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## Second Order Linear Equations

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### 3.1

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1. Let  $y = e^{rt}$ , so that  $y' = r e^{rt}$  and  $y'' = r^2 e^{rt}$ . Direct substitution into the differential equation yields  $(r^2 + 2r - 3)e^{rt} = 0$ . Canceling the exponential, the characteristic equation is  $r^2 + 2r - 3 = 0$ . The roots of the equation are  $r = -3, 1$ . Hence the general solution is  $y = c_1 e^t + c_2 e^{-3t}$ .

2. Let  $y = e^{rt}$ . Substitution of the assumed solution results in the characteristic equation  $r^2 + 3r + 2 = 0$ . The roots of the equation are  $r = -2, -1$ . Hence the general solution is  $y = c_1 e^{-t} + c_2 e^{-2t}$ .

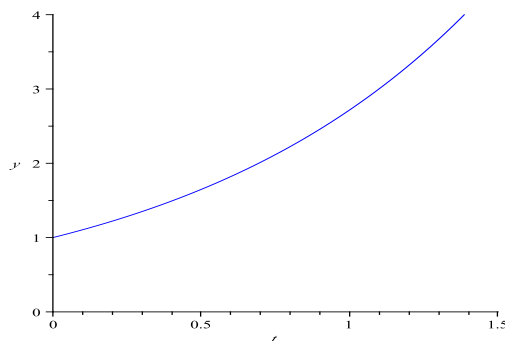
4. Substitution of the assumed solution  $y = e^{rt}$  results in the characteristic equation  $2r^2 - 3r + 1 = 0$ . The roots of the equation are  $r = 1/2, 1$ . Hence the general solution is  $y = c_1 e^{t/2} + c_2 e^t$ .

6. The characteristic equation is  $4r^2 - 9 = 0$ , with roots  $r = \pm 3/2$ . Therefore the general solution is  $y = c_1 e^{-3t/2} + c_2 e^{3t/2}$ .

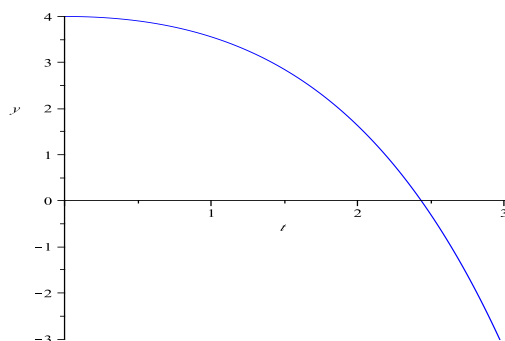
8. The characteristic equation is  $r^2 - 2r - 2 = 0$ , with roots  $r = 1 \pm \sqrt{3}$ . Hence the general solution is  $y = c_1 e^{(1-\sqrt{3})t} + c_2 e^{(1+\sqrt{3})t}$ .

9. Substitution of the assumed solution  $y = e^{rt}$  results in the characteristic equation  $r^2 + r - 2 = 0$ . The roots of the equation are  $r = -2, 1$ . Hence the general solution is  $y = c_1 e^{-2t} + c_2 e^t$ . Its derivative is  $y' = -2c_1 e^{-2t} + c_2 e^t$ . Based on the

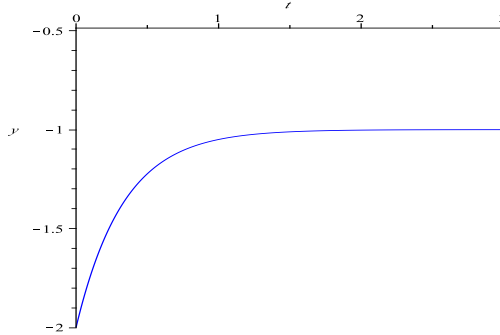
first condition,  $y(0) = 1$ , we require that  $c_1 + c_2 = 1$ . In order to satisfy  $y'(0) = 1$ , we find that  $-2c_1 + c_2 = 1$ . Solving for the constants,  $c_1 = 0$  and  $c_2 = 1$ . Hence the specific solution is  $y(t) = e^t$ . It clearly increases without bound as  $t \rightarrow \infty$ .



