/8$.

(b) Dividing both sides of the given equation by 3, the integrating factor is $\mu(t) = e^{-2t/3}$. Multiplying both sides of the original differential equation by $\mu(t)$ and integrating results in $y(t) = (2\, e^{2t/3} - 2\, e^{-\pi t/2} + a(4+3\pi)\, e^{2t/3})/(4+3\pi)$. The qualitative behavior of the solution is determined by the terms containing $e^{2t/3}$: $2\, e^{2t/3} + a(4+3\pi)\, e^{2t/3}$. The nature of the solutions will change when $2 + a(4+3\pi) = 0$. Thus the critical initial value is $a_0 = -2/(4+3\pi)$.

(c) In addition to the behavior described in part (a), when $y(0) = -2/(4+3\pi)$, the solution is $y(t) = (-2\, e^{-\pi t/2})/(4+3\pi)$, and that specific solution will converge to $y = 0$.

24.(a)



As $t \to 0$, solutions increase without bound if $y(1) = a > 0.4$, and solutions decrease without bound if $y(1) = a < 0.4$.

(b) The integrating factor is $\mu(t) = e^{\int (t+1)/t\, dt} = t\, e^t$. The general solution of the differential equation is $y(t) = t\, e^{-t} + c\, e^{-t}/t$. Since $y(1) = a$, we have that $1 + c = ae$. That is, $c = ae - 1$. Hence the solution can also be expressed as $y(t) = t\, e^{-t} + (ae - 1)\, e^{-t}/t$. For small values of t, the second term is dominant. Setting $ae - 1 = 0$, the critical value of the parameter is $a_0 = 1/e$.

(c) When $a = 1/e$, the solution is $y(t) = t\, e^{-t}$, which approaches 0 as $t \to 0$.

27. The integrating factor is $\mu(t) = e^{\int (1/2)\, dt} = e^{t/2}$. Therefore the general solution is $y(t) = (4\cos t + 8\sin t)/5 + c\, e^{-t/2}$. Invoking the initial condition, the specific solution is $y(t) = (4\cos t + 8\sin t - 9\, e^{-t/2})/5$. Differentiating, it follows that $y'(t) = (-4\sin t + 8\cos t + 4.5\, e^{-t/2})/5$ and $y''(t) = (-4\cos t - 8\sin t - 2.25\, e^{-t/2})/5$. Setting $y'(t) = 0$, the first solution is $t_1 = 1.3643$, which gives the location of the first stationary point. Since $y''(t_1) < 0$, the first stationary point in a local maximum. The coordinates of the point are $(1.3643\,, 0.82008)$.

28. The integrating factor is $\mu(t) = e^{\int (2/3)\, dt} = e^{2t/3}$, and the differential equation can be written as $(e^{2t/3}\, y)' = e^{2t/3} - t\, e^{2t/3}/2$. The general solution is $y(t) = (21 - 6t)/8 + c\, e^{-2t/3}$. Imposing the initial condition, we have $y(t) = (21 - 6t)/8 + (y_0 - 21/8)e^{-2t/3}$. Since the solution is smooth, the desired intersection will be a point of tangency. Taking the derivative, $y'(t) = -3/4 - (2y_0 - 21/4)e^{-2t/3}/3$. Setting $y'(t) = 0$, the solution is $t_1 = (3/2)\ln[(21 - 8y_0)/9]$. Substituting into the solution, the respective value at the stationary point is $y(t_1) = 3/2 + (9/4)\ln 3 - (9/8)\ln(21 - 8y_0)$. Setting this result equal to zero, we obtain the required initial value $y_0 = (21 - 9\, e^{4/3})/8 \approx -1.643$.

29.(a) The integrating factor is $\mu(t) = e^{t/4}$, and the differential equation can be written as $(e^{t/4}\, y)' = 3\, e^{t/4} + 2\, e^{t/4}\cos 2t$. After integration, we get that the general solution is $y(t) = 12 + (8\cos 2t + 64\sin 2t)/65 + ce^{-t/4}$. Invoking the initial condition, $y(0) = 0$, the specific solution is $y(t) = 12 + (8\cos 2t + 64\sin 2t - 788\, e^{-t/4})/65$. As $t \to \infty$, the exponential term will decay, and the solution will oscillate about an average value of 12, with an amplitude of $8/\sqrt{65}$.

(b) Solving $y(t) = 12$, we obtain the desired value $t \approx 10.0658$.

31. The integrating factor is $\mu(t) = e^{-3t/2}$, and the differential equation can be written as $(e^{-3t/2}\,y)' = 3t\,e^{-3t/2} + 2\,e^{-t/2}$. The general solution is $y(t) = -2t - 4/3 - 4\,e^{t} + c\,e^{3t/2}$. Imposing the initial condition, $y(t) = -2t - 4/3 - 4\,e^{t} + (y_0 + 16/3)\,e^{3t/2}$. Now as $t \to \infty$, the term containing $e^{3t/2}$ will dominate the solution. Its sign will determine the divergence properties. Hence the critical value of the initial condition is $y_0 = -16/3$. The corresponding solution, $y(t) = -2t - 4/3 - 4\,e^{t}$, will also decrease without bound.

Note on Problems 34-37:

Let $g(t)$ be given, and consider the function $y(t) = y_1(t) + g(t)$, in which $y_1(t) \to 0$ as $t \to \infty$. Differentiating, $y'(t) = y_1'(t) + g'(t)$. Letting a be a constant, it follows that $y'(t) + ay(t) = y_1'(t) + ay_1(t) + g'(t) + ag(t)$. Note that the hypothesis on the function $y_1(t)$ will be satisfied, if $y_1'(t) + ay_1(t) = 0$. That is, $y_1(t) = c\,e^{-at}$. Hence $y(t) = c\,e^{-at} + g(t)$, which is a solution of the equation $y' + ay = g'(t) + ag(t)$. For convenience, choose $a = 1$.

34. Here $g(t) = 3$, and we consider the linear equation $y' + y = 3$. The integrating factor is $\mu(t) = e^{t}$, and the differential equation can be written as $(e^{t}\,y)' = 3e^{t}$. The general solution is $y(t) = 3 + c\,e^{-t}$.

36. Here $g(t) = 2t - 5$. Consider the linear equation $y' + y = 2 + 2t - 5$. The integrating factor is $\mu(t) = e^{t}$, and the differential equation can be written as $(e^{t}\,y)' = (2t - 3)e^{t}$. The general solution is $y(t) = 2t - 5 + c\,e^{-t}$.

37. $g(t) = 4 - t^2$. Consider the linear equation $y' + y = 4 - 2t - t^2$. The integrating factor is $\mu(t) = e^{t}$, and the equation can be written as $(e^{t}\,y)' = (4 - 2t - t^2)e^{t}$. The general solution is $y(t) = 4 - t^2 + c\,e^{-t}$.

38.(a) Differentiating y and using the fundamental theorem of calculus we obtain that $y' = Ae^{-\int p(t)dt} \cdot (-p(t))$, and then $y' + p(t)y = 0$.

(b) Differentiating y we obtain that

$$y' = A'(t)e^{-\int p(t)dt} + A(t)e^{-\int p(t)dt} \cdot (-p(t)).$$

If this satisfies the differential equation then

$$y' + p(t)y = A'(t)e^{-\int p(t)dt} = g(t)$$

and the required condition follows.

(c) Let us denote $\mu(t) = e^{\int p(t)dt}$. Then clearly $A(t) = \int \mu(t)g(t)dt$, and after substitution $y = \int \mu(t)g(t)dt \cdot (1/\mu(t))$, which is just Eq. (33).

40. We assume a solution of the form $y = A(t)e^{-\int(1/t)\,dt} = A(t)e^{-\ln t} = A(t)t^{-1}$, where $A(t)$ satisfies $A'(t) = 3t\cos 2t$. This implies that

$$A(t) = \frac{3\cos 2t}{4} + \frac{3t\sin 2t}{2} + c$$

and the solution is

$$y = \frac{3\cos 2t}{4t} + \frac{3\sin 2t}{2} + \frac{c}{t}.$$

41. First rewrite the differential equation as

$$y' + \frac{2}{t}\,y = \frac{\sin\,t}{t}\,.$$

Assume a solution of the form $y = A(t)e^{-\int(2/t)\,dt} = A(t)t^{-2}$, where $A(t)$ satisfies the ODE $A'(t) = t\,\sin\,t$. It follows that $A(t) = \sin\,t\, - t\,\cos\,t\, + c$ and thus $y = (\sin\,t\, - t\,\cos\,t\, + c)/t^2$.

2.2

Problems 1 through 20 follow the pattern of the examples worked in this section. The first eight problems, however, do not have an initial condition, so the integration constant c cannot be found.

2. For $x \neq -1$, the differential equation may be written as $y\,dy = \left[x^2/(1+x^3)\right]dx$. Integrating both sides, with respect to the appropriate variables, we obtain the relation $y^2/2 = (1/3)\ln\left|1+x^3\right| + c$. That is, $y(x) = \pm\sqrt{(2/3)\ln\left|1+x^3\right| + c}$.

3. The differential equation may be written as $y^{-2}dy = -\sin\,x\,dx$. Integrating both sides of the equation, with respect to the appropriate variables, we obtain the relation $-y^{-1} = \cos\,x + c$. That is, $(c - \,\cos\,x)y = 1$, in which c is an arbitrary constant. Solving for the dependent variable, explicitly, $y(x) = 1/(c - \,\cos\,x)$.

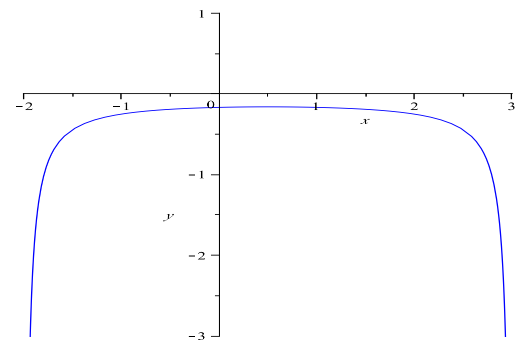
5. Write the differential equation as $\cos^{-2}\,2y\,dy = \cos^2 x\,dx$, which also can be written as $\sec^2\,2y\,dy = \cos^2 x\,dx$. Integrating both sides of the equation, with respect to the appropriate variables, we obtain the relation $\tan\,2y = \sin\,x\,\cos\,x + x + c$.

7. The differential equation may be written as $(y + e^y)dy = (x - e^{-x})dx$. Integrating both sides of the equation, with respect to the appropriate variables, we obtain the relation $y^2 + 2\,e^y = x^2 + 2\,e^{-x} + c$.

8. Write the differential equation as $(1+y^2)dy = x^2\,dx$. Integrating both sides of the equation, we obtain the relation $y + y^3/3 = x^3/3\, + c$.

9.(a) The differential equation is separable, with $y^{-2}dy = (1-2x)dx$. Integration yields $-y^{-1} = x - x^2 + c$. Substituting $x = 0$ and $y = -1/6$, we find that $c = 6$. Hence the specific solution is $y = 1/(x^2 - x - 6)$.

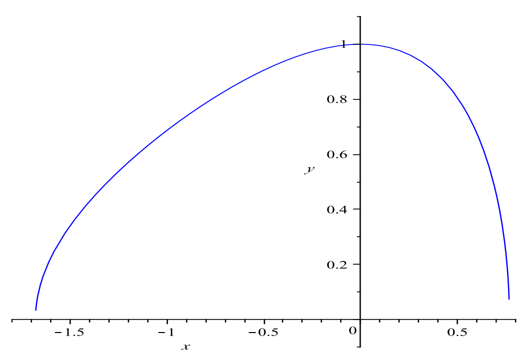
(b)



(c) Note that $x^2 - x - 6 = (x+2)(x-3)$. Hence the solution becomes singular at $x = -2$ and $x = 3$, so the interval of existence is $(-2, 3)$.

11.(a) Rewrite the differential equation as $x\,e^x dx = -y\,dy$. Integrating both sides of the equation results in $x\,e^x - e^x = -y^2/2 + c$. Invoking the initial condition, we obtain $c = -1/2$. Hence $y^2 = 2e^x - 2x\,e^x - 1$. The explicit form of the solution is $y(x) = \sqrt{2e^x - 2x\,e^x - 1}$. The positive sign is chosen, since $y(0) = 1$.

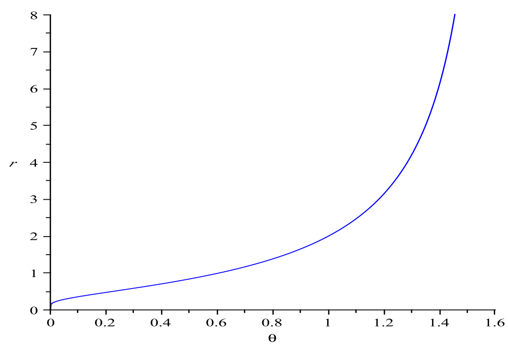
(b)



(c) The function under the radical becomes negative near $x \approx -1.7$ and $x \approx 0.77$.

12.(a) Write the differential equation as $r^{-2}dr = \theta^{-1}\,d\theta$. Integrating both sides of the equation results in the relation $-r^{-1} = \ln\,\theta + c$. Imposing the condition $r(1) = 2$, we obtain $c = -1/2$. The explicit form of the solution is $r = 2/(1 - 2\,\ln\,\theta)$.

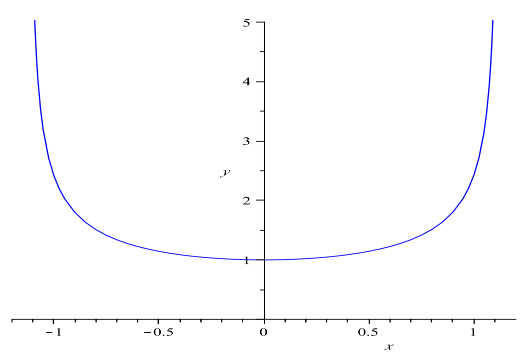
(b)



(c) Clearly, the solution makes sense only if $\theta > 0$. Furthermore, the solution becomes singular when $\ln\theta = 1/2$, that is, $\theta = \sqrt{e}$.

14.(a) Write the differential equation as $y^{-3}dy = x(1+x^2)^{-1/2}\,dx$. Integrating both sides of the equation, with respect to the appropriate variables, we obtain the relation $-y^{-2}/2 = \sqrt{1+x^2} + c$. Imposing the initial condition, we obtain $c = -3/2$. Hence the specific solution can be expressed as $y^{-2} = 3 - 2\sqrt{1+x^2}$. The explicit form of the solution is $y(x) = 1/\sqrt{3 - 2\sqrt{1+x^2}}$. The positive sign is chosen to satisfy the initial condition.

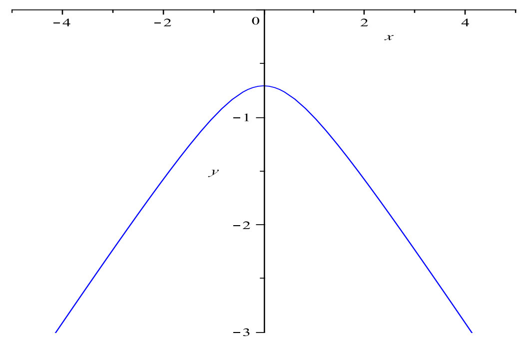
(b)



(c) The solution becomes singular when $2\sqrt{1+x^2} = 3$. That is, at $x = \pm\sqrt{5}/2$.

16.(a) Rewrite the differential equation as $4y^3dy = x(x^2+1)dx$. Integrating both sides of the equation results in $y^4 = (x^2+1)^2/4 + c$. Imposing the initial condition, we obtain $c = 0$. Hence the solution may be expressed as $(x^2+1)^2 - 4y^4 = 0$. The explicit form of the solution is $y(x) = -\sqrt{(x^2+1)/2}$. The sign is chosen based on $y(0) = -1/\sqrt{2}$.

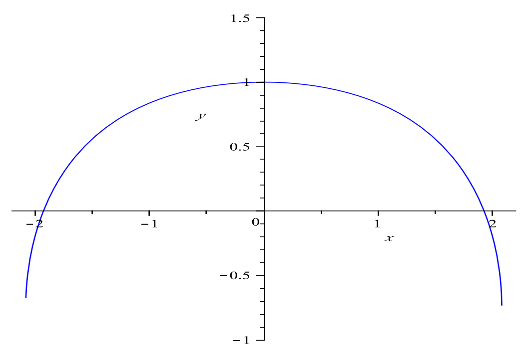
(b)



(c) The solution is valid for all $x \in \mathbb{R}$.

18.(a) Write the differential equation as $(3+4y)dy = (e^{-x} - e^{x})dx$. Integrating both sides of the equation, with respect to the appropriate variables, we obtain the relation $3y + 2y^2 = -(e^{x} + e^{-x}) + c$. Imposing the initial condition, $y(0) = 1$, we obtain $c = 7$. Thus, the solution can be expressed as $3y + 2y^2 = -(e^{x} + e^{-x}) + 7$. Now by completing the square on the left hand side, $2(y + 3/4)^2 = -(e^{x} + e^{-x}) + 65/8$. Hence the explicit form of the solution is $y(x) = -3/4 + \sqrt{65/16 - \cosh x}$.

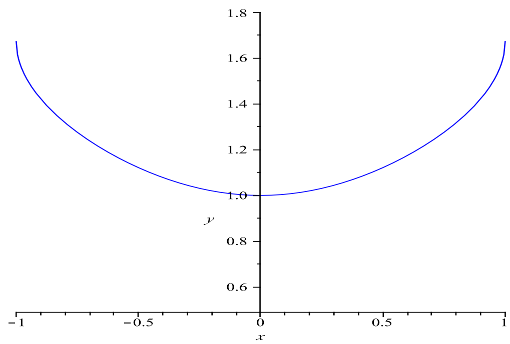
(b)



(c) Note the $65 - 16\cosh x \geq 0$ as long as $|x| > 2.1$ (approximately). Hence the solution is valid on the interval $-2.1 < x < 2.1$.