11. Substitution of the assumed solution  $y = e^{rt}$  results in the characteristic equation  $6r^2 - 5r + 1 = 0$ . The roots of the equation are  $r = 1/3, 1/2$ . Hence the general solution is  $y = c_1 e^{t/3} + c_2 e^{t/2}$ . Its derivative is  $y' = c_1 e^{t/3}/3 + c_2 e^{t/2}/2$ . Based on the first condition,  $y(0) = 1$ , we require that  $c_1 + c_2 = 4$ . In order to satisfy the condition  $y'(0) = 1$ , we find that  $c_1/3 + c_2/2 = 0$ . Solving for the constants,  $c_1 = 12$  and  $c_2 = -8$ . Hence the specific solution is  $y(t) = 12 e^{t/3} - 8 e^{t/2}$ . It clearly decreases without bound as  $t \rightarrow \infty$ .



12. The characteristic equation is  $r^2 + 3r = 0$ , with roots  $r = -3, 0$ . Therefore the general solution is  $y = c_1 + c_2 e^{-3t}$ , with derivative  $y' = -3c_2 e^{-3t}$ . In order to satisfy the initial conditions, we find that  $c_1 + c_2 = -2$ , and  $-3c_2 = 3$ . Hence the specific solution is  $y(t) = -1 - e^{-3t}$ . This converges to  $-1$  as  $t \rightarrow \infty$ .



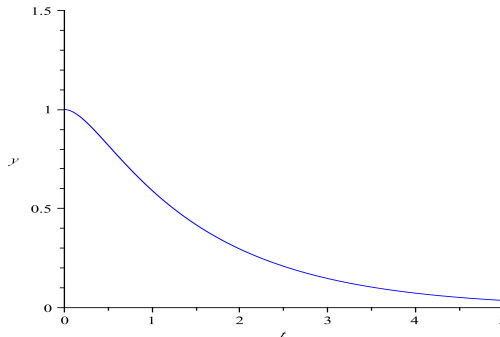
13. The characteristic equation is  $r^2 + 5r + 3 = 0$ , with roots  $r = (-5 \pm \sqrt{13})/2$ . The general solution is  $y = c_1 e^{(-5-\sqrt{13})t/2} + c_2 e^{(-5+\sqrt{13})t/2}$ , with derivative

$$y' = \frac{-5 - \sqrt{13}}{2} c_1 e^{(-5-\sqrt{13})t/2} + \frac{-5 + \sqrt{13}}{2} c_2 e^{(-5+\sqrt{13})t/2}.$$

In order to satisfy the initial conditions, we require that

$$c_1 + c_2 = 1 \quad \text{and} \quad \frac{-5 - \sqrt{13}}{2} c_1 + \frac{-5 + \sqrt{13}}{2} c_2 = 0.$$

Solving for the coefficients,  $c_1 = (1 - 5/\sqrt{13})/2$  and  $c_2 = (1 + 5/\sqrt{13})/2$ . The solution clearly converges to 0 as  $t \rightarrow \infty$ .



14. The characteristic equation is  $2r^2 + r - 4 = 0$ , with roots  $r = (-1 \pm \sqrt{33})/4$ . The general solution is  $y = c_1 e^{(-1-\sqrt{33})t/4} + c_2 e^{(-1+\sqrt{33})t/4}$ , with derivative

$$y' = \frac{-1 - \sqrt{33}}{4} c_1 e^{(-1-\sqrt{33})t/4} + \frac{-1 + \sqrt{33}}{4} c_2 e^{(-1+\sqrt{33})t/4}.$$

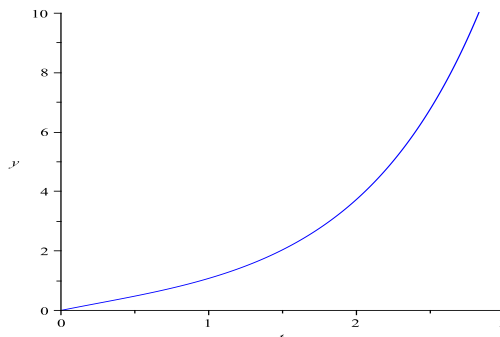
In order to satisfy the initial conditions, we require that

$$c_1 + c_2 = 0 \quad \text{and} \quad \frac{-1 - \sqrt{33}}{4} c_1 + \frac{-1 + \sqrt{33}}{4} c_2 = 1.$$

Solving for the coefficients,  $c_1 = -2/\sqrt{33}$  and  $c_2 = 2/\sqrt{33}$ . The specific solution is

$$y(t) = -2 \left[ e^{(-1-\sqrt{33})t/4} - e^{(-1+\sqrt{33})t/4} \right] / \sqrt{33}.$$

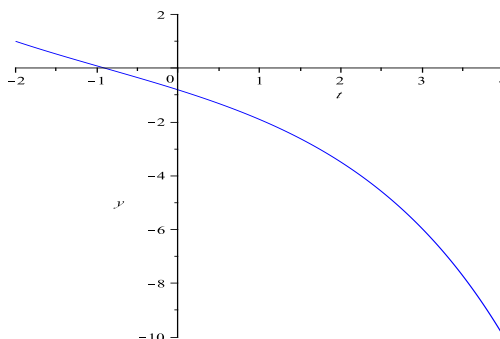
It clearly increases without bound as  $t \rightarrow \infty$ .



16. The characteristic equation is  $4r^2 - 1 = 0$ , with roots  $r = \pm 1/2$ . Therefore the general solution is  $y = c_1 e^{-t/2} + c_2 e^{t/2}$ . Since the initial conditions are specified at  $t = -2$ , is more convenient to write  $y = d_1 e^{-(t+2)/2} + d_2 e^{(t+2)/2}$ . The derivative is given by  $y' = -[d_1 e^{-(t+2)/2}]/2 + [d_2 e^{(t+2)/2}]/2$ . In order to satisfy the initial conditions, we find that  $d_1 + d_2 = 1$ , and  $-d_1/2 + d_2/2 = -1$ . Solving for the coefficients,  $d_1 = 3/2$ , and  $d_2 = -1/2$ . The specific solution is

$$y(t) = \frac{3}{2} e^{-(t+2)/2} - \frac{1}{2} e^{(t+2)/2} = \frac{3}{2e} e^{-t/2} - \frac{e}{2} e^{t/2}.$$

It clearly decreases without bound as  $t \rightarrow \infty$ .



18. An algebraic equation with roots  $-2$  and  $-1/2$  is  $2r^2 + 5r + 2 = 0$ . This is the characteristic equation for the differential equation  $2y'' + 5y' + 2y = 0$ .

20. The characteristic equation is  $2r^2 - 3r + 1 = 0$ , with roots  $r = 1/2, 1$ . Therefore the general solution is  $y = c_1 e^{t/2} + c_2 e^t$ , with derivative  $y' = c_1 e^{t/2}/2 + c_2 e^t$ . In order to satisfy the initial conditions, we require  $c_1 + c_2 = 2$  and  $c_1/2 + c_2 = 1/2$ . Solving for the coefficients,  $c_1 = 3$ , and  $c_2 = -1$ . The specific solution is  $y(t) = 3e^{t/2} - e^t$ . To find the stationary point, set  $y' = 3e^{t/2}/2 - e^t = 0$ . There is a unique solution, with  $t_1 = \ln(9/4)$ . The maximum value is then  $y(t_1) = 9/4$ . To find the  $x$ -intercept, solve the equation  $3e^{t/2} - e^t = 0$ . The solution is readily found to be  $t_2 = \ln 9 \approx 2.1972$ .

22. The characteristic equation is  $4r^2 - 1 = 0$ , with roots  $r = \pm 1/2$ . Hence the general solution is  $y = c_1 e^{-t/2} + c_2 e^{t/2}$  and  $y' = -c_1 e^{-t/2}/2 + c_2 e^{t/2}/2$ . Invoking the initial conditions, we require that  $c_1 + c_2 = 2$  and  $-c_1 + c_2 = 2\beta$ . The specific solution is  $y(t) = (1 - \beta)e^{-t/2} + (1 + \beta)e^{t/2}$ . Based on the form of the solution, it is evident that as  $t \rightarrow \infty$ ,  $y(t) \rightarrow 0$  as long as  $\beta = -1$ .

23. The characteristic equation is  $r^2 - (2\alpha - 1)r + \alpha(\alpha - 1) = 0$ . Examining the coefficients, the roots are  $r = \alpha, \alpha - 1$ . Hence the general solution of the differential equation is  $y(t) = c_1 e^{\alpha t} + c_2 e^{(\alpha-1)t}$ . Assuming  $\alpha \in \mathbb{R}$ , all solutions will tend to zero as long as  $\alpha < 0$ . On the other hand, all solutions will become unbounded as long as  $\alpha - 1 > 0$ , that is,  $\alpha > 1$ .