20.(a) Rewrite the differential equation as $y^2 dy = \arcsin x/\sqrt{1-x^2}\, dx$. Integrating both sides of the equation results in $y^3/3 = (\arcsin x)^2/2 + c$. Imposing the condition $y(0) = 1$, we obtain $c = 1/3$. The explicit form of the solution is $y(x) = (3(\arcsin x)^2/2 + 1)^{1/3}$.

(b)



(c) Since $\arcsin x$ is defined for $-1 \leq x \leq 1$, this is the interval of existence.

22. The differential equation can be written as $(3y^2 - 4)dy = 3x^2 dx$. Integrating both sides, we obtain $y^3 - 4y = x^3 + c$. Imposing the initial condition, the specific solution is $y^3 - 4y = x^3 - 1$. Referring back to the differential equation, we find that $y' \to \infty$ as $y \to \pm 2/\sqrt{3}$. The respective values of the abscissas are $x \approx -1.276$, 1.598. Hence the solution is valid for $-1.276 < x < 1.598$.

24. Write the differential equation as $(3 + 2y)dy = (2 - e^x)dx$. Integrating both sides, we obtain $3y + y^2 = 2x - e^x + c$. Based on the specified initial condition, the solution can be written as $3y + y^2 = 2x - e^x + 1$. Completing the square, it follows that $y(x) = -3/2 + \sqrt{2x - e^x + 13/4}$. The solution is defined if $2x - e^x + 13/4 \geq 0$, that is, $-1.5 \leq x \leq 2$ (approximately). In that interval, $y' = 0$ for $x = \ln 2$. It can be verified that $y''(\ln 2) < 0$. In fact, $y''(x) < 0$ on the interval of definition. Hence the solution attains a global maximum at $x = \ln 2$.

26. The differential equation can be written as $(1 + y^2)^{-1}dy = 2(1 + x)dx$. Integrating both sides of the equation, we obtain $\arctan y = 2x + x^2 + c$. Imposing the given initial condition, the specific solution is $\arctan y = 2x + x^2$. Therefore, $y = \tan(2x + x^2)$. Observe that the solution is defined as long as $-\pi/2 < 2x + x^2 < \pi/2$. It is easy to see that $2x + x^2 \geq -1$. Furthermore, $2x + x^2 = \pi/2$ for $x \approx -2.6$ and 0.6. Hence the solution is valid on the interval $-2.6 < x < 0.6$. Referring back to the differential equation, the solution is stationary at $x = -1$. Since $y''(-1) > 0$, the solution attains a global minimum at $x = -1$.

28.(a) Write the differential equation as $y^{-1}(4 - y)^{-1}dy = t(1 + t)^{-1}dt$. Integrating both sides of the equation, we obtain $\ln |y| - \ln |y - 4| = 4t - 4\ln|1 + t| + c$. Taking the exponential of both sides $|y/(y - 4)| = c\, e^{4t}/(1 + t)^4$. It follows that as $t \to \infty$, $|y/(y - 4)| = |1 + 4/(y - 4)| \to \infty$. That is, $y(t) \to 4$.

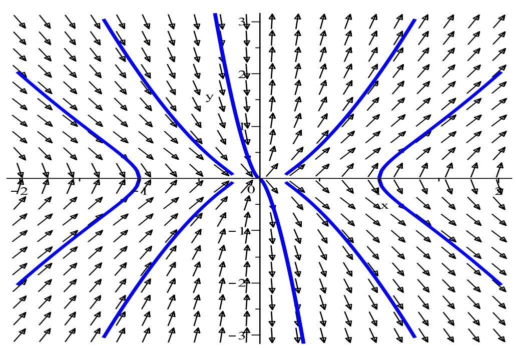
(b) Setting $y(0) = 2$, we obtain that $c = 1$. Based on the initial condition, the solution may be expressed as $y/(y - 4) = -e^{4t}/(1 + t)^4$. Note that $y/(y - 4) < 0$, for all $t \geq 0$. Hence $y < 4$ for all $t \geq 0$. Referring back to the differential equation, it follows that y' is always positive. This means that the solution is monotone increasing. We find that the root of the equation $e^{4t}/(1+t)^4 = 399$ is near $t = 2.844$.

(c) Note the $y(t) = 4$ is an equilibrium solution. Examining the local direction field we see that if $y(0) > 0$, then the corresponding solutions converge to $y = 4$. Referring back to part (a), we have $y/(y-4) = [y_0/(y_0-4)]\, e^{4t}/(1+t)^4$, for $y_0 \neq 4$. Setting $t = 2$, we obtain $y_0/(y_0-4) = (3/e^2)^4 y(2)/(y(2)-4)$. Now since the function $f(y) = y/(y-4)$ is monotone for $y < 4$ and $y > 4$, we need only solve the equations $y_0/(y_0-4) = -399(3/e^2)^4$ and $y_0/(y_0-4) = 401(3/e^2)^4$. The respective solutions are $y_0 = 3.6622$ and $y_0 = 4.4042$.

32.(a) Observe that $(x^2+3y^2)/2xy = (1/2)(y/x)^{-1} + (3/2)(y/x)$. Hence the differential equation is homogeneous.

(b) The substitution $y = x\,v$ results in $v + x\,v' = (x^2+3x^2v^2)/2x^2v$. The transformed equation is $v' = (1+v^2)/2xv$. This equation is separable, with general solution $v^2 + 1 = c\,x$. In terms of the original dependent variable, the solution is $x^2 + y^2 = c\,x^3$.

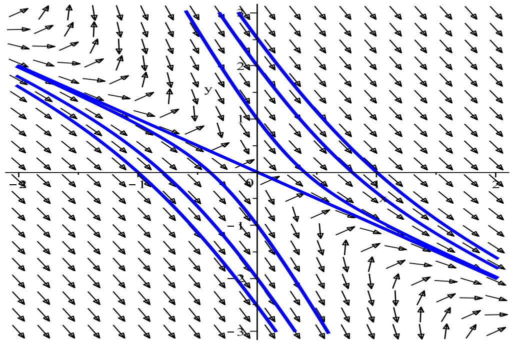
(c) The integral curves are symmetric with respect to the origin.



34.(a) Observe that $-(4x+3y)/(2x+y) = -2 - (y/x)\,[2+(y/x)]^{-1}$. Hence the differential equation is homogeneous.

(b) The substitution $y = x\,v$ results in $v + x\,v' = -2 - v/(2+v)$. The transformed equation is $v' = -(v^2+5v+4)/(2+v)x$. This equation is separable, with general solution $(v+4)^2\,|v+1| = c/x^3$. In terms of the original dependent variable, the solution is $(4x+y)^2\,|x+y| = c$.

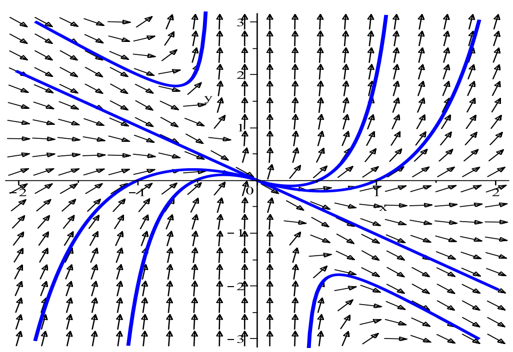
(c) The integral curves are symmetric with respect to the origin.



36.(a) Divide by x^2 to see that the equation is homogeneous. Substituting $y = x\,v$, we obtain $x\,v' = (1+v)^2$. The resulting differential equation is separable.

(b) Write the equation as $(1+v)^{-2}dv = x^{-1}dx$. Integrating both sides of the equation, we obtain the general solution $-1/(1+v) = \ln|x| + c$. In terms of the original dependent variable, the solution is $y = x\,(c - \ln|x|)^{-1} - x$.

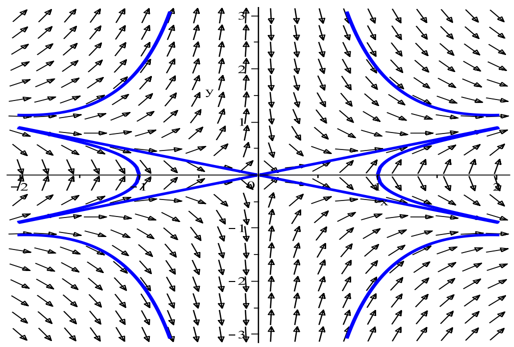
(c) The integral curves are symmetric with respect to the origin.



37.(a) The differential equation can be expressed as $y' = (1/2)(y/x)^{-1} - (3/2)(y/x)$. Hence the equation is homogeneous. The substitution $y = xv$ results in $xv' = (1-5v^2)/2v$. Separating variables, we have $2vdv/(1-5v^2) = dx/x$.

(b) Integrating both sides of the transformed equation yields $-(\ln|1-5v^2|)/5 = \ln|x| + c$, that is, $1 - 5v^2 = c/\left|x\right|^5$. In terms of the original dependent variable, the general solution is $5y^2 = x^2 - c/\left|x\right|^3$.

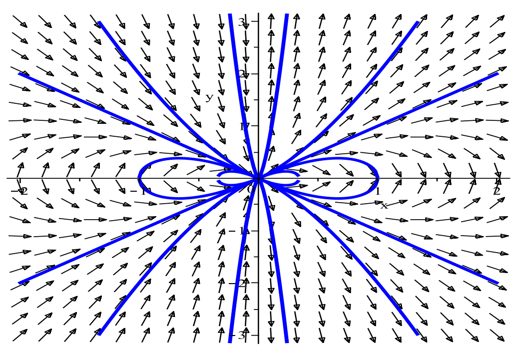
(c) The integral curves are symmetric with respect to the origin.



38.(a) The differential equation can be expressed as $y' = (3/2)(y/x) - (1/2)(y/x)^{-1}$. Hence the equation is homogeneous. The substitution $y = x\,v$ results in $x\,v' = (v^2 - 1)/2v$, that is, $2vdv/(v^2 - 1) = dx/x$.

(b) Integrating both sides of the transformed equation yields $\ln\left|v^2 - 1\right| = \ln|x| + c$, that is, $v^2 - 1 = c\,|x|$. In terms of the original dependent variable, the general solution is $y^2 = c\,x^2\,|x| + x^2$.

(c) The integral curves are symmetric with respect to the origin.



2.3

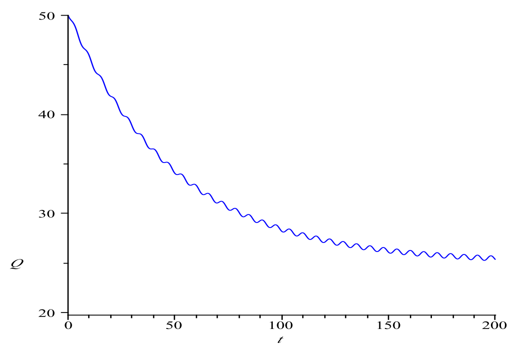
1. Let $Q(t)$ be the amount of dye in the tank at time t. Clearly, $Q(0) = 200$ g. The differential equation governing the amount of dye is $Q'(t) = -2Q(t)/200$. The solution of this separable equation is $Q(t) = Q(0)e^{-t/100} = 200e^{-t/100}$. We need the time T such that $Q(T) = 2$ g. This means we have to solve $2 = 200e^{-T/100}$ and we obtain that $T = -100\ln(1/100) = 100\ln 100 \approx 460.5$ min.

5.(a) Let Q be the amount of salt in the tank. Salt enters the tank of water at a rate of $2\,(1/4)(1 + (1/2)\sin\,t) = 1/2 + (1/4)\sin\,t$ oz/min. It leaves the tank at a rate of $2\,Q/100$ oz/min. Hence the differential equation governing the amount of salt at any time is

$$\frac{dQ}{dt} = \frac{1}{2} + \frac{1}{4}\sin\,t - \frac{Q}{50}\,.$$

The initial amount of salt is $Q_0 = 50$ oz. The governing differential equation is linear, with integrating factor $\mu(t) = e^{t/50}$. Write the equation as $(e^{t/50}Q)' = e^{t/50}(1/2 + (1/4)\sin\,t)$. The specific solution is $Q(t) = 25 + (12.5\sin\,t - 625\cos\,t + 63150\,e^{-t/50})/2501$ oz.

(b)



(c) The amount of salt approaches a steady state, which is an oscillation of approximate amplitude 1/4 about a level of 25 oz.

6.(a) Using the Principle of Conservation of Energy, the speed v of a particle falling from a height h is given by

$$\frac{1}{2}mv^2 = mgh\,.$$

(b) The outflow rate is (outflow cross-section area)×(outflow velocity): $\alpha\,a\sqrt{2gh}$. At any instant, the volume of water in the tank is $V(h) = \int_0^h A(u)du$. The time rate of change of the volume is given by $dV/dt = (dV/dh)(dh/dt) = A(h)dh/dt$. Since the volume is decreasing, $dV/dt = -\alpha\,a\sqrt{2gh}$.

(c) With $A(h) = \pi$, $a = 0.01\,\pi$, $\alpha = 0.6$, the differential equation for the water level h is $\pi(dh/dt) = -0.006\,\pi\sqrt{2gh}$, with solution $h(t) = 0.000018gt^2 - 0.006\sqrt{2gh(0)}\,t + h(0)$. Setting $h(0) = 3$ and $g = 9.8$, $h(t) = 0.0001764\,t^2 - 0.046\,t + 3$, resulting in $h(t) = 0$ for $t \approx 130.4$ s.

7.(a) The equation governing the value of the investment is $dS/dt = r\,S$. The value of the investment, at any time, is given by $S(t) = S_0e^{rt}$. Setting $S(T) = 2S_0$, the required time is $T = \ln(2)/r$.

(b) For the case $r = .07$, $T \approx 9.9$ yr.

(c) Referring to part (a), $r = \ln(2)/T$. Setting $T = 8$, the required interest rate is to be approximately $r = 8.66\%$.

12.(a) Using Eq.(15) we have $dS/dt - 0.005S = -(800 + 10t)$, $S(0) = 150,000$. Using an integrating factor and integration by parts we obtain that $S(t) = 560,000 - 410,000e^{0.005t} + 2000t$. Setting $S(t) = 0$ and solving numerically for t yields $t = 146.54$ months.

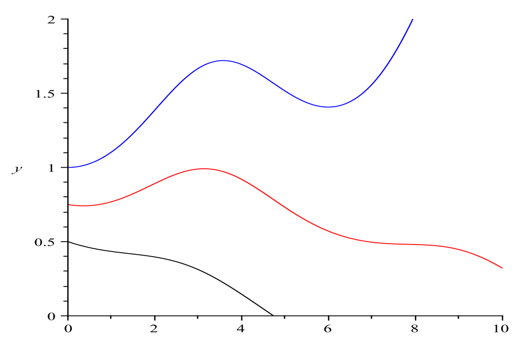
(b) The solution we obtained in part (a) with a general initial condition $S(0) = S_0$ is $S(t) = 560,000 - 560,000e^{0.005t} + S_0e^{0.005t} + 2000t$. Solving the equation $S(240) = 0$ yields $S_0 = 246,758$.

13.(a) Let $Q' = -r\,Q$. The general solution is $Q(t) = Q_0e^{-rt}$. Based on the definition of half-life, consider the equation $Q_0/2 = Q_0e^{-5730\,r}$. It follows that $-5730\,r = \ln(1/2)$, that is, $r = 1.2097 \times 10^{-4}$ per year.

(b) The amount of carbon-14 is given by $Q(t) = Q_0\, e^{-1.2097\times10^{-4}t}$.

(c) Given that $Q(T) = Q_0/5$, we have the equation $1/5 = e^{-1.2097\times10^{-4}T}$. Solving for the decay time, the apparent age of the remains is approximately $T = 13,305$ years.

15.(a) The differential equation $dy/dt = r(t)\,y - k$ is linear, with integrating factor $\mu(t) = e^{-\int r(t)dt}$. Write the equation as $(\mu\, y)' = -k\,\mu(t)$. Integration of both sides yields the general solution $y = \left[-k\int \mu(\tau)d\tau + y_0\,\mu(0)\right]/\mu(t)$. In this problem, the integrating factor is $\mu(t) = e^{(\cos\, t - t)/5}$.



(b) The population becomes extinct, if $y(t^*) = 0$, for some $t = t^*$. Referring to part (a), we find that $y(t^*) = 0$ when

$$\int_0^{t^*} e^{(\cos\,\tau - \tau)/5} d\tau = 5\, e^{1/5} y_c .$$

It can be shown that the integral on the left hand side increases monotonically, from zero to a limiting value of approximately 5.0893 . Hence extinction can happen only if $5\, e^{1/5} y_0 < 5.0893$. Solving $5e^{1/5}y_c = 5.0893$ yields $y_c = 0.8333$.

(c) Repeating the argument in part (b), it follows that $y(t^*) = 0$ when

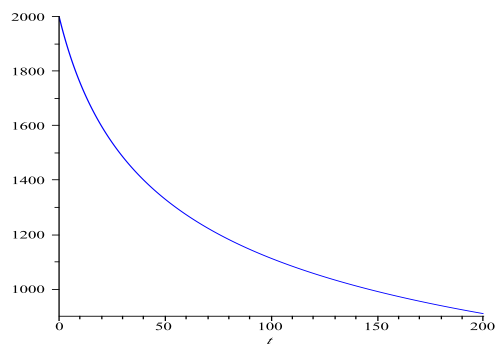
$$\int_0^{t^*} e^{(\cos\tau - \tau)/5} d\tau = \frac{1}{k}\, e^{1/5} y_c.$$

Hence extinction can happen only if $e^{1/5} y_0/k < 5.0893$, so $y_c = 4.1667\, k$.

(d) Evidently, y_c is a linear function of the parameter k.

17.(a) The solution of the governing equation satisfies $u^3 = u_0^3/(\,3\,\alpha\, u_0^3\, t + 1\,)$. With the given data, it follows that $u(t) = 2000/\sqrt[3]{6\,t/125 + 1}$.

(b)

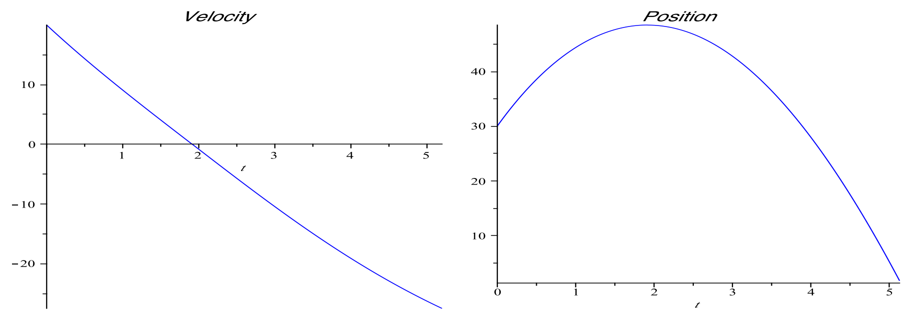


(c) Numerical evaluation results in $u(t) = 600$ for $t \approx 750.77$ s.

22.(a) The differential equation for the upward motion is $mdv/dt = -\mu v^2 - mg$, in which $\mu = 1/1325$. This equation is separable, with $m/(\mu\, v^2 + mg)\, dv = -dt$. Integrating both sides and invoking the initial condition, $v(t) = 44.133\,\tan(0.425 - 0.222\,t)$. Setting $v(t_1) = 0$, the ball reaches the maximum height at $t_1 = 1.916$ s. Integrating $v(t)$, the position is given by $x(t) = 198.75\,\ln[\cos(0.222\,t - 0.425)] + 48.57$. Therefore the maximum height is $x(t_1) = 48.56$ m.

(b) The differential equation for the downward motion is $m\,dv/dt = +\mu v^2 - mg$. This equation is also separable, with $m/(mg - \mu\, v^2)\, dv = -dt$. For convenience, set $t = 0$ at the top of the trajectory. The new initial condition becomes $v(0) = 0$. Integrating both sides and invoking the initial condition, we obtain $\ln((44.13 - v)/(44.13 + v)) = t/2.25$. Solving for the velocity, $v(t) = 44.13(1 - e^{t/2.25})/(1 + e^{t/2.25})$. Integrating $v(t)$, we obtain $x(t) = 99.29\,\ln(e^{t/2.25}/(1 + e^{t/2.25})^2) + 186.2$. To estimate the duration of the downward motion, set $x(t_2) = 0$, resulting in $t_2 = 3.276$ s. Hence the total time that the ball spends in the air is $t_1 + t_2 = 5.192$ s.

(c)



24.(a) Setting $-\mu v^2 = v(dv/dx)$, we obtain $dv/dx = -\mu v$.

(b) The speed v of the sled satisfies $\ln(v/v_0) = -\mu x$. Noting that the unit conversion factors cancel, solution of $\ln(15/150) = -2000\,\mu$ results in $\mu = \ln(10)/2000\ \text{ft}^{-1} \approx 0.00115\ \text{ft}^{-1} \approx 6.0788\ \text{mi}^{-1}$.

(c) Solution of $dv/dt = -\mu v^2$ can be expressed as $1/v - 1/v_0 = \mu t$. Noting that $1\,\text{mi/hr} = 5280/3600\,\text{ft/s}$, the elapsed time is

$$t = (1/15 - 1/150)/((5280/3600)(\ln(10)/2000)) \approx 35.53\,\text{s}.$$

25.(a) Measure the positive direction of motion upward. The equation of motion is given by $m dv/dt = -k\,v - mg$. The initial value problem is $dv/dt = -kv/m - g$, with $v(0) = v_0$. The solution is $v(t) = -mg/k + (v_0 + mg/k)e^{-kt/m}$. Setting $v(t_m) = 0$, the maximum height is reached at time $t_m = (m/k)\ln\left[(mg + k\,v_0)/mg\right]$. Integrating the velocity, the position of the body is

$$x(t) = -mg\,t/k + \left[(\frac{m}{k})^2 g + \frac{m\,v_0}{k}\right](1 - e^{-kt/m}).$$

Hence the maximum height reached is

$$x_m = x(t_m) = \frac{m\,v_0}{k} - g(\frac{m}{k})^2 \ln\left[\frac{mg + k\,v_0}{mg}\right].$$

(b) Recall that for $\delta \ll 1$, $\ln(1+\delta) = \delta - \delta^2/2 + \delta^3/3 - \delta^4/4 + \ldots$.

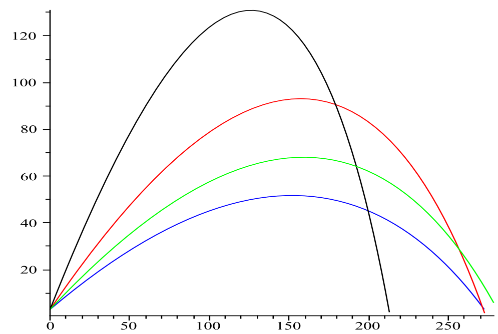
(c) The dimensions of the quantities involved are $[k] = MT^{-1}$, $[v_0] = LT^{-1}$, $[m] = M$ and $[g] = LT^{-2}$. This implies that kv_0/mg is dimensionless.

31.(a) Both equations are linear and separable. Initial conditions: $v(0) = u\cos A$ and $w(0) = u\sin A$. We obtain the solutions $v(t) = (u\cos A)e^{-rt}$ and $w(t) = -g/r + (u\sin A + g/r)e^{-rt}$.

(b) Integrating the solutions in part (a), and invoking the initial conditions, the coordinates are $x(t) = u\cos A(1 - e^{-rt})/r$ and

$$y(t) = -\frac{gt}{r} + \frac{g + ur\,\sin\,A + hr^2}{r^2} - (\frac{u}{r}\sin\,A + \frac{g}{r^2})e^{-rt}.$$

(c)



(d) Let T be the time that it takes the ball to go 350 ft horizontally. Then from above, $e^{-T/5} = (u \cos A - 70)/u \cos A$. At the same time, the height of the ball is given by

$$y(T) = -160T + 803 + 5u \sin A - \frac{(800 + 5u \sin A)(u \cos A - 70)}{u \cos A}.$$

Hence A and u must satisfy the equality

$$800 \ln \left[\frac{u \cos A - 70}{u \cos A} \right] + 803 + 5u \sin A - \frac{(800 + 5u \sin A)(u \cos A - 70)}{u \cos A} = 10$$

for the ball to touch the top of the wall. To find the optimal values for u and A, consider u as a function of A and use implicit differentiation in the above equation to find that

$$\frac{du}{dA} = -\frac{u(u^2 \cos A - 70u - 11200 \sin A)}{11200 \cos A}.$$

Solving this equation simultaneously with the above equation yields optimal values for u and A: $u \approx 145.3\,\text{ft/s}$, $A \approx 0.644\,\text{rad}$.

32.(a) Solving equation (i), $y'(x) = \left[(k^2 - y)/y\right]^{1/2}$. The positive answer is chosen, since y is an increasing function of x.

(b) Let $y = k^2 \sin^2 t$. Then $dy = 2k^2 \sin t \cos t dt$. Substituting into the equation in part (a), we find that

$$\frac{2k^2 \sin t \cos t dt}{dx} = \frac{\cos t}{\sin t}.$$

Hence $2k^2 \sin^2 t dt = dx$.

(c) Setting $\theta = 2t$, we further obtain $k^2 \sin^2(\theta/2)\, d\theta = dx$. Integrating both sides of the equation and noting that $t = \theta = 0$ corresponds to the origin, we obtain the solutions $x(\theta) = k^2(\theta - \sin \theta)/2$ and (from part (b)) $y(\theta) = k^2(1 - \cos \theta)/2$.

(d) Note that $y/x = (1-\cos\theta)/(\theta-\sin\theta)$. Setting $x = 1$, $y = 2$, the solution of the equation $(1-\cos\theta)/(\theta-\sin\theta) = 2$ is $\theta \approx 1.401$. Substitution into either of the expressions yields $k \approx 2.193$.

2.4

2. Rewrite the differential equation as $y' + 1/(t(t-4))\,y = 0$. It is evident that the coefficient $1/t(t-4)$ is continuous everywhere except at $t = 0, 4$. Since the initial condition is specified at $t = 2$, Theorem 2.4.1 assures the existence of a unique solution on the interval $0 < t < 4$.

3. The function $\tan t$ is discontinuous at odd multiples of $\pi/2$. Since $\pi/2 < \pi < 3\pi/2$, the initial value problem has a unique solution on the interval $(\pi/2, 3\pi/2)$.

5. $p(t) = 2t/(4-t^2)$ and $g(t) = 3t^2/(4-t^2)$. These functions are discontinuous at $x = \pm 2$. The initial value problem has a unique solution on the interval $(-2, 2)$.

6. The function $\ln t$ is defined and continuous on the interval $(0, \infty)$. At $t = 1$, $\ln t = 0$, so the normal form of the differential equation has a singularity there. Also, $\cot t$ is not defined at integer multiples of π, so the initial value problem will have a solution on the interval $(1, \pi)$.

7. The function $f(t, y)$ is continuous everywhere on the plane, except along the straight line $y = -2t/5$. The partial derivative $\partial f/\partial y = -7t/(2t+5y)^2$ has the same region of continuity.

9. The function $f(t, y)$ is discontinuous along the coordinate axes, and on the hyperbola $t^2 - y^2 = 1$. Furthermore,

$$\frac{\partial f}{\partial y} = \frac{\pm 1}{y(1-t^2+y^2)} - 2\frac{y\ln|ty|}{(1-t^2+y^2)^2}$$

has the same points of discontinuity.

10. $f(t, y)$ is continuous everywhere on the plane. The partial derivative $\partial f/\partial y$ is also continuous everywhere.

12. The function $f(t, y)$ is discontinuous along the lines $t = \pm k\pi$ for $k = 0, 1, 2, \ldots$ and $y = -1$. The partial derivative $\partial f/\partial y = \cot t/(1+y)^2$ has the same region of continuity.

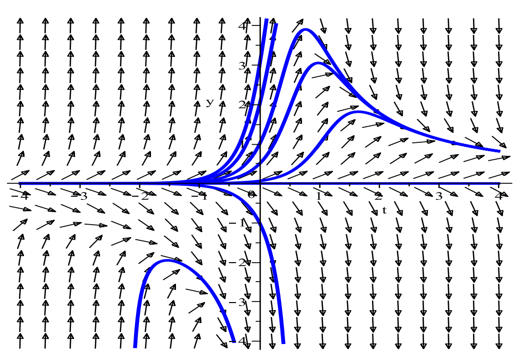
14. The equation is separable, with $dy/y^2 = 2t\,dt$. Integrating both sides, the solution is given by $y(t) = y_0/(1-y_0t^2)$. For $y_0 > 0$, solutions exist as long as $t^2 < 1/y_0$. For $y_0 \le 0$, solutions are defined for all t.

15. The equation is separable, with $dy/y^3 = -\,dt$. Integrating both sides and invoking the initial condition, $y(t) = y_0/\sqrt{2y_0^2t+1}$. Solutions exist as long as

$2y_0^2t + 1 > 0$, that is, $2y_0^2t > -1$. If $y_0 \neq 0$, solutions exist for $t > -1/2y_0^2$. If $y_0 = 0$, then the solution $y(t) = 0$ exists for all t.

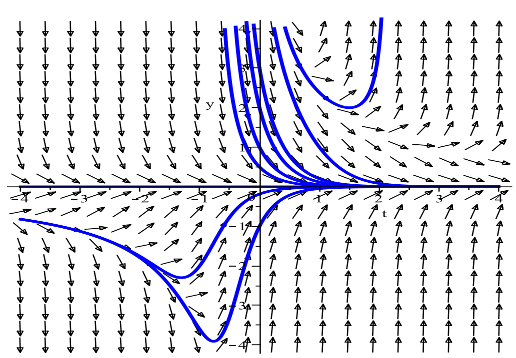
16. The function $f(t, y)$ is discontinuous along the straight lines $t = -1$ and $y = 0$. The partial derivative $\partial f/\partial y$ is discontinuous along the same lines. The equation is separable, with $y\,dy = t^2\,dt/(1 + t^3)$. Integrating and invoking the initial condition, the solution is $y(t) = \left[(2/3)\ln\left|1 + t^3\right| + y_0^2\right]^{1/2}$. Solutions exist as long as $(2/3)\ln\left|1 + t^3\right| + y_0^2 \geq 0$, that is, $y_0^2 \geq -(2/3)\ln\left|1 + t^3\right|$. For all y_0 (it can be verified that $y_0 = 0$ yields a valid solution, even though Theorem 2.4.2 does not guarantee one), solutions exist as long as $\left|1 + t^3\right| \geq e^{-3y_0^2/2}$. From above, we must have $t > -1$. Hence the inequality may be written as $t^3 \geq e^{-3y_0^2/2} - 1$. It follows that the solutions are valid for $(e^{-3y_0^2/2} - 1)^{1/3} < t < \infty$.

18.



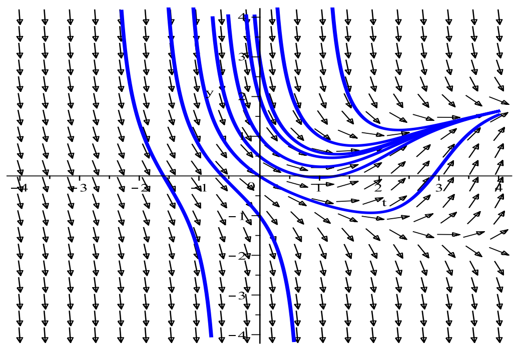
Based on the direction field, and the differential equation, for $y_0 < 0$, the slopes eventually become negative, and hence solutions tend to $-\infty$. For $y_0 > 0$, solutions increase without bound if $t_0 < 0$. Otherwise, the slopes eventually become negative, and solutions tend to zero. Furthermore, $y_0 = 0$ is an equilibrium solution. Note that slopes are zero along the curves $y = 0$ and $ty = 3$.

19.



For initial conditions (t_0, y_0) satisfying $ty < 3$, the respective solutions all tend to zero. For $y_0 \leq 9$, the solutions tend to 0; for $y_0 > 9$, the solutions tend to ∞. Also, $y_0 = 0$ is an equilibrium solution.

20.



Solutions with $t_0 < 0$ all tend to $-\infty$. Solutions with initial conditions (t_0, y_0) to the right of the parabola $t = 1 + y^2$ asymptotically approach the parabola as $t \to \infty$. Integral curves with initial conditions above the parabola (and $y_0 > 0$) also approach the curve. The slopes for solutions with initial conditions below the parabola (and $y_0 < 0$) are all negative. These solutions tend to $-\infty$.

21.(a) No. There is no value of $t_0 \geq 0$ for which $(2/3)(t - t_0)^{2/3}$ satisfies the condition $y(1) = 1$.

(b) Yes. Let $t_0 = 1/2$ in Eq.(19).

(c) For $t_0 > 0$, $|y(2)| \leq (4/3)^{3/2} \approx 1.54$.

24. The assumption is $\phi'(t) + p(t)\phi(t) = 0$. But then $c\phi'(t) + p(t)c\phi(t) = 0$ as well.

26.(a) Recalling Eq.(33) in Section 2.1,

$$y = \frac{1}{\mu(t)} \int_{t_0}^{t} \mu(s)g(s)\,ds + \frac{c}{\mu(t)}.$$

It is evident that $y_1(t) = 1/\mu(t)$ and $y_2(t) = (1/\mu(t)) \int_{t_0}^{t} \mu(s)g(s)\,ds$.

(b) By definition, $1/\mu(t) = e^{-\int p(t)dt}$. Hence $y_1' = -p(t)/\mu(t) = -p(t)y_1$. That is, $y_1' + p(t)y_1 = 0$.

(c) $y_2' = (-p(t)/\mu(t)) \int_0^t \mu(s)g(s)\,ds + \mu(t)g(t)/\mu(t) = -p(t)y_2 + g(t)$. This implies that $y_2' + p(t)y_2 = g(t)$.

30. Since $n = 3$, set $v = y^{-2}$. It follows that $v' = -2y^{-3}y'$ and $y' = -(y^3/2)v'$. Substitution into the differential equation yields $-(y^3/2)v' - \varepsilon y = -\sigma y^3$, which further results in $v' + 2\varepsilon v = 2\sigma$. The latter differential equation is linear, and can be written as $(ve^{2\varepsilon t})' = 2\sigma e^{2\varepsilon t}$. The solution is given by $v(t) = \sigma/\varepsilon + ce^{-2\varepsilon t}$. Converting back to the original dependent variable, $y = \pm v^{-1/2} = \pm(\sigma/\varepsilon + ce^{-2\varepsilon t})^{-1/2}$.