26.(a) The characteristic roots are  $r = -3, -2$ . The solution of the initial value problem is  $y(t) = (6 + \beta)e^{-2t} - (4 + \beta)e^{-3t}$ .

(b) The maximum point has coordinates  $t_0 = \ln[(3(4 + \beta))/(2(6 + \beta))]$ ,  $y_0 = 4(6 + \beta)^3/(27(4 + \beta)^2)$ .

(c)  $y_0 = 4(6 + \beta)^3/(27(4 + \beta)^2) \geq 4$ , as long as  $\beta \geq 6 + 6\sqrt{3}$ .

(d)  $\lim_{\beta \rightarrow \infty} t_0 = \ln(3/2)$ ,  $\lim_{\beta \rightarrow \infty} y_0 = \infty$ .

27.(a) Assuming that  $y$  is a constant, the differential equation reduces to  $cy = d$ . Hence the only equilibrium solution is  $y = d/c$ .

(b) Setting  $y = Y + d/c$ , substitution into the differential equation results in the equation  $aY'' + bY' + c(Y + d/c) = d$ . The equation satisfied by  $Y$  is  $aY'' + bY' + cY = 0$ .

## 3.2

1.

$$W(e^{2t}, e^{-3t/2}) = \begin{vmatrix} e^{2t} & e^{-3t/2} \\ 2e^{2t} & -\frac{3}{2}e^{-3t/2} \end{vmatrix} = -\frac{7}{2}e^{t/2}.$$

3.

$$W(e^{-2t}, te^{-2t}) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & (1 - 2t)e^{-2t} \end{vmatrix} = e^{-4t}.$$

5.

$$W(e^t \sin t, e^t \cos t) = \begin{vmatrix} e^t \sin t & e^t \cos t \\ e^t(\sin t + \cos t) & e^t(\cos t - \sin t) \end{vmatrix} = -e^{2t}.$$

6.

$$W(\cos^2 \theta, 1 + \cos 2\theta) = \begin{vmatrix} \cos^2 \theta & 1 + \cos 2\theta \\ -2 \sin \theta \cos \theta & -2 \sin 2\theta \end{vmatrix} = 0.$$

7. Write the equation as  $y'' + (3/t)y' = 1$ .  $p(t) = 3/t$  is continuous for all  $t > 0$ . Since  $t_0 > 0$ , the IVP has a unique solution for all  $t > 0$ .

9. Write the equation as  $y'' + (3/(t-4))y' + (4/(t-4))y = 2/t(t-4)$ . The coefficients are not continuous at  $t = 0$  and  $t = 4$ . Since  $t_0 \in (0, 4)$ , the largest interval is  $0 < t < 4$ .

10. The coefficient  $3 \ln |t|$  is discontinuous at  $t = 0$ . Since  $t_0 > 0$ , the largest interval of existence is  $0 < t < \infty$ .

11. Write the equation as  $y'' + (x/(x-3))y' + (\ln |x|/(x-3))y = 0$ . The coefficients are discontinuous at  $x = 0$  and  $x = 3$ . Since  $x_0 \in (0, 3)$ , the largest interval is  $0 < x < 3$ .

13.  $y_1'' = 2$ . We see that  $t^2(2) - 2(t^2) = 0$ .  $y_2'' = 2t^{-3}$ , with  $t^2(y_2'') - 2(y_2) = 0$ . Let  $y_3 = c_1 t^2 + c_2 t^{-1}$ , then  $y_3'' = 2c_1 + 2c_2 t^{-3}$ . It is evident that  $y_3$  is also a solution.

16. No. Substituting  $y = \sin(t^2)$  into the differential equation,

$$-4t^2 \sin(t^2) + 2 \cos(t^2) + 2t \cos(t^2)p(t) + \sin(t^2)q(t) = 0.$$

At  $t = 0$ , this equation becomes  $2 = 0$  (if we suppose that  $p(t)$  and  $q(t)$  are continuous), which is impossible.

17.  $W(e^{2t}, g(t)) = e^{2t}g'(t) - 2e^{2t}g(t) = 3e^{4t}$ . Dividing both sides by  $e^{2t}$ , we find that  $g$  must satisfy the ODE  $g' - 2g = 3e^{2t}$ . Hence  $g(t) = 3te^{2t} + ce^{2t}$ .