31. Since $n = 3$, set $v = y^{-2}$. It follows that $v' = -2y^{-3}y'$ and $y' = -(y^3/2)v'$. The differential equation is written as $-(y^3/2)v' - (\Gamma\cos t + T)y = \sigma y^3$, which upon

further substitution is $v' + 2(\Gamma\cos t + T)v = 2$. This ODE is linear, with integrating factor $\mu(t) = e^{2\int(\Gamma\cos t + T)dt} = e^{2\Gamma\sin t + 2Tt}$. The solution is

$$v(t) = 2e^{-(2\Gamma\sin t + 2Tt)}\int_0^t e^{2\Gamma\sin\tau + 2T\tau}d\tau + ce^{-(2\Gamma\sin t + 2Tt)}.$$

Converting back to the original dependent variable, $y = \pm v^{-1/2}$.

33. The solution of the initial value problem $y_1' + 2y_1 = 0$, $y_1(0) = 1$ is $y_1(t) = e^{-2t}$. Therefore $y(1^-) = y_1(1) = e^{-2}$. On the interval $(1, \infty)$, the differential equation is $y_2' + y_2 = 0$, with $y_2(t) = ce^{-t}$. Therefore $y(1^+) = y_2(1) = ce^{-1}$. Equating the limits $y(1^-) = y(1^+)$, we require that $c = e^{-1}$. Hence the global solution of the initial value problem is

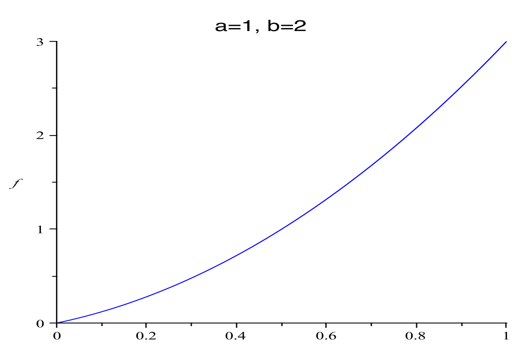
$$y(t) = \begin{cases} e^{-2t}, & 0 \le t \le 1 \\ e^{-1-t}, & t > 1 \end{cases}.$$

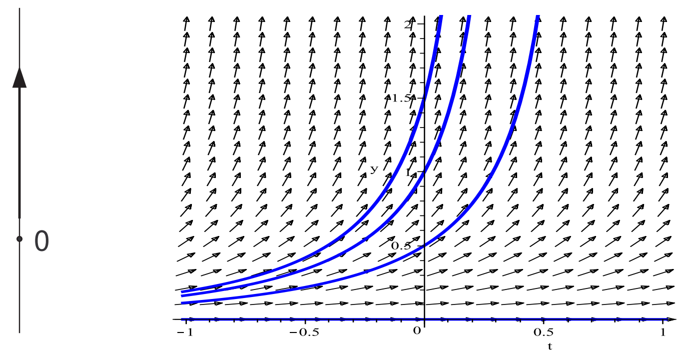
Note the discontinuity of the derivative

$$y'(t) = \begin{cases} -2e^{-2t}, & 0 < t < 1 \\ -e^{-1-t}, & t > 1 \end{cases}.$$

2.5

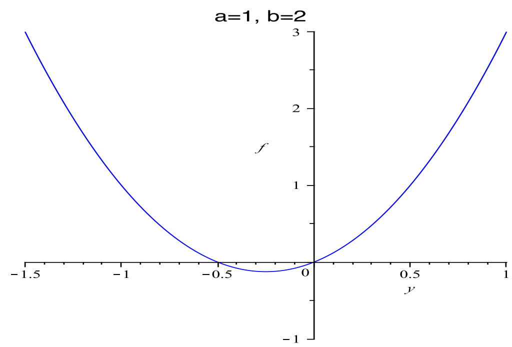
1.

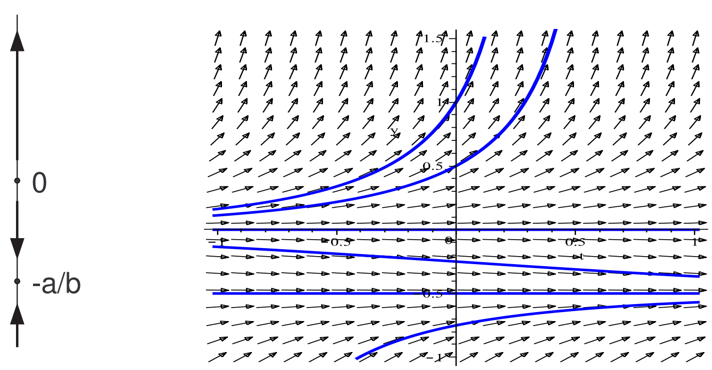




For $y_0 \geq 0$, the only equilibrium point is $y^* = 0$, and $y' = ay + by^2 > 0$ when $y > 0$, hence the equilibrium solution $y = 0$ is unstable.

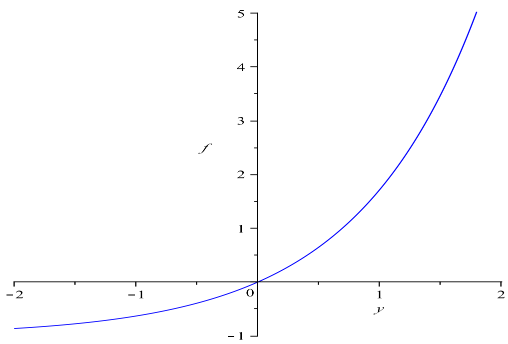
2.

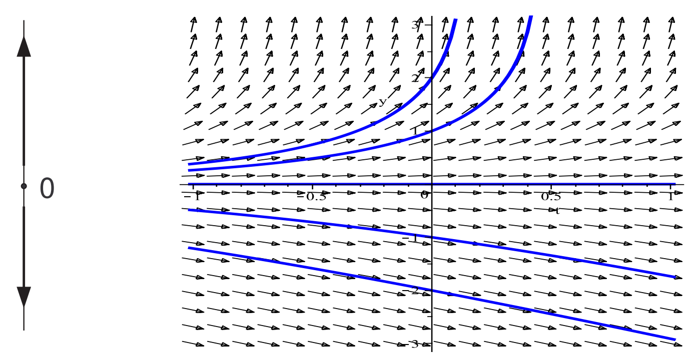




The equilibrium points are $y^* = -a/b$ and $y^* = 0$, and $y' > 0$ when $y > 0$ or $y < -a/b$, and $y' < 0$ when $-a/b < y < 0$, therefore the equilibrium solution $y = -a/b$ is asymptotically stable and the equilibrium solution $y = 0$ is unstable.

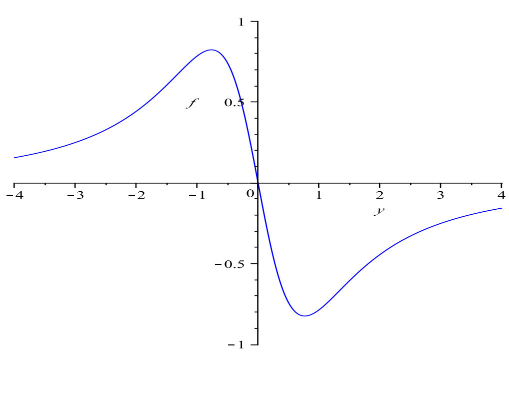
4.

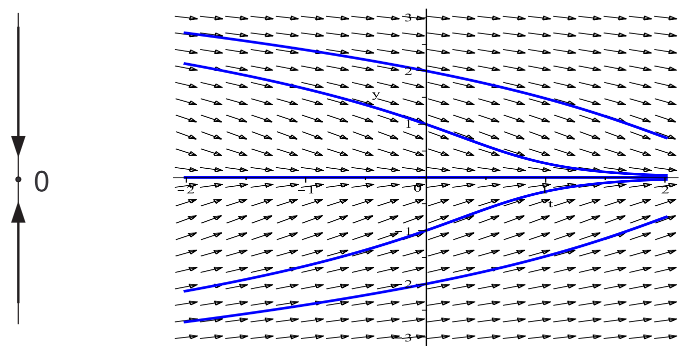




The only equilibrium point is $y^* = 0$, and $y' > 0$ when $y > 0$, $y' < 0$ when $y < 0$, hence the equilibrium solution $y = 0$ is unstable.

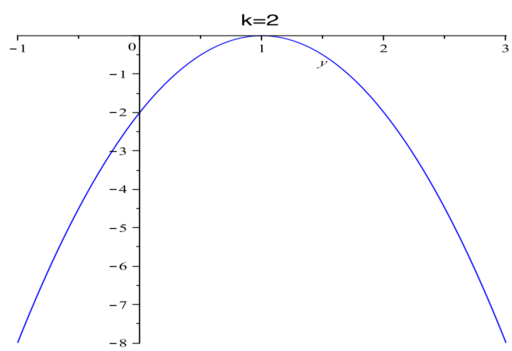
6.

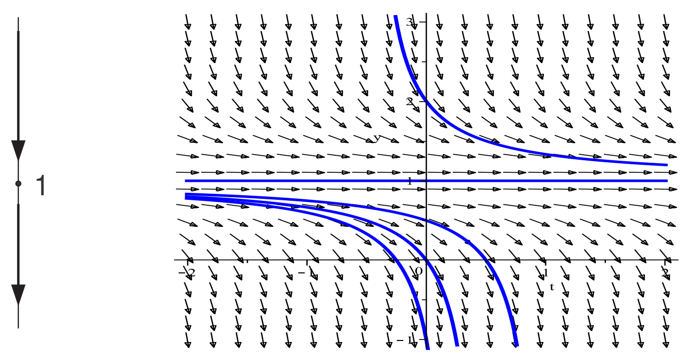




The only equilibrium point is $y^* = 0$, and $y' > 0$ when $y < 0$, $y' < 0$ when $y > 0$, hence the equilibrium solution $y = 0$ is asymptotically stable.

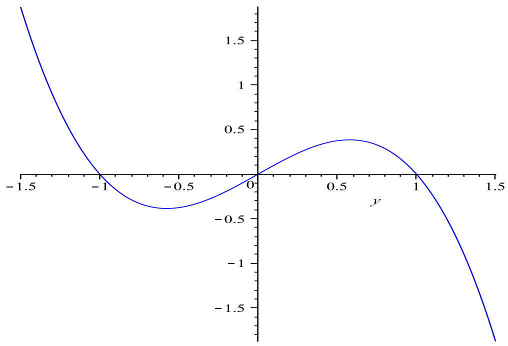
8.

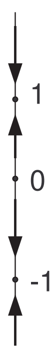


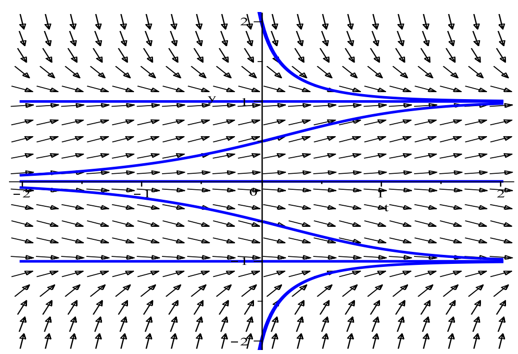


The only equilibrium point is $y^* = 1$, and $y' < 0$ for $y \neq 1$. As long as $y_0 \neq 1$, the corresponding solution is monotone decreasing. Hence the equilibrium solution $y = 1$ is semistable.

10.

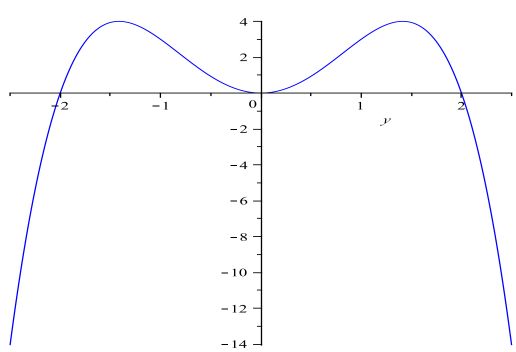


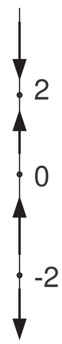


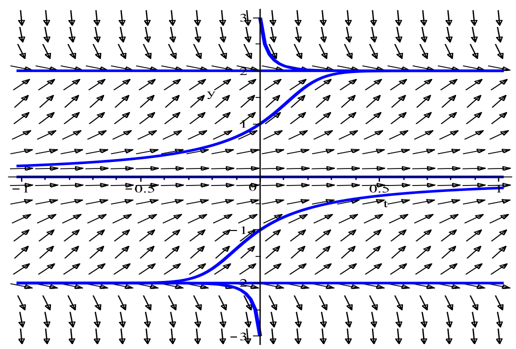


The equilibrium points are $y^* = 0, \pm 1$, and $y' > 0$ for $y < -1$ or $0 < y < 1$ and $y' < 0$ for $-1 < y < 0$ or $y > 1$. The equilibrium solution $y = 0$ is unstable, and the remaining two are asymptotically stable.

12.

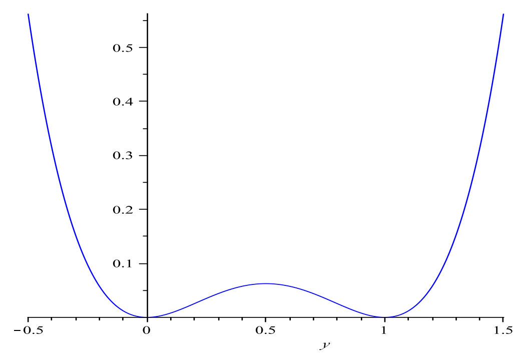


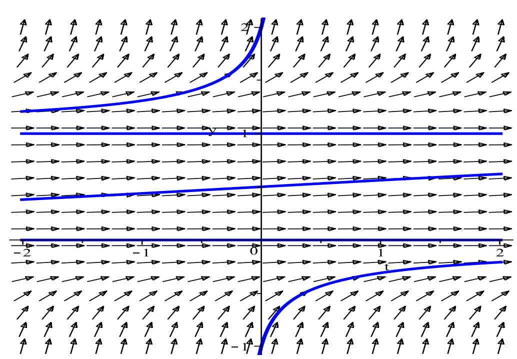




The equilibrium points are $y^* = 0, \pm 2$, and $y' < 0$ when $y < -2$ or $y > 2$, and $y' > 0$ for $-2 < y < 0$ or $0 < y < 2$. The equilibrium solutions $y = -2$ and $y = 2$ are unstable and asymptotically stable, respectively. The equilibrium solution $y = 0$ is semistable.

13.





The equilibrium points are $y^* = 0, 1$. $y' > 0$ for all y except $y = 0$ and $y = 1$. Both equilibrium solutions are semistable.

15.(a) Inverting Eq.(11), Eq.(13) shows t as a function of the population y and the carrying capacity K. With $y_0 = K/3$,

$$t = -\frac{1}{r}\ln\left|\frac{(1/3)\,[1-(y/K)]}{(y/K)\,[1-(1/3)]}\right|.$$

Setting $y = 2y_0$,

$$\tau = -\frac{1}{r}\ln\left|\frac{(1/3)\,[1-(2/3)]}{(2/3)\,[1-(1/3)]}\right|.$$

That is, $\tau = (\ln 4)/r$. If $r = 0.025$ per year, $\tau \approx 55.45$ years.

(b) In Eq.(13), set $y_0/K = \alpha$ and $y/K = \beta$. As a result, we obtain

$$T = -\frac{1}{r}\ln\left|\frac{\alpha\,[1-\beta]}{\beta\,[1-\alpha]}\right|.$$

Given $\alpha = 0.1$, $\beta = 0.9$ and $r = 0.025$ per year, $\tau \approx 175.78$ years.

19.(a) The rate of increase of the volume is given by rate of flow in$-$rate of flow out. That is, $dV/dt = k - \alpha a\sqrt{2gh}$. Since the cross section is constant, $dV/dt = Adh/dt$. Hence the governing equation is $dh/dt = (k - \alpha a\sqrt{2gh})/A$.

(b) Setting $dh/dt = 0$, the equilibrium height is $h_e = (1/2g)(k/\alpha a)^2$. Furthermore, since $dh/dt < 0$ for $h > h_e$ and $dh/dt > 0$ for $h < h_e$, it follows that the equilibrium height is asymptotically stable.

22.(a) The equilibrium points are at $y^* = 0$ and $y^* = 1$. Since $f'(y) = \alpha - 2\alpha y$, the equilibrium solution $y = 0$ is unstable and the equilibrium solution $y = 1$ is asymptotically stable.

(b) The differential equation is separable, with $[y(1-y)]^{-1}\,dy = \alpha\,dt$. Integrating both sides and invoking the initial condition, the solution is

$$y(t) = \frac{y_0\,e^{\alpha t}}{1 - y_0 + y_0\,e^{\alpha t}} = \frac{y_0}{y_0 + (1-y_0)e^{-\alpha t}}.$$

It is evident that (independent of y_0) $\lim_{t\to-\infty} y(t) = 0$ and $\lim_{t\to\infty} y(t) = 1$.