19.  $W(f, g) = fg' - f'g$ . Also,  $W(u, v) = W(2f - g, f + 2g)$ . Upon evaluation,  $W(u, v) = 5fg' - 5f'g = 5W(f, g)$ .

20.  $W(f, g) = fg' - f'g = t \cos t - \sin t$ , and  $W(u, v) = -4fg' + 4f'g$ . Hence  $W(u, v) = -4t \cos t + 4 \sin t$ .

21. We compute

$$\begin{aligned} W(a_1y_1 + a_2y_2, b_1y_1 + b_2y_2) &= \begin{vmatrix} a_1y_1 + a_2y_2 & b_1y_1 + b_2y_2 \\ a_1y_1' + a_2y_2' & b_1y_1' + b_2y_2' \end{vmatrix} = \\ &= (a_1y_1 + a_2y_2)(b_1y_1' + b_2y_2') - (b_1y_1 + b_2y_2)(a_1y_1' + a_2y_2') = \\ &= a_1b_2(y_1y_2' - y_1'y_2) - a_2b_1(y_1y_2' - y_1'y_2) = (a_1b_2 - a_2b_1)W(y_1, y_2). \end{aligned}$$

This now readily shows that  $y_3$  and  $y_4$  form a fundamental set of solutions if and only if  $a_1b_2 - a_2b_1 \neq 0$ .

23. The general solution is  $y = c_1e^{-3t} + c_2e^{-t}$ .  $W(e^{-3t}, e^{-t}) = 2e^{-4t}$ , and hence the exponentials form a fundamental set of solutions. On the other hand, the fundamental solutions must also satisfy the conditions  $y_1(1) = 1$ ,  $y_1'(1) = 0$ ;  $y_2(1) = 0$ ,  $y_2'(1) = 1$ . For  $y_1$ , the initial conditions require  $c_1 + c_2 = e$ ,  $-3c_1 - c_2 = 0$ . The

coefficients are  $c_1 = -e^3/2$ ,  $c_2 = 3e/2$ . For the solution  $y_2$ , the initial conditions require  $c_1 + c_2 = 0$ ,  $-3c_1 - c_2 = e$ . The coefficients are  $c_1 = -e^3/2$ ,  $c_2 = e/2$ . Hence the fundamental solutions are

$$y_1 = -\frac{1}{2}e^{-3(t-1)} + \frac{3}{2}e^{-(t-1)} \quad \text{and} \quad y_2 = -\frac{1}{2}e^{-3(t-1)} + \frac{1}{2}e^{-(t-1)}.$$

24. Yes.  $y_1'' = -4 \cos 2t$ ;  $y_2'' = -4 \sin 2t$ .  $W(\cos 2t, \sin 2t) = 2$ .

25. Clearly,  $y_1 = e^t$  is a solution.  $y_2' = (1+t)e^t$ ,  $y_2'' = (2+t)e^t$ . Substitution into the ODE results in  $(2+t)e^t - 2(1+t)e^t + te^t = 0$ . Furthermore,  $W(e^t, te^t) = e^{2t}$ . Hence the solutions form a fundamental set of solutions.

27. Clearly,  $y_1 = x$  is a solution.  $y_2' = \cos x$ ,  $y_2'' = -\sin x$ . Substitution into the ODE results in  $(1 - x \cot x)(-\sin x) - x(\cos x) + \sin x = 0$ . We can compute that  $W(y_1, y_2) = x \cos x - \sin x$ , which is nonzero for  $0 < x < \pi$ . Hence  $\{x, \sin x\}$  is a fundamental set of solutions.

30. Writing the equation in standard form, we find that  $P(t) = \sin t / \cos t$ . Hence the Wronskian is  $W(t) = ce^{-\int (\sin t / \cos t) dt} = ce^{\ln|\cos t|} = c \cos t$ , in which  $c$  is some constant.

31. After writing the equation in standard form, we have  $P(x) = 1/x$ . The Wronskian is  $W(x) = ce^{-\int (1/x) dx} = ce^{-\ln|x|} = c/x$ , in which  $c$  is some constant.

32. Writing the equation in standard form, we find that  $P(x) = -2x/(1-x^2)$ . The Wronskian is  $W(x) = ce^{-\int -2x/(1-x^2) dx} = ce^{-\ln|1-x^2|} = c/(1-x^2)$ , in which  $c$  is some constant.