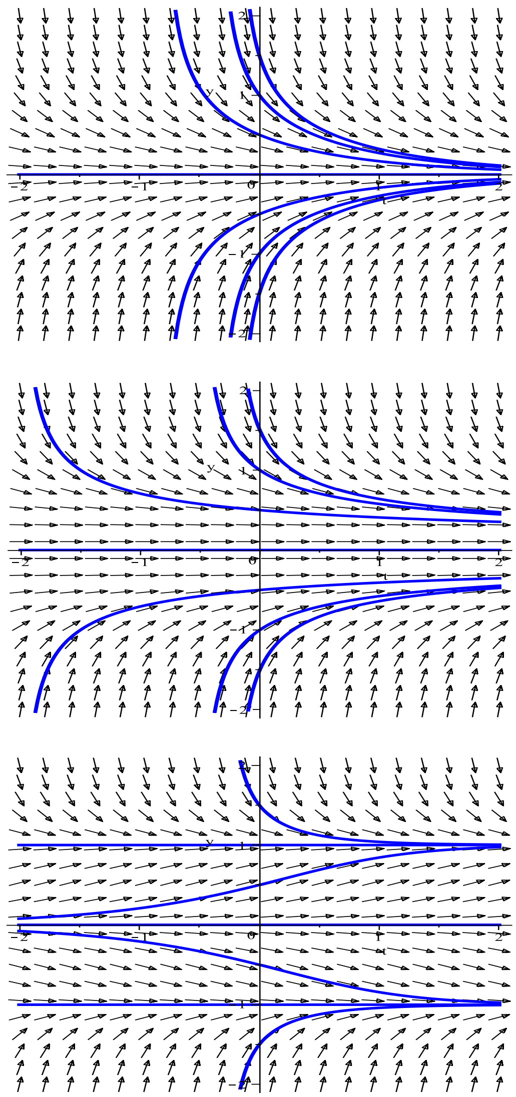
23.(a) $y(t) = y_0\,e^{-\beta t}$.

(b) From part (a), $dx/dt = -\alpha x y_0 e^{-\beta t}$. Separating variables, $dx/x = -\alpha y_0 e^{-\beta t} dt$. Integrating both sides, the solution is $x(t) = x_0\,e^{-\alpha\,y_0(1-e^{-\beta t})/\beta}$.

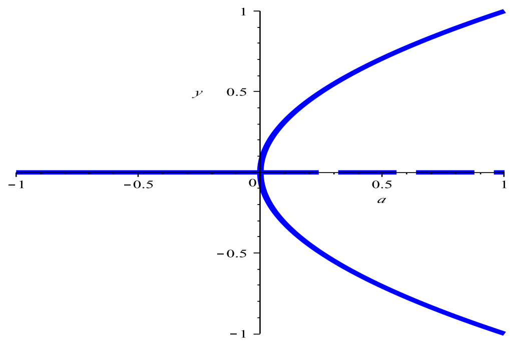
(c) As $t \to \infty$, $y(t) \to 0$ and $x(t) \to x_0\,e^{-\alpha\,y_0/\beta}$. Over a long period of time, the proportion of carriers vanishes. Therefore the proportion of the population that escapes the epidemic is the proportion of susceptibles left at that time, $x_0\,e^{-\alpha\,y_0/\beta}$.

26.(a) For $a < 0$, the only critical point is at $y = 0$, which is asymptotically stable. For $a = 0$, the only critical point is at $y = 0$, which is asymptotically stable. For $a > 0$, the three critical points are at $y = 0$, $\pm\sqrt{a}$. The critical point at $y = 0$ is unstable, whereas the other two are asymptotically stable.

(b) Below, we graph solutions in the case $a = -1$, $a = 0$ and $a = 1$ respectively.

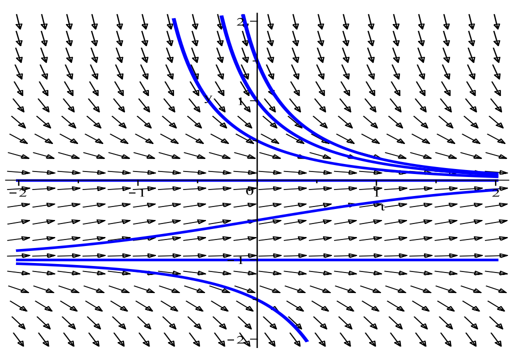


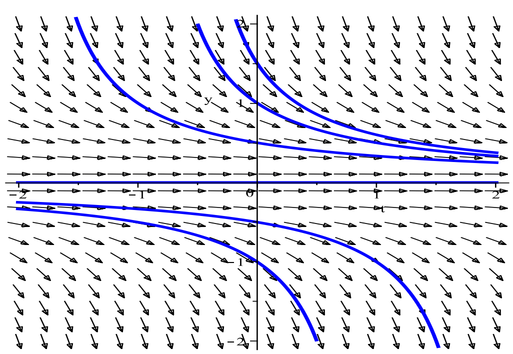
(c)

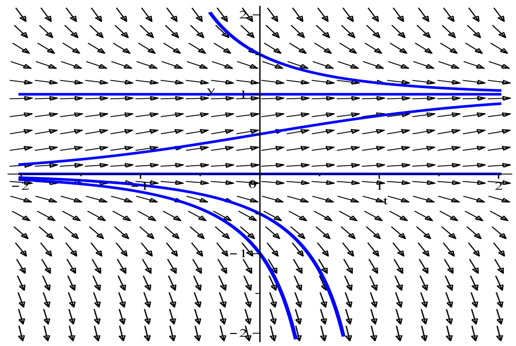


27.(a) $f(y) = y(a-y)$; $f'(y) = a - 2y$. For $a < 0$, the critical points are at $y = a$ and $y = 0$. Observe that $f'(a) > 0$ and $f'(0) < 0$. Hence $y = a$ is unstable and $y = 0$ asymptotically stable. For $a = 0$, the only critical point is at $y = 0$, which is semistable since $f(y) = -y^2$ is concave down. For $a > 0$, the critical points are at $y = 0$ and $y = a$. Observe that $f'(0) > 0$ and $f'(a) < 0$. Hence $y = 0$ is unstable and $y = a$ asymptotically stable.

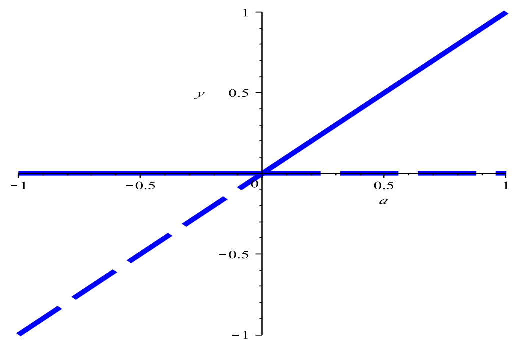
(b) Below, we graph solutions in the case $a = -1$, $a = 0$ and $a = 1$ respectively.







(c)



2.6

1. $M(x,y) = 2x+3$ and $N(x,y) = 2y-2$. Since $M_y = N_x = 0$, the equation is exact. Integrating M with respect to x, while holding y constant, yields $\psi(x,y) = x^2+3x+h(y)$. Now $\psi_y = h'(y)$, and equating with N results in the possible function $h(y) = y^2-2y$. Hence $\psi(x,y) = x^2+3x+y^2-2y$, and the solution is defined implicitly as $x^2+3x+y^2-2y = c$.

2. $M(x,y) = 2x+4y$ and $N(x,y) = 2x-2y$. Note that $M_y \neq N_x$, and hence the differential equation is not exact.

4. First divide both sides by $(2xy+2)$. We now have $M(x,y) = y$ and $N(x,y) = x$. Since $M_y = N_x = 0$, the resulting equation is exact. Integrating M with respect to x, while holding y constant, results in $\psi(x,y) = xy + h(y)$. Differentiating with respect to y, $\psi_y = x + h'(y)$. Setting $\psi_y = N$, we find that $h'(y) = 0$, and hence $h(y) = 0$ is acceptable. Therefore the solution is defined implicitly as $xy = c$. Note that if $xy+1 = 0$, the equation is trivially satisfied.

6. Write the equation as $(ax-by)dx + (bx-cy)dy = 0$. Now $M(x,y) = ax-by$ and $N(x,y) = bx-cy$. Since $M_y \neq N_x$, the differential equation is not exact.

8. $M(x,y) = e^x \sin y + 3y$ and $N(x,y) = -3x + e^x \sin y$. Note that $M_y \neq N_x$, and hence the differential equation is not exact.

10. $M(x,y) = y/x + 6x$ and $N(x,y) = \ln x - 2$. Since $M_y = N_x = 1/x$, the given equation is exact. Integrating N with respect to y, while holding x constant, results in $\psi(x,y) = y \ln x - 2y + h(x)$. Differentiating with respect to x, $\psi_x = y/x + h'(x)$. Setting $\psi_x = M$, we find that $h'(x) = 6x$, and hence $h(x) = 3x^2$. Therefore the solution is defined implicitly as $3x^2 + y \ln x - 2y = c$.

11. $M(x,y) = x \ln y + xy$ and $N(x,y) = y \ln x + xy$. Note that $M_y \neq N_x$, and hence the differential equation is not exact.

13. $M(x,y) = 2x - y$ and $N(x,y) = 2y - x$. Since $M_y = N_x = -1$, the equation is exact. Integrating M with respect to x, while holding y constant, yields $\psi(x,y) = x^2 - xy + h(y)$. Now $\psi_y = -x + h'(y)$. Equating ψ_y with N results in $h'(y) = 2y$, and hence $h(y) = y^2$. Thus $\psi(x,y) = x^2 - xy + y^2$, and the solution is given implicitly as $x^2 - xy + y^2 = c$. Invoking the initial condition $y(1) = 3$, the specific solution is $x^2 - xy + y^2 = 7$. The explicit form of the solution is $y(x) = (x + \sqrt{28 - 3x^2}\,)/2$. Hence the solution is valid as long as $3x^2 \leq 28$.

16. $M(x,y) = y\,e^{2xy} + x$ and $N(x,y) = bx\,e^{2xy}$. Note that $M_y = e^{2xy} + 2xy\,e^{2xy}$, and $N_x = b\,e^{2xy} + 2bxy\,e^{2xy}$. The given equation is exact, as long as $b = 1$. Integrating N with respect to y, while holding x constant, results in $\psi(x,y) = e^{2xy}/2 + h(x)$. Now differentiating with respect to x, $\psi_x = y\,e^{2xy} + h'(x)$. Setting $\psi_x = M$, we find that $h'(x) = x$, and hence $h(x) = x^2/2$. We conclude that $\psi(x,y) = e^{2xy}/2 + x^2/2$. Hence the solution is given implicitly as $e^{2xy} + x^2 = c$.

17. Note that ψ is of the form $\psi(x\,,y) = f(x) + g(y)$, since each of the integrands is a function of a single variable. It follows that $\psi_x = f'(x)$ and $\psi_y = g'(y)$. That is, $\psi_x = M(x\,,y_0)$ and $\psi_y = N(x_0\,,y)$. Furthermore,

$$\frac{\partial^2\psi}{\partial x \partial y}(x_0\,,y_0\,) = \frac{\partial M}{\partial y}(x_0\,,y_0\,) \text{ and } \frac{\partial^2\psi}{\partial y \partial x}(x_0\,,y_0\,) = \frac{\partial N}{\partial x}(x_0\,,y_0\,),$$

based on the hypothesis and the fact that the point (x_0, y_0) is arbitrary, $\psi_{xy} = \psi_{yx}$ and $M_y(x,y) = N_x(x,y)$.

18. Observe that $(M(x))_y = (N(y))_x = 0$.

20. $M_y = y^{-1}\cos y - y^{-2}\sin y$ and $N_x = -2\,e^{-x}(\cos x + \sin x)/y$. Multiplying both sides by the integrating factor $\mu(x,y) = y\,e^x$, the given equation can be written as $(e^x \sin y - 2y \sin x)dx + (e^x \cos y + 2\cos x)dy = 0$. Let $\tilde{M} = \mu M$ and $\tilde{N} = \mu N$. Observe that $\tilde{M}_y = \tilde{N}_x$, and hence the latter ODE is exact. Integrating $\tilde{N}$ with respect to y, while holding x constant, results in $\psi(x,y) = e^x \sin y + 2y \cos x + h(x)$. Now differentiating with respect to x, $\psi_x = e^x \sin y - 2y \sin x + h'(x)$. Setting $\psi_x = \tilde{M}$, we find that $h'(x) = 0$, and hence $h(x) = 0$ is feasible. Hence the solution of the given equation is defined implicitly by $e^x \sin y + 2y \cos x = c$.

21. $M_y = 1$ and $N_x = 2$. Multiply both sides by the integrating factor $\mu(x,y) = y$ to obtain $y^2dx + (2xy - y^2e^y)dy = 0$. Let $\tilde{M} = yM$ and $\tilde{N} = yN$. It is easy to see that $\tilde{M}_y = \tilde{N}_x$, and hence the latter ODE is exact. Integrating $\tilde{M}$ with respect to x yields $\psi(x,y) = xy^2 + h(y)$. Equating ψ_y with $\tilde{N}$ results in $h'(y) = -y^2e^y$, and hence $h(y) = -e^y(y^2 - 2y + 2)$. Thus $\psi(x,y) = xy^2 - e^y(y^2 - 2y + 2)$, and the solution is defined implicitly by $xy^2 - e^y(y^2 - 2y + 2) = c$.

24. The equation $\mu M + \mu N y' = 0$ has an integrating factor if $(\mu M)_y = (\mu N)_x$, that is, $\mu_y M - \mu_x N = \mu N_x - \mu M_y$. Suppose that $N_x - M_y = R\,(xM - yN)$, in which R is some function depending only on the quantity $z = xy$. It follows that the modified form of the equation is exact, if $\mu_y M - \mu_x N = \mu R\,(xM - yN) = R\,(\mu\, xM - \mu\, yN)$. This relation is satisfied if $\mu_y = (\mu\, x)R$ and $\mu_x = (\mu\, y)R$. Now consider $\mu = \mu(xy)$. Then the partial derivatives are $\mu_x = \mu' y$ and $\mu_y = \mu' x$. Note that $\mu' = d\mu/dz$. Thus μ must satisfy $\mu'(z) = R(z)$. The latter equation is separable, with $d\mu = R(z)dz$, and $\mu(z) = \int R(z)dz$. Therefore, given $R = R(xy)$, it is possible to determine $\mu = \mu(xy)$ which becomes an integrating factor of the differential equation.

28. The equation is not exact, since $N_x - M_y = 2y - 1$. However, $(N_x - M_y)/M = (2y-1)/y$ is a function of y alone. Hence there exists $\mu = \mu(y)$, which is a solution of the differential equation $\mu' = (2 - 1/y)\mu$. The latter equation is separable, with $d\mu/\mu = 2 - 1/y$. One solution is $\mu(y) = e^{2y - \ln\, y} = e^{2y}/y$. Now rewrite the given ODE as $e^{2y}dx + (2x\, e^{2y} - 1/y)dy = 0$. This equation is exact, and it is easy to see that $\psi(x,y) = x\, e^{2y} - \ln|y|$. Therefore the solution of the given equation is defined implicitly by $x\, e^{2y} - \ln|y| = c$.

30. The given equation is not exact, since $N_x - M_y = 8x^3/y^3 + 6/y^2$. But note that $(N_x - M_y)/M = 2/y$ is a function of y alone, and hence there is an integrating factor $\mu = \mu(y)$. Solving the equation $\mu' = (2/y)\mu$, an integrating factor is $\mu(y) = y^2$. Now rewrite the differential equation as $(4x^3 + 3y)dx + (3x + 4y^3)dy = 0$. By inspection, $\psi(x,y) = x^4 + 3xy + y^4$, and the solution of the given equation is defined implicitly by $x^4 + 3xy + y^4 = c$.

32. Multiplying both sides of the ODE by $\mu = [xy(2x+y)]^{-1}$, the given equation is equivalent to $\left[(3x+y)/(2x^2+xy)\right] dx + \left[(x+y)/(2xy+y^2)\right] dy = 0$. Rewrite the differential equation as

$$\left[\frac{2}{x} + \frac{2}{2x+y}\right] dx + \left[\frac{1}{y} + \frac{1}{2x+y}\right] dy = 0\,.$$

It is easy to see that $M_y = N_x$. Integrating M with respect to x, while keeping y constant, results in $\psi(x,y) = 2\ln|x| + \ln|2x+y| + h(y)$. Now taking the partial derivative with respect to y, $\psi_y = (2x+y)^{-1} + h'(y)$. Setting $\psi_y = N$, we find that $h'(y) = 1/y$, and hence $h(y) = \ln\,|y|$. Therefore $\psi(x,y) = 2\ln|x| + \ln|2x+y| + \ln\,|y|$, and the solution of the given equation is defined implicitly by $2x^3y + x^2y^2 = c$.

2.7

2. The Euler formula is given by $y_{n+1} = y_n + h(2y_n - 1) = (1 + 2h)y_n - h$.

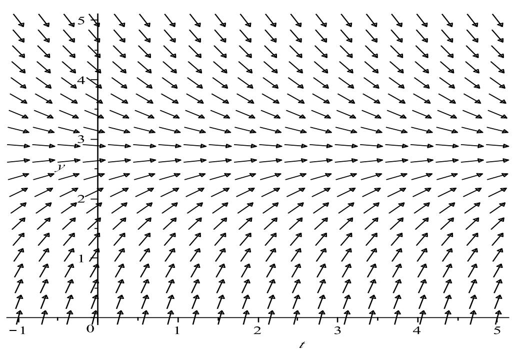
(a) 1.1, 1.22, 1.364, 1.5368

(b) 1.105, 1.23205, 1.38578, 1.57179

(c) 1.10775, 1.23873, 1.39793, 1.59144

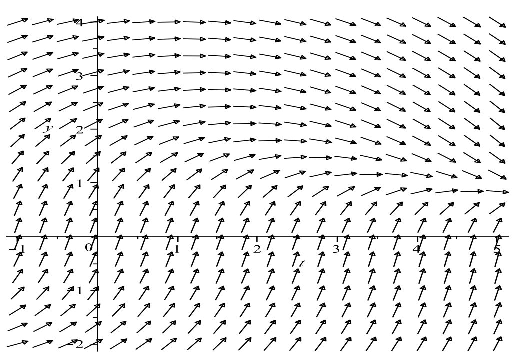
(d) The differential equation is linear with solution $y(t) = (1 + e^{2t})/2$. The values are 1.1107, 1.24591, 1.41106, 1.61277.

5.



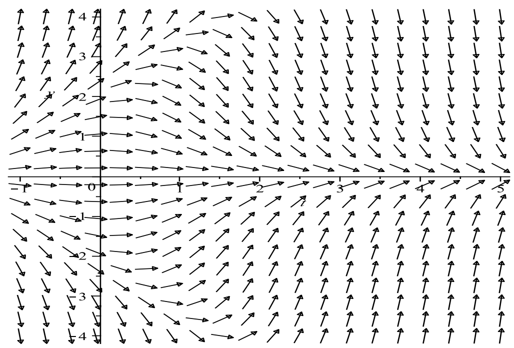
All solutions seem to converge to $y = 25/9$.

7.



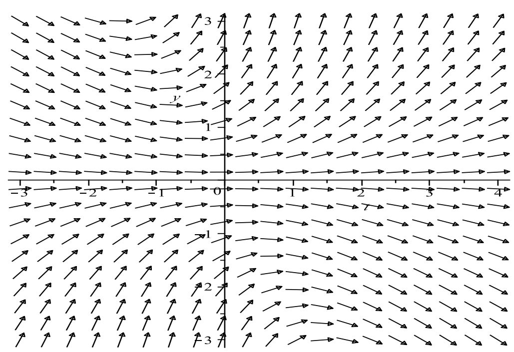
All solutions seem to converge to a specific function.

8.



Solutions with initial conditions $|y(0)| > 2.5$ seem to diverge. On the other hand, solutions with initial conditions $|y(0)| < 2.5$ seem to converge to zero. Also, $y = 0$ is an equilibrium solution.

10.



Solutions with positive initial conditions increase without bound. Solutions with negative initial conditions decrease without bound. Note that $y = 0$ is an equilibrium solution.

11. The Euler formula is $y_{n+1} = y_n - 3h\sqrt{y_n} + 5h$. The initial value is $y_0 = 2$.

(a) 2.30800, 2.49006, 2.60023, 2.66773, 2.70939, 2.73521

(b) 2.30167, 2.48263, 2.59352, 2.66227, 2.70519, 2.73209

(c) 2.29864, 2.47903, 2.59024, 2.65958, 2.70310, 2.73053

(d) 2.29686, 2.47691, 2.58830, 2.65798, 2.70185, 2.72959

12. The Euler formula is $y_{n+1} = (1 + 3h)y_n - ht_n y_n^2$. The initial value is $(t_0, y_0) = (0, 0.5)$.

(a) 1.70308, 3.06605, 2.44030, 1.77204, 1.37348, 1.11925

(b) 1.79548, 3.06051, 2.43292, 1.77807, 1.37795, 1.12191

(c) 1.84579, 3.05769, 2.42905, 1.78074, 1.38017, 1.12328

(d) 1.87734, 3.05607, 2.42672, 1.78224, 1.38150, 1.12411

14. The Euler formula is $y_{n+1} = (1 - ht_n)y_n + hy_n^3/10$, with $(t_0, y_0) = (0, 1)$.

(a) 0.950517, 0.687550, 0.369188, 0.145990, 0.0421429, 0.00872877

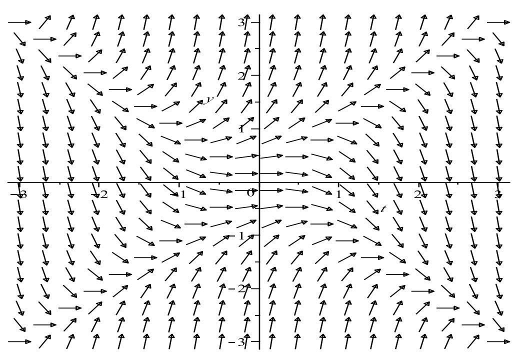
(b) 0.938298, 0.672145, 0.362640, 0.147659, 0.0454100, 0.0104931

(c) 0.932253, 0.664778, 0.359567, 0.148416, 0.0469514, 0.0113722

(d) 0.928649, 0.660463, 0.357783, 0.148848, 0.0478492, 0.0118978

17. The Euler formula is $y_{n+1} = y_n + h(y_n^2 + 2t_n\, y_n)/(3 + t_n^2)$. The initial point is $(t_0, y_0) = (1\,, 2)$. Using this iteration formula with the specified h values, the value of the solution at $t = 2.5$ is somewhere between 18 and 19. At $t = 3$ there is no reliable estimate.

19.(a)



(b) The iteration formula is $y_{n+1} = y_n + h\,y_n^2 \; - h\,t_n^2$. The critical value α_0 appears to be between 0.67 and 0.68. For $y_0 > \alpha_0$, the iterations diverge.

20.(a) The ODE is linear, with general solution $y(t) = t + ce^t$. Invoking the specified initial condition, $y(t_0) = y_0$, we have $y_0 = t_0 + ce^{t_0}$. Hence $c = (y_0 - t_0)e^{-t_0}$. Thus the solution is given by $\phi(t) = (y_0 - t_0)e^{t - t_0} + t$.

(b) The Euler formula is $y_{n+1} = (1 + h)y_n + h - h\,t_n$. Now set $k = n + 1$.

(c) We have $y_1 = (1 + h)y_0 + h - ht_0 = (1 + h)y_0 + (t_1 - t_0) - ht_0$. Rearranging the terms, $y_1 = (1 + h)(y_0 - t_0) + t_1$. Now suppose that $y_k = (1 + h)^k(y_0 - t_0) + t_k$, for some $k \geq 1$. Then $y_{k+1} = (1 + h)y_k + h - ht_k$. Substituting for y_k, we find that

$$y_{k+1} = (1+h)^{k+1}(y_0 - t_0) + (1+h)t_k + h - ht_k = (1+h)^{k+1}(y_0 - t_0) + t_k + h.$$

Noting that $t_{k+1} = t_k + h$, the result is verified.

(d) Substituting $h = (t - t_0)/n$, with $t_n = t$, $y_n = (1 + (t - t_0)/n)^n (y_0 - t_0) + t$. Taking the limit of both sides, and using the fact that $\lim_{n\to\infty}(1 + a/n)^n = e^a$, pointwise convergence is proved.

21. The exact solution is $y(t) = e^t$. The Euler formula is $y_{n+1} = (1+h)y_n$. It is easy to see that $y_n = (1+h)^n y_0 = (1+h)^n$. Given $t > 0$, set $h = t/n$. Taking the limit, we find that $\lim_{n\to\infty} y_n = \lim_{n\to\infty}(1 + t/n)^n = e^t$.

23. The exact solution is $y(t) = t/2 + e^{2t}$. The Euler formula is $y_{n+1} = (1 + 2h)y_n + h/2 - h\,t_n$. Since $y_0 = 1$, $y_1 = (1+2h) + h/2 = (1+2h) + t_1/2$. It is easy to show by mathematical induction, that $y_n = (1+2h)^n + t_n/2$. For $t > 0$, set $h = t/n$ and thus $t_n = t$. Taking the limit, we find that $\lim_{n\to\infty} y_n = \lim_{n\to\infty}\left[(1 + 2t/n)^n + t/2\right] = e^{2t} + t/2$. Hence pointwise convergence is proved.

2.8

2. Let $z = y - 3$ and $\tau = t + 1$. It follows that $dz/d\tau = (dz/dt)(dt/d\tau) = dz/dt$. Furthermore, $dz/dt = dy/dt = 1 - y^3$. Hence $dz/d\tau = 1 - (z+3)^3$. The new initial condition is $z(0) = 0$.

3.(a) The approximating functions are defined recursively by

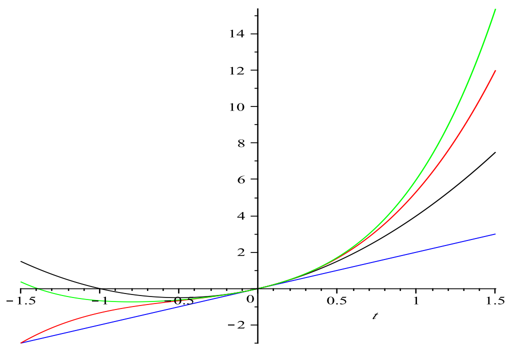
$$\phi_{n+1}(t) = \int_0^t 2\left[\phi_n(s) + 1\right] ds\,.$$

Setting $\phi_0(t) = 0$, $\phi_1(t) = 2t$. Continuing, $\phi_2(t) = 2t^2 + 2t$, $\phi_3(t) = 4t^3/3 + 2t^2 + 2t$, $\phi_4(t) = 2t^4/3 + 4t^3/3 + 2t^2 + 2t$, ... Based upon these we conjecture that $\phi_n(t) = \sum_{k=1}^{n} 2^k t^k/k!$ and use mathematical induction to verify this form for $\phi_n(t)$. First, let $n = 1$, then $\phi_n(t) = 2t$, so it is certainly true for $n = 1$. Then, using Eq.(7) again we have

$$\phi_{n+1}(t) = \int_0^t 2\left[\phi_n(s) + 1\right] ds = \int_0^t 2\left[\sum_{k=1}^{n} \frac{2^k}{k\,!} s^k + 1\right] ds = \sum_{k=1}^{n+1} \frac{2^k}{k\,!} t^k,$$

and we have verified our conjecture.

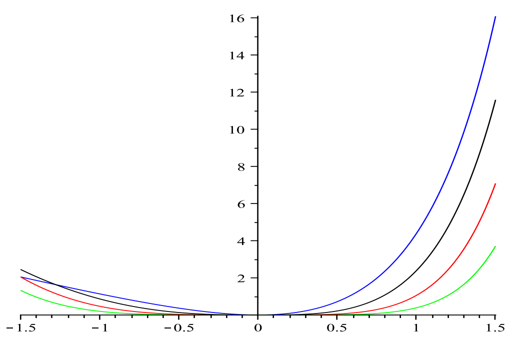
(b)



(c) Recall from calculus that $e^{at} = 1 + \sum_{k=1}^{\infty} a^k t^k / k!$. Thus

$$\phi(t) = \sum_{k=1}^{\infty} \frac{2^k}{k\,!} t^k = e^{2t} - 1\,.$$

(d)



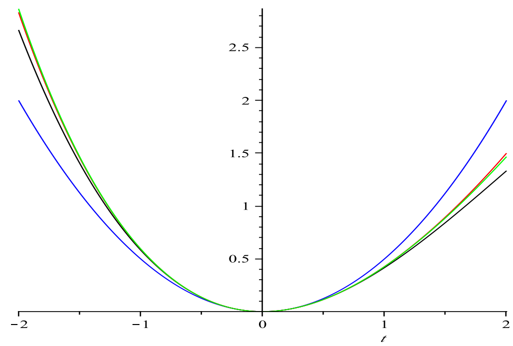
From the plot it appears that ϕ_4 is a good estimate for $|t| < 1/2$.

5.(a) The approximating functions are defined recursively by

$$\phi_{n+1}(t) = \int_0^t \left[-\phi_n(s)/2 + s\right] ds\,.$$

Setting $\phi_0(t) = 0$, $\phi_1(t) = t^2/2$. Continuing, $\phi_2(t) = t^2/2 - t^3/12$, $\phi_3(t) = t^2/2 - t^3/12 + t^4/96$, $\phi_4(t) = t^2/2 - t^3/12 + t^4/96 - t^5/960$, ... Based upon these we conjecture that $\phi_n(t) = \sum_{k=1}^{n} 4(-1/2)^{k+1} t^{k+1}/(k+1)!$ and use mathematical induction to verify this form for $\phi_n(t)$.

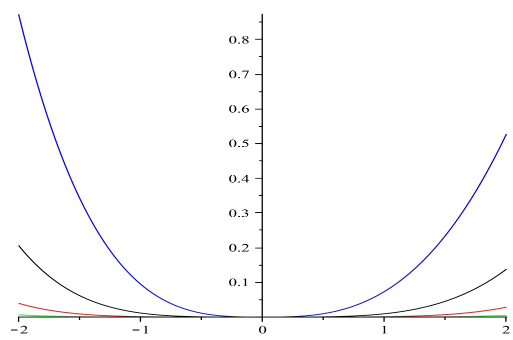
(b)



(c) Recall from calculus that $e^{at} = 1 + \sum_{k=1}^{\infty} a^k t^k / k!$. Thus

$$\phi(t) = \sum_{k=1}^{\infty} 4 \frac{(-1/2)^{k+1}}{k+1\,!} t^{k+1} = 4e^{-t/2} + 2t - 4\,.$$

(d)



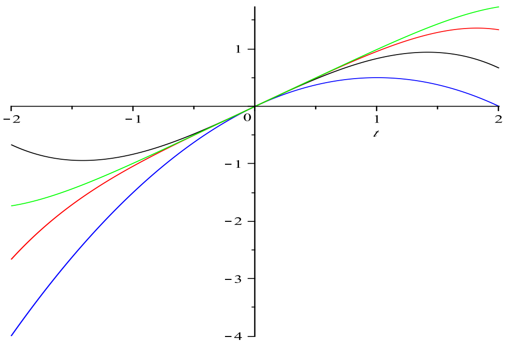
From the plot it appears that ϕ_4 is a good estimate for $|t| < 2$.

6.(a) The approximating functions are defined recursively by

$$\phi_{n+1}(t) = \int_0^t [\phi_n(s) + 1 - s]\, ds\,.$$

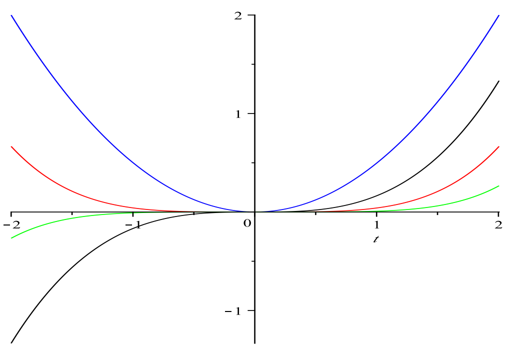
Setting $\phi_0(t) = 0$, $\phi_1(t) = t - t^2/2$, $\phi_2(t) = t - t^3/6$, $\phi_3(t) = t - t^4/24$, $\phi_4(t) = t - t^5/120$, ... Based upon these we conjecture that $\phi_n(t) = t - t^{n+1}/(n+1)!$ and use mathematical induction to verify this form for $\phi_n(t)$.

(b)



(c) Clearly $\phi(t) = t$.

(d)



From the plot it appears that ϕ_4 is a good estimate for $|t| < 1$.

8.(a) The approximating functions are defined recursively by

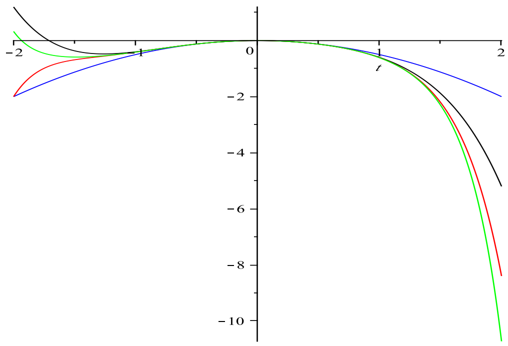
$$\phi_{n+1}(t) = \int_0^t \left[s^2\phi_n(s) - s\right] ds\,.$$

Set $\phi_0(t) = 0$. The iterates are given by $\phi_1(t) = -t^2/2\,,\ \phi_2(t) = -t^2/2 - t^5/10\,,$ $\phi_3(t) = -t^2/2 - t^5/10 - t^8/80\,, \phi_4(t) = -t^2/2 - t^5/10 - t^8/80 - t^{11}/880\,, \ldots.$ Upon inspection, it becomes apparent that

$$\phi_n(t) = -t^2\left[\frac{1}{2} + \frac{t^3}{2\cdot 5} + \frac{t^6}{2\cdot 5\cdot 8} + \ldots + \frac{(t^3)^{n-1}}{2\cdot 5\cdot 8\ldots[2+3(n-1)]}\right] =$$

$$= -t^2\sum_{k=1}^{n}\frac{(t^3)^{k-1}}{2\cdot 5\cdot 8\ldots[2+3(k-1)]}.$$

(b)



(c) Using the identity $\phi_n(t) = \phi_1(t) + [\phi_2(t) - \phi_1(t)] + [\phi_3(t) - \phi_2(t)] + \ldots + [\phi_n(t) - \phi_{n-1}(t)]$, consider the series $\phi_1(t) + \sum_{k=1}^{\infty}[\phi_{k+1}(t) - \phi_k(t)]$. Fix any t value now. We use the Ratio Test to prove the convergence of this series:

$$\left|\frac{\phi_{k+1}(t) - \phi_k(t)}{\phi_k(t) - \phi_{k-1}(t)}\right| = \left|\frac{\frac{(-t^2)(t^3)^k}{2\cdot5\cdots(2+3k)}}{\frac{(-t^2)(t^3)^{k-1}}{2\cdot5\cdots(2+3(k-1))}}\right| = \frac{|t|^3}{2+3k}.$$

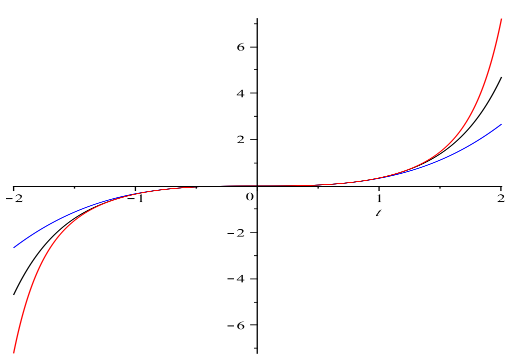
The limit of this quantity is 0 for any fixed t as $k \to \infty$, and we obtain that $\phi_n(t)$ is convergent for any t.