33. Rewrite the equation as  $p(t)y'' + p'(t)y' + q(t)y = 0$ . After writing the equation in standard form, we have  $P(t) = p'(t)/p(t)$ . Hence the Wronskian is

$$W(t) = ce^{-\int p'(t)/p(t) dt} = ce^{-\ln p(t)} = c/p(t).$$

35. The Wronskian associated with the solutions of the differential equation is given by  $W(t) = ce^{-\int -2/t^2 dt} = ce^{-2/t}$ . Since  $W(2) = 3$ , it follows that for the hypothesized set of solutions,  $c = 3e$ . Hence  $W(4) = 3\sqrt{e}$ .

36. For the given differential equation, the Wronskian satisfies the first order differential equation  $W' + p(t)W = 0$ . Given that  $W$  is constant, it is necessary that  $p(t) \equiv 0$ .

37. Direct calculation shows that  $W(fg, fh) = (fg)(fh)' - (fg)'(fh) = (fg)(f'h + fh') - (f'g + fg')(fh) = f^2 W(g, h)$ .

39. Since  $y_1$  and  $y_2$  are solutions, they are differentiable. The hypothesis can thus be restated as  $y_1'(t_0) = y_2'(t_0) = 0$  at some point  $t_0$  in the interval of definition.

This implies that  $W(y_1, y_2)(t_0) = 0$ . But  $W(y_1, y_2)(t_0) = c e^{-\int p(t) dt}$ , which cannot be equal to zero, unless  $c = 0$ . Hence  $W(y_1, y_2) \equiv 0$ , which is ruled out for a fundamental set of solutions.

42.  $P = 1$ ,  $Q = x$ ,  $R = 1$ . We have  $P'' - Q' + R = 0$ . The equation is exact. Note that  $(y')' + (xy)' = 0$ . Hence  $y' + xy = c_1$ . This equation is linear, with integrating factor  $\mu = e^{x^2/2}$ . Therefore the general solution is

$$y(x) = c_1 e^{-x^2/2} \int_{x_0}^x e^{u^2/2} du + c_2 e^{-x^2/2}.$$

43.  $P = 1$ ,  $Q = 3x^2$ ,  $R = x$ . Note that  $P'' - Q' + R = -5x$ , and therefore the differential equation is not exact.

45.  $P = x^2$ ,  $Q = x$ ,  $R = -1$ . We have  $P'' - Q' + R = 0$ . The equation is exact. Write the equation as  $(x^2 y')' - (xy)' = 0$ . After integration, we conclude that  $x^2 y' - xy = c$ . Divide both sides of the differential equation by  $x^2$ . The resulting equation is linear, with integrating factor  $\mu = 1/x$ . Hence  $(y/x)' = c x^{-3}$ . The solution is  $y(x) = c_1 x^{-1} + c_2 x$ .

47.  $P = x^2$ ,  $Q = x$ ,  $R = x^2 - \nu^2$ . Hence the coefficients are  $2P' - Q = 3x$  and  $P'' - Q' + R = x^2 + 1 - \nu^2$ . The adjoint of the original differential equation is given by  $x^2 \mu'' + 3x \mu' + (x^2 + 1 - \nu^2) \mu = 0$ .

49.  $P = 1$ ,  $Q = 0$ ,  $R = -x$ . Hence the coefficients are given by  $2P' - Q = 0$  and  $P'' - Q' + R = -x$ . Therefore the adjoint of the original equation is  $\mu'' - x \mu = 0$ .

### 3.3

2.  $e^{2-3i} = e^2 e^{-3i} = e^2 (\cos 3 - i \sin 3)$ .

3.  $e^{i\pi} = \cos \pi + i \sin \pi = -1$ .

4.  $e^{2-(\pi/2)i} = e^2 (\cos(\pi/2) - i \sin(\pi/2)) = -e^2 i$ .

6.  $\pi^{-1+2i} = e^{(-1+2i) \ln \pi} = e^{-\ln \pi} e^{2 \ln \pi i} = (\cos(2 \ln \pi) + i \sin(2 \ln \pi)) / \pi$ .

8. The characteristic equation is  $r^2 - 2r + 6 = 0$ , with roots  $r = 1 \pm i\sqrt{5}$ . Hence the general solution is  $y = c_1 e^t \cos \sqrt{5}t + c_2 e^t \sin \sqrt{5}t$ .