9.(a) The approximating functions are defined recursively by

$$\phi_{n+1}(t) = \int_0^t \left[s^2 + \phi_n^2(s)\right] ds\,.$$

Set $\phi_0(t) = 0$. The first three iterates are given by $\phi_1(t) = t^3/3$, $\phi_2(t) = t^3/3 + t^7/63$, $\phi_3(t) = t^3/3 + t^7/63 + 2t^{11}/2079 + t^{15}/59535\,.$

(b)



The iterates appear to be converging.

12.(a) The approximating functions are defined recursively by

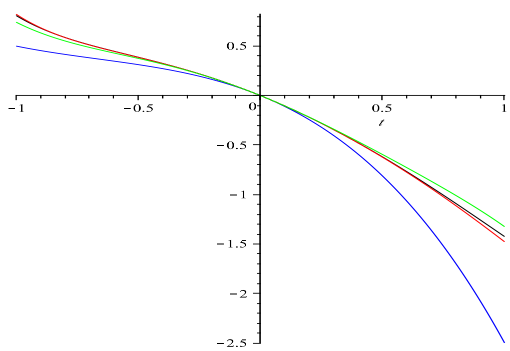
$$\phi_{n+1}(t) = \int_0^t \left[\frac{3s^2+4s+2}{2(\phi_n(s)-1)}\right] ds\,.$$

Note that $1/(2y-2) = -(1/2)\sum_{k=0}^{6} y^k + O(y^7)$. For computational purposes, use the geometric series sum to replace the above iteration formula by

$$\phi_{n+1}(t) = -\frac{1}{2}\int_0^t \left[(3s^2+4s+2)\sum_{k=0}^{6}\phi_n^k(s)\right] ds\,.$$

Set $\phi_0(t) = 0$. The first four approximations are given by $\phi_1(t) = -t - t^2 - t^3/2$, $\phi_2(t) = -t - t^2/2 + t^3/6 + t^4/4 - t^5/5 - t^6/24 + \ldots$, $\phi_3(t) = -t - t^2/2 + t^4/12 - 3t^5/20 + 4t^6/45 + \ldots$, $\phi_4(t) = -t - t^2/2 + t^4/8 - 7t^5/60 + t^6/15 + \ldots$

(b)



The approximations appear to be converging to the exact solution, which can be found by separating the variables: $\phi(t) = 1 - \sqrt{1+2t+2t^2+t^3}\,$.

14.(a) $\phi_n(0) = 0$, for every $n \geq 1$. Let $a \in (0\,,1]$. Then $\phi_n(a) = 2na\,e^{-na^2} = 2na/e^{na^2}$. Using l'Hospital's rule, $\lim_{z\to\infty} 2az/e^{az^2} = \lim_{z\to\infty} 1/ze^{az^2} = 0$. Hence $\lim_{n\to\infty}\phi_n(a) = 0\,$.

(b) $\int_0^1 2nx\,e^{-nx^2}dx = -e^{-nx^2}\big|_0^1 = 1 - e^{-n}$. Therefore,

$$\lim_{n\to\infty}\int_0^1 \phi_n(x)dx \neq \int_0^1 \lim_{n\to\infty}\phi_n(x)dx\,.$$

15. Let t be fixed, such that $(t\,,y_1), (t\,,y_2) \in D$. Without loss of generality, assume that $y_1 < y_2$. Since f is differentiable with respect to y, the mean value theorem asserts that there exists $\xi \in (y_1\,,y_2)$ such that $f(t\,,y_1) - f(t\,,y_2) = f_y(t\,,\xi)(y_1 - y_2)$. This means that $|f(t\,,y_1) - f(t\,,y_2)| = |f_y(t\,,\xi)|\;|y_1 - y_2|$. Since, by assumption, $\partial f/\partial y$ is continuous in D, f_y attains a maximum K on any closed and bounded subset of D. Hence $|f(t\,,y_1) - f(t\,,y_2)| \leq K\;|y_1 - y_2|$.

16. For a sufficiently small interval of t, $\phi_{n-1}(t)$, $\phi_n(t) \in D$. Since f satisfies a Lipschitz condition, $|f(t,\phi_n(t)) - f(t,\phi_{n-1}(t))| \leq K\,|\phi_n(t) - \phi_{n-1}(t)|$. Here $K = \max|f_y|$.

17.(a) $\phi_1(t) = \int_0^t f(s\,,0)ds$. Hence $|\phi_1(t)| \le \int_0^{|t|} |f(s\,,0)|\,ds \le \int_0^{|t|} M ds = M\,|t|$, in which M is the maximum value of $|f(t\,,y)|$ on D.

(b) By definition, $\phi_2(t) - \phi_1(t) = \int_0^t [f(s\,,\phi_1(s)) - f(s\,,0)]\,ds$. Taking the absolute value of both sides, $|\phi_2(t) - \phi_1(t)| \le \int_0^{|t|} |[f(s\,,\phi_1(s)) - f(s\,,0)]|\,ds$. Based on the results in Problems 16 and 17,

$$|\phi_2(t) - \phi_1(t)| \le \int_0^{|t|} K\,|\phi_1(s) - 0|\,ds \le KM \int_0^{|t|} |s|\,ds\,.$$

Evaluating the last integral, we obtain that $|\phi_2(t) - \phi_1(t)| \le MK\,|t|^2/2$.

(c) Suppose that

$$|\phi_i(t) - \phi_{i-1}(t)| \le \frac{MK^{i-1}\,|t|^i}{i!}$$

for some $i \ge 1$. By definition,

$$\phi_{i+1}(t) - \phi_i(t) = \int_0^t [f(s,\phi_i(s)) - f(s,\phi_{i-1}(s))]\,ds\,.$$

It follows that

$$|\phi_{i+1}(t) - \phi_i(t)| \le \int_0^{|t|} |f(s,\phi_i(s)) - f(s,\phi_{i-1}(s))|\,ds$$

$$\le \int_0^{|t|} K\,|\phi_i(s) - \phi_{i-1}(s)|\,ds \le \int_0^{|t|} K\frac{MK^{\,i-1}\,|s|^i}{i!}ds =$$

$$= \frac{MK^{\,i}\,|t|^{i+1}}{(i+1)!} \le \frac{MK^{\,i}h^{i+1}}{(i+1)!}\,.$$

Hence, by mathematical induction, the assertion is true.

18.(a) Use the triangle inequality, $|a+b| \le |a| + |b|$.

(b) For $|t| \le h$, $|\phi_1(t)| \le Mh$, and $|\phi_n(t) - \phi_{n-1}(t)| \le MK^{\,n-1}h^n/(n\,!)$. Hence

$$|\phi_n(t)| \le M\sum_{i=1}^{n} \frac{K^{\,i-1}h^i}{i\,!} = \frac{M}{K}\sum_{i=1}^{n}\frac{(Kh)^i}{i\,!}\,.$$

(c) The sequence of partial sums in (b) converges to $M(e^{Kh} - 1)/K$. By the comparison test, the sums in (a) also converge. Since individual terms of a convergent series must tend to zero, $|\phi_n(t) - \phi_{n-1}(t)| \to 0$, and it follows that the sequence $|\phi_n(t)|$ is convergent.

19.(a) Let $\phi(t) = \int_0^t f(s\,,\phi(s))ds$ and $\psi(t) = \int_0^t f(s\,,\psi(s))ds$. Then by linearity of the integral, $\phi(t) - \psi(t) = \int_0^t [f(s\,,\phi(s)) - f(s\,,\psi(s))]\,ds$.

(b) It follows that $|\phi(t) - \psi(t)| \le \int_0^t |f(s\,,\phi(s)) - f(s\,,\psi(s))|\,ds$.

(c) We know that f satisfies a Lipschitz condition, $|f(t\,,y_1)-f(t\,,y_2)|\le K\ |y_1-y_2|$, based on $|\partial f/\partial y|\le K$ in D. Therefore,

$$|\phi(t)-\psi(t)|\le\int_0^t|f(s\,,\phi(s))-f(s\,,\psi(s))|\,ds\le\int_0^t K\,|\phi(s)-\psi(s)|\,ds.$$

2.9

1. Writing the equation for each $n\ge 0$, $y_1=-0.9\,y_0$, $y_2=-0.9\,y_1=(-0.9^2)y_0$, $y_3=-0.9\,y_2=(-0.9)^3y_0$ and so on, it is apparent that $y_n=(-0.9)^n\,y_0$. The terms constitute an alternating series, which converge to zero, regardless of y_0.

3. Write the equation for each $n\ge 0$, $y_1=\sqrt{3}\,y_0$, $y_2=\sqrt{4/2}\,y_1$, $y_3=\sqrt{5/3}\,y_2$, ... Upon substitution, we find that $y_2=\sqrt{(4\cdot 3)/2}\,y_1$, $y_3=\sqrt{(5\cdot 4\cdot 3)/(3\cdot 2)}\,y_0$, ... It can be proved by mathematical induction, that

$$y_n=\frac{1}{\sqrt{2}}\sqrt{\frac{(n+2)!}{n!}}\;y_0\;=\frac{1}{\sqrt{2}}\sqrt{(n+1)(n+2)}\;y_0\,.$$

This sequence is divergent, except for $y_0=0$.

4. Writing the equation for each $n\ge 0$, $y_1=-y_0$, $y_2=y_1$, $y_3=-y_2$, $y_4=y_3$, and so on. It can be shown that

$$y_n=\begin{cases}\;\;y_0, & \text{for } n=4k \text{ or } n=4k-1\\ -y_0, & \text{for } n=4k-2 \text{ or } n=4k-3\end{cases}$$

The sequence is convergent only for $y_0=0$.

6. Writing the equation for each $n\ge 0$,

$$\begin{aligned}
y_1&=-0.5\,y_0+6\\
y_2&=-0.5\,y_1+6=-0.5(-0.5\,y_0+6)+6=(-0.5)^2y_0+6+(-0.5)6\\
y_3&=-0.5\,y_2+6=-0.5(-0.5\,y_1+6)+6=(-0.5)^3y_0+6\left[1+(-0.5)+(-0.5)^2\right]\\
&\;\;\vdots\\
y_n&=(-0.5)^ny_0+4\,[1-(-0.5)^n\;]
\end{aligned}$$

which follows from Eq.(13) and (14). The sequence is convergent for all y_0, and in fact $y_n\to 4$.

8. Let y_n be the balance at the end of the nth month. Then $y_{n+1}=(1+r/12)y_n+25$. We have $y_n=\rho^n[y_0-25/(1-\rho)]+25/(1-\rho)$, in which $\rho=(1+r/12)$. Here r is the annual interest rate, given as 8%. Thus $y_{36}=(1.0066)^{36}\,[1000+12\cdot 25/r]-12\cdot 25/r=\$2,283.63$.

9. Let y_n be the balance due at the end of the nth month. The appropriate difference equation is $y_{n+1}=(1+r/12)\,y_n-P$. Here r is the annual interest rate

and P is the monthly payment. The solution, in terms of the amount borrowed, is given by $y_n = \rho^n[y_0 + P/(1-\rho)] - P/(1-\rho)$, in which $\rho = (1+r/12)$ and $y_0 = 8,000$. To figure out the monthly payment P, we require that $y_{36} = 0$. That is, $\rho^{36}[y_0 + P/(1-\rho)] = P/(1-\rho)$. After the specified amounts are substituted, we find that $P = \$258.14$.

11. Let y_n be the balance due at the end of the nth month. The appropriate difference equation is $y_{n+1} = (1+r/12)\,y_n - P$, in which $r = .09$ and P is the monthly payment. The initial value of the mortgage is $y_0 = \$100,000$. Then the balance due at the end of the n-th month is $y_n = \rho^n[y_0 + P/(1-\rho)] - P/(1-\rho)$, where $\rho = (1+r/12)$. In terms of the specified values, $y_n = (1.0075)^n[10^5 - 12P/r] + 12P/r$. Setting $n = 30 \cdot 12 = 360$, and $y_{360} = 0$, we find that $P = \$804.62$. For the monthly payment corresponding to a 20 year mortgage, set $n = 240$ and $y_{240} = 0$ to find that $P = \$899.73$. The total amount paid during the term of the loan is $360 \times 804.62 = \$289,663.20$ for the 30-year loan and is $240 \times 899.73 = \$215,935.20$ for the 20-year loan.

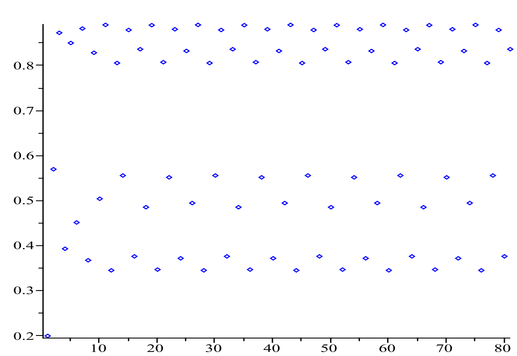
12. Let y_n be the balance due at the end of the nth month, with y_0 the initial value of the mortgage. The appropriate difference equation is $y_{n+1} = (1+r/12)\,y_n - P$, in which $r = 0.1$ and $P = \$1000$ is the maximum monthly payment. Given that the life of the mortgage is 20 years, we require that $y_{240} = 0$. The balance due at the end of the n-th month is $y_n = \rho^n[y_0 + P/(1-\rho)] - P/(1-\rho)$. In terms of the specified values for the parameters, the solution of $(1.00833)^{240}[y_0 - 12 \cdot 1000/0.1] = -12 \cdot 1000/0.1$ is $y_0 = \$103,624.62$.

19.(a) $\delta_2 = (\rho_2 - \rho_1)/(\rho_3 - \rho_2) = (3.449 - 3)/(3.544 - 3.449) = 4.7263$.

(b) diff$= (|\delta - \delta_2|/\delta) \cdot 100 = (|4.6692 - 4.7363|/4.6692) \cdot 100 \approx 1.22\%$.

(c) Assuming $(\rho_3 - \rho_2)/(\rho_4 - \rho_3) = \delta$, $\rho_4 \approx 3.5643$

(d) A period 16 solution appears near $\rho \approx 3.565$.



(e) Note that $(\rho_{n+1} - \rho_n) = \delta_n^{-1}(\rho_n - \rho_{n-1})$. With the assumption that $\delta_n = \delta$, we have $(\rho_{n+1} - \rho_n) = \delta^{-1}(\rho_n - \rho_{n-1})$, which is of the form $y_{n+1} = \alpha\, y_n$, $n \geq 3$. It follows that $(\rho_k - \rho_{k-1}) = \delta^{3-k}(\rho_3 - \rho_2)$ for $k \geq 4$. Then

$$\begin{aligned}
\rho_n &= \rho_1 + (\rho_2 - \rho_1) + (\rho_3 - \rho_2) + (\rho_4 - \rho_3) + \ldots + (\rho_n - \rho_{n-1}) \\
&= \rho_1 + (\rho_2 - \rho_1) + (\rho_3 - \rho_2)\left[1 + \delta^{-1} + \delta^{-2} + \ldots + \delta^{3-n}\right] \\
&= \rho_1 + (\rho_2 - \rho_1) + (\rho_3 - \rho_2)\left[\frac{1 - \delta^{4-n}}{1 - \delta^{-1}}\right].
\end{aligned}$$

Hence $\lim_{n\to\infty} \rho_n = \rho_2 + (\rho_3 - \rho_2)\left[\frac{\delta}{\delta - 1}\right]$. Substitution of the appropriate values yields

$$\lim_{n\to\infty} \rho_n = 3.5699$$

PROBLEMS

1. The equation is *linear*. It can be written in the form $y' + 2y/x = x^2$, and the integrating factor is $\mu(x) = e^{\int (2/x)\,dx} = e^{2\ln x} = x^2$. Multiplication by $\mu(x)$ yields $x^2y' + 2yx = (yx^2)' = x^4$. Integration with respect to x and division by x^2 gives that $y = x^3/5 + c/x^2$.

5. The equation is *exact*. Algebraic manipulations give the symmetric form of the equation, $(2xy + y^2 + 1)dx + (x^2 + 2xy)dy = 0$. We can check that $M_y = 2x + 2y = N_x$, so the equation is really exact. Integrating M with respect to x gives that $\psi(x, y) = x^2y + xy^2 + x + g(y)$, then $\psi_y = x^2 + 2xy + g'(y) = x^2 + 2xy$, so we get that $g'(y) = 0$, so we obtain that $g(y) = 0$ is acceptable. Therefore the solution is defined implicitly as $x^2y + xy^2 + x = c$.

6. The equation is *linear*. It can be written in the form $y' + (1 + (1/x))y = 1/x$ and the integrating factor is $\mu(x) = e^{\int 1 + (1/x)\,dx} = e^{x + \ln x} = xe^x$. Multiplication by $\mu(x)$ yields $xe^xy' + (xe^x + e^x)y = (xe^xy)' = e^x$. Integration with respect to x and division by xe^x shows that the general solution of the equation is $y = 1/x + c/(xe^x)$. The initial condition implies that $0 = 1 + c/e$, which means that $c = -e$ and the solution is $y = 1/x - e/(xe^x) = x^{-1}(1 - e^{1-x})$.

7. The equation is *separable*. Separation of variables gives the differential equation $y(2 + 3y)dy = (4x^3 + 1)dx$, and then after integration we obtain that the solution is $x^4 + x - y^2 - y^3 = c$.

8. The equation is *linear*. It can be written in the form $y' + 2y/x = \sin x/x^2$ and the integrating factor is $\mu(x) = e^{\int (2/x)\,dx} = e^{2\ln x} = x^2$. Multiplication by $\mu(x)$ gives $x^2y' + 2xy = (x^2y)' = \sin x$, and after integration with respect to x and division by x^2 we obtain the general solution $y = (c - \cos x)/x^2$. The initial condition implies that $c = 4 + \cos 2$ and the solution becomes $y = (4 + \cos 2 - \cos x)/x^2$.

11. The equation is *exact*. It is easy to check that $M_y = 1 = N_x$. Integrating M with respect to x gives that $\psi(x, y) = x^3/3 + xy + g(y)$, then $\psi_y = x + g'(y) =$

$x + e^y$, which means that $g'(y) = e^y$, so we obtain that $g(y) = e^y$. Therefore the solution is defined implicitly as $x^3/3 + xy + e^y = c$.

13. The equation is *separable*. Factoring the right hand side leads to the equation $y' = (1 + y^2)(1 + 2x)$. We separate the variables to obtain $dy/(1 + y^2) = (1 + 2x)dx$, then integration gives us $\arctan y = x + x^2 + c$. The solution is $y = \tan(x + x^2 + c)$.

14. The equation is *exact*. We can check that $M_y = 1 = N_x$. Integrating M with respect to x gives that $\psi(x, y) = x^2/2 + xy + g(y)$, then $\psi_y = x + g'(y) = x + 2y$, which means that $g'(y) = 2y$, so we obtain that $g(y) = y^2$. Therefore the general solution is defined implicitly as $x^2/2 + xy + y^2 = c$. The initial condition gives us $c = 17$, so the solution is $x^2 + 2xy + 2y^2 = 34$.

15. The equation is *separable*. Separation of variables leads us to the equation

$$\frac{dy}{y} = \frac{1 - e^x}{1 + e^x} dx.$$

Note that $1 + e^x - 2e^x = 1 - e^x$. We obtain that

$$\ln|y| = \int \frac{1 - e^x}{1 + e^x} dx = \int 1 - \frac{2e^x}{1 + e^x} dx = x - 2\ln(1 + e^x) + \tilde{c}.$$

This means that $y = ce^x(1 + e^x)^{-2}$, which also can be written as $y = c/\cosh^2(x/2)$ after some algebraic manipulations.

16. The equation is *exact*. The symmetric form is $(-e^{-x}\cos y + e^{2y}\cos x)dx + (-e^{-x}\sin y + 2e^{2y}\sin x)dy = 0$. We can check that $M_y = e^{-x}\sin y + 2e^{2y}\cos x = N_x$. Integrating M with respect to x gives that $\psi(x, y) = e^{-x}\cos y + e^{2y}\sin x + g(y)$, then $\psi_y = -e^{-x}\sin y + 2e^{2y}\sin x + g'(y) = -e^{-x}\sin y + 2e^{2y}\sin x$, so we get that $g'(y) = 0$, so we obtain that $g(y) = 0$ is acceptable. Therefore the solution is defined implicitly as $e^{-x}\cos y + e^{2y}\sin x = c$.

17. The equation is *linear*. The integrating factor is $\mu(x) = e^{-\int 3\,dx} = e^{-3x}$, which turns the equation into $e^{-3x}y' - 3e^{-3x}y = (e^{-3x}y)' = e^{-x}$. We integrate with respect to x to obtain $e^{-3x}y = -e^{-x} + c$, and the solution is $y = ce^{3x} - e^{2x}$ after multiplication by e^{3x}.

18. The equation is *linear*. The integrating factor is $\mu(x) = e^{\int 2\,dx} = e^{2x}$, which gives us $e^{2x}y' + 2e^{2x}y = (e^{2x}y)' = e^{-x^2}$. The antiderivative of the function on the right hand side can not be expressed in a closed form using elementary functions, so we have to express the solution using integrals. Let us integrate both sides of this equation from 0 to x. We obtain that the left hand side turns into

$$\int_0^x (e^{2s}y(s))'ds = e^{2x}y(x) - e^0y(0) = e^{2x}y - 3.$$

The right hand side gives us $\int_0^x e^{-s^2}\,ds$. So we found that

$$y = e^{-2x}\int_0^x e^{-s^2}\,ds + 3e^{-2x}.$$

19. The equation is *exact*. Algebraic manipulations give us the symmetric form $(y^3 + 2y - 3x^2)dx + (2x + 3xy^2)dy = 0$. We can check that $M_y = 3y^2 + 2 = N_x$. Integrating M with respect to x gives that $\psi(x, y) = xy^3 + 2xy - x^3 + g(y)$, then $\psi_y = 3xy^2 + 2x + g'(y) = 2x + 3xy^2$, which means that $g'(y) = 0$, so we obtain that $g(y) = 0$ is acceptable. Therefore the solution is $xy^3 + 2xy - x^3 = c$.

20. The equation is *separable*, because $y' = e^{x+y} = e^x e^y$. Separation of variables yields the equation $e^{-y}dy = e^x dx$, which turns into $-e^{-y} = e^x + c$ after integration and we obtain the implicitly defined solution $e^x + e^{-y} = c$.

22. The equation is *separable*. Separation of variables turns the equation into $(y^2 + 1)dy = (x^2 - 1)dx$, which, after integration, gives $y^3/3 + y = x^3/3 - x + c$. The initial condition yields $c = 2/3$, and the solution is $y^3 + 3y - x^3 + 3x = 2$.

23. The equation is *linear*. Division by t gives $y' + (1 + (1/t))y = e^{2t}/t$, so the integrating factor is $\mu(t) = e^{\int (1+(1/t))dt} = e^{t+\ln t} = te^t$. The equation turns into $te^t y' + (te^t + e^t)y = (te^t y)' = e^{3t}$. Integration therefore leads to $te^t y = e^{3t}/3 + c$ and the solution is $y = e^{2t}/(3t) + ce^{-t}/t$.

24. The equation is *exact*. We can check that $M_y = 2\cos y \sin x \cos x = N_x$. Integrating M with respect to x gives that $\psi(x, y) = \sin y \sin^2 x + g(y)$, then $\psi_y = \cos y \sin^2 x + g'(y) = \cos y \sin^2 x$, which means that $g'(y) = 0$, so we obtain that $g(y) = 0$ is acceptable. Therefore the solution is defined implicitly as $\sin y \sin^2 x = c$.

25. The equation is *exact*. We can check that

$$M_y = -\frac{2x}{y^2} - \frac{x^2 - y^2}{(x^2 + y^2)^2} = N_x.$$

Integrating M with respect to x gives that $\psi(x, y) = x^2/y + \arctan(y/x) + g(y)$, then $\psi_y = -x^2/y^2 + x/(x^2 + y^2) + g'(y) = x/(x^2 + y^2) - x^2/y^2$, which means that $g'(y) = 0$, so we obtain that $g(y) = 0$ is acceptable. Therefore the solution is defined implicitly as $x^2/y + \arctan(y/x) = c$.

28. The equation can be made *exact* by choosing an appropriate integrating factor. We can check that $(M_y - N_x)/N = (2 - 1)/x = 1/x$ depends only on x, so $\mu(x) = e^{\int (1/x)dx} = e^{\ln x} = x$ is an integrating factor. After multiplication, the equation becomes $(2yx + 3x^2)dx + x^2 dy = 0$. This equation is exact now, because $M_y = 2x = N_x$. Integrating M with respect to x gives that $\psi(x, y) = yx^2 + x^3 + g(y)$, then $\psi_y = x^2 + g'(y) = x^2$, which means that $g'(y) = 0$, so we obtain that $g(y) = 0$ is acceptable. Therefore the solution is defined implicitly as $x^3 + x^2 y = c$.

29. The equation is *homogeneous*. (See Section 2.2, Problem 30) We can see that

$$y' = \frac{x + y}{x - y} = \frac{1 + (y/x)}{1 - (y/x)}.$$

We substitute $u = y/x$, which means also that $y = ux$ and then $y' = u'x + u =$

$(1+u)/(1-u)$, which implies that

$$u'x = \frac{1+u}{1-u} - u = \frac{1+u^2}{1-u},$$

a separable equation. Separating the variables yields

$$\frac{1-u}{1+u^2}du = \frac{dx}{x},$$

and then integration gives $\arctan u - \ln(1+u^2)/2 = \ln|x| + c$. Substituting $u = y/x$ back into this expression and using that

$$-\ln(1+(y/x)^2)/2 - \ln|x| = -\ln(|x|\sqrt{1+(y/x)^2}) = -\ln(\sqrt{x^2+y^2})$$

we obtain that the solution is $\arctan(y/x) - \ln(\sqrt{x^2+y^2}) = c$.

30. The equation is *homogeneous*. (See Section 2.2, Problem 30) Algebraic manipulations show that it can be written in the form

$$y' = \frac{3y^2+2xy}{2xy+x^2} = \frac{3(y/x)^2+2(y/x)}{2(y/x)+1}.$$

Substituting $u = y/x$ gives that $y = ux$ and then

$$y' = u'x + u = \frac{3u^2+2u}{2u+1},$$

which implies that

$$u'x = \frac{3u^2+2u}{2u+1} - u = \frac{u^2+u}{2u+1},$$

a separable equation. We obtain that $(2u+1)du/(u^2+u) = dx/x$, which in turn means that $\ln(u^2+u) = \ln|x| + \tilde{c}$. Therefore, $u^2+u = cx$ and then substituting $u = y/x$ gives us the solution $(y^2/x^3) + (y/x^2) = c$.

31. The equation can be made *exact* by choosing an appropriate integrating factor. We can check that $(M_y - N_x)/M = -(3x^2+y)/(y(3x^2+y)) = -1/y$ depends only on y, so $\mu(y) = e^{\int(1/y)dy} = e^{\ln y} = y$ is an integrating factor. After multiplication, the equation becomes $(3x^2y^2+y^3)dx + (2x^3y+3xy^2)dy = 0$. This equation is exact now, because $M_y = 6x^2y + 3y^2 = N_x$. Integrating M with respect to x gives that $\psi(x,y) = x^3y^2 + y^3x + g(y)$, then $\psi_y = 2x^3y + 3y^2x + g'(y) = 2x^3y + 3xy^2$, which means that $g'(y) = 0$, so we obtain that $g(y) = 0$ is acceptable. Therefore the general solution is defined implicitly as $x^3y^2 + xy^3 = c$. The initial condition gives us $4 - 8 = c = -4$, and the solution is $x^3y^2 + xy^3 = -4$.

33. Let y_1 be a solution, i.e. $y_1' = q_1 + q_2y_1 + q_3y_1^2$. Now let $y = y_1 + (1/v)$ also be a solution. Differentiating this expression with respect to t and using that y is also a solution we obtain $y' = y_1' - (1/v^2)v' = q_1 + q_2y + q_3y^2 = q_1 + q_2(y_1 + (1/v)) + q_3(y_1 + (1/v))^2$. Now using that y_1 was also a solution we get that $-(1/v^2)v' = q_2(1/v) + 2q_3(y_1/v) + q_3(1/v^2)$, which, after some simple algebraic manipulations turns into $v' = -(q_2 + 2q_3y_1)v - q_3$.

35.(a) The equation is $y' = (1-y)(x+by) = x+(b-x)y-by^2$. We set $y = 1+(1/v)$ and differentiate: $y' = -v^{-2}v' = x+(b-x)(1+(1/v))-b(1+(1/v))^2$, which, after simplification, turns into $v' = (b+x)v+b$.

(b) When $x = at$, the equation is $v' - (b+at)v = b$, so the integrating factor is $\mu(t) = e^{-bt-at^2/2}$. This turns the equation into $(v\mu(t))' = b\mu(t)$, so $v\mu(t) = \int b\mu(t)dt$, and then $v = (b\int \mu(t)dt)/\mu(t)$.

36. Substitute $v = y'$, then $v' = y''$. The equation turns into $t^2v' + 2tv = (t^2v)' = 1$, which yields $t^2v = t + c_1$, so $y' = v = (1/t) + (c_1/t^2)$. Integrating this expression gives us the solution $y = \ln t - (c_1/t) + c_2$.

37. Set $v = y'$, then $v' = y''$. The equation with this substitution is $tv' + v = (tv)' = 1$, which gives $tv = t + c_1$, so $y' = v = 1 + (c_1/t)$. Integrating this expression yields the solution $y = t + c_1 \ln t + c_2$.

38. Set $v = y'$, so $v' = y''$. The equation is $v' + tv^2 = 0$, which is a separable equation. Separating the variables we obtain $dv/v^2 = -tdt$, so $-1/v = -t^2/2 + c$, and then $y' = v = 2/(t^2 + c_1)$. Now depending on the value of c_1, we have the following possibilities: when $c_1 = 0$, then $y = -2/t + c_2$, when $0 < c_1 = k^2$, then $y = (2/k)\arctan(t/k) + c_2$, and when $0 > c_1 = -k^2$ then

$$y = (1/k)\ln|(t-k)/(t+k)| + c_2.$$

We also divided by $v = y'$ when we separated the variables, and $v = 0$ (which is $y = c$) is also a solution.

39. Substitute $v = y'$ and $v' = y''$. The equation is $2t^2v' + v^3 = 2tv$. This is a *Bernoulli* equation (See Section 2.4, Problem 27), so the substitution $z = v^{-2}$ yields $z' = -2v^{-3}v'$, and the equation turns into $2t^2v'v^3 + 1 = 2t/v^2$, i.e. into $-2t^2z'/2 + 1 = 2tz$, which in turn simplifies to $t^2z' + 2tz = (t^2z)' = 1$. Integration yields $t^2z = t + c$, which means that $z = (1/t) + (c/t^2)$. Now $y' = v = \pm\sqrt{1/z} = \pm t/\sqrt{t+c_1}$ and another integration gives

$$y = \pm\frac{2}{3}(t - 2c_1)\sqrt{t+c_1} + c_2.$$

The substitution also loses the solution $v = 0$, i.e. $y = c$.

40. Set $v = y'$, then $v' = y''$. The equation reads $v' + v = e^{-t}$, which is a linear equation with integrating factor $\mu(t) = e^t$. This turns the equation into $e^tv' + e^tv = (e^tv)' = 1$, which means that $e^tv = t + c$ and then $y' = v = te^{-t} + ce^{-t}$. Another integration yields the solution $y = -te^{-t} + c_1e^{-t} + c_2$.

41. Let $v = y'$ and $v' = y''$. The equation is $t^2v' = v^2$, which is a separable equation. Separating the variables we obtain $dv/v^2 = dt/t^2$, which gives us $-1/v = -(1/t) + c_1$, and then $y' = v = t/(1 + c_1t)$. Now when $c_1 = 0$, then $y = t^2/2 + c_2$, and when $c_1 \neq 0$, then $y = t/c_1 - (\ln|1 + c_1t|)/c_1^2 + c_2$. Also, at the separation we divided by $v = 0$, which also gives us the solution $y = c$.

43. Set $y' = v(y)$. Then $y'' = v'(y)(dy/dt) = v'(y)v(y)$. We obtain the equation $v'v + y = 0$, where the differentiation is with respect to y. This is a separable equation which simplifies to $vdv = -ydy$. We obtain that $v^2/2 = -y^2/2 + c$, so $y' = v(y) = \pm\sqrt{c - y^2}$. We separate the variables again to get $dy/\sqrt{c - y^2} = \pm dt$, so $\arcsin(y/\sqrt{c}) = t + d$, which means that $y = \sqrt{c}\sin(\pm t + d) = c_1 \sin(t + c_2)$.

44. Set $y' = v(y)$. Then $y'' = v'(y)(dy/dt) = v'(y)v(y)$. We obtain the equation $v'v + yv^3 = 0$, where the differentiation is with respect to y. Separation of variables turns this into $dv/v^2 = -ydy$, which gives us $y' = v = 2/(c_1 + y^2)$. This implies that $(c_1 + y^2)dy = 2dt$ and then the solution is defined implicitly as $c_1 y + y^3/3 = 2t + c_2$. Also, $y = c$ is a solution which we lost when divided by $y' = v = 0$.

46. Set $y' = v(y)$. Then $y'' = v'(y)(dy/dt) = v'(y)v(y)$. We obtain the equation $yv'v - v^3 = 0$, where the differentiation is with respect to y. This separable equation gives us $dv/v^2 = dy/y$, which means that $-1/v = \ln|y| + c$, and then $y' = v = 1/(c - \ln|y|)$. We separate variables again to obtain $(c - \ln|y|)dy = dt$, and then integration yields the implicitly defined solution $cy - (y\ln|y| - y) = t + d$. Also, $y = c$ is a solution which we lost when we divided by $v = 0$.

49. Set $y' = v(y)$. Then $y'' = v'(y)(dy/dt) = v'(y)v(y)$. We obtain the equation $v'v - 3y^2 = 0$, where the differentiation is with respect to y. Separation of variables gives $vdv = 3y^2dy$, and after integration this turns into $v^2/2 = y^3 + c$. The initial conditions imply that $c = 0$ here, so $(y')^2 = v^2 = 2y^3$. This implies that $y' = \sqrt{2}y^{3/2}$ (the sign is determined by the initial conditions again), and this separable equation now turns into $y^{-3/2}dy = \sqrt{2}dt$. Integration yields $-2y^{-1/2} = \sqrt{2}t + d$, and the initial conditions at this point give that $d = -\sqrt{2}$. Algebraic manipulations find that $y = 2(1 - t)^{-2}$.

50. Set $v = y'$, then $v' = y''$. The equation with this substitution turns into the equation $(1 + t^2)v' + 2tv = ((1 + t^2)v)' = -3t^{-2}$. Integrating this we get that $(1 + t^2)v = 3t^{-1} + c$, and $c = -5$ from the initial conditions. This means that $y' = v = 3/(t(1 + t^2)) - 5/(1 + t^2)$. The partial fraction decomposition of the first expression shows that $y' = 3/t - 3t/(1 + t^2) - 5/(1 + t^2)$ and then another integration here gives us that $y = 3\ln t - (3/2)\ln(1 + t^2) - 5\arctan t + d$. The initial conditions identify $d = 2 + (3/2)\ln 2 + 5\pi/4$, and we obtained the solution.