9. The characteristic equation is  $r^2 + 2r - 8 = 0$ , with roots  $r = -4, 2$ . The roots are real and different, hence the general solution is  $y = c_1 e^{-4t} + c_2 e^{2t}$ .

10. The characteristic equation is  $r^2 + 2r + 2 = 0$ , with roots  $r = -1 \pm i$ . Hence the general solution is  $y = c_1 e^{-t} \cos t + c_2 e^{-t} \sin t$ .



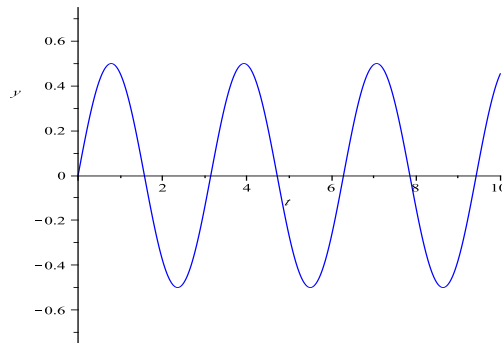
12. The characteristic equation is  $4r^2 + 9 = 0$ , with roots  $r = \pm(3/2)i$ . Hence the general solution is  $y = c_1 \cos(3t/2) + c_2 \sin(3t/2)$ .

13. The characteristic equation is  $r^2 + 2r + 1.25 = 0$ , with roots  $r = -1 \pm i/2$ . Hence the general solution is  $y = c_1 e^{-t} \cos(t/2) + c_2 e^{-t} \sin(t/2)$ .

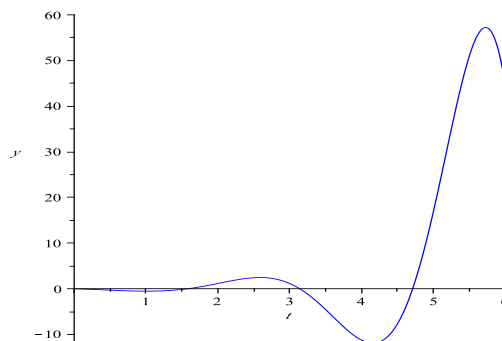
15. The characteristic equation is  $r^2 + r + 1.25 = 0$ , with roots  $r = -(1/2) \pm i$ . Hence the general solution is  $y = c_1 e^{-t/2} \cos t + c_2 e^{-t/2} \sin t$ .

16. The characteristic equation is  $r^2 + 4r + 6.25 = 0$ , with roots  $r = -2 \pm (3/2)i$ . Hence the general solution is  $y = c_1 e^{-2t} \cos(3t/2) + c_2 e^{-2t} \sin(3t/2)$ .

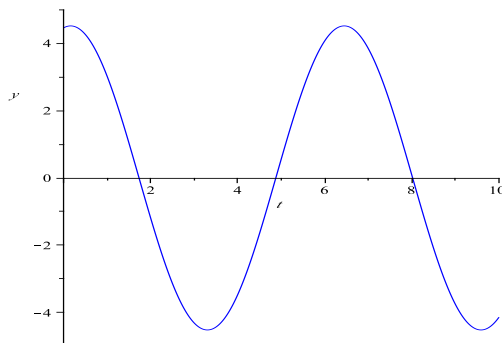
17. The characteristic equation is  $r^2 + 4 = 0$ , with roots  $r = \pm 2i$ . Hence the general solution is  $y = c_1 \cos 2t + c_2 \sin 2t$ . Now  $y' = -2c_1 \sin 2t + 2c_2 \cos 2t$ . Based on the first condition,  $y(0) = 0$ , we require that  $c_1 = 0$ . In order to satisfy the condition  $y'(0) = 1$ , we find that  $2c_2 = 1$ . The constants are  $c_1 = 0$  and  $c_2 = 1/2$ . Hence the specific solution is  $y(t) = \sin 2t/2$ . The solution is periodic.



19. The characteristic equation is  $r^2 - 2r + 5 = 0$ , with roots  $r = 1 \pm 2i$ . Hence the general solution is  $y = c_1 e^t \cos 2t + c_2 e^t \sin 2t$ . Based on the initial condition  $y(\pi/2) = 0$ , we require that  $c_1 = 0$ . It follows that  $y = c_2 e^t \sin 2t$ , and so the first derivative is  $y' = c_2 e^t \sin 2t + 2c_2 e^t \cos 2t$ . In order to satisfy the condition  $y'(\pi/2) = 2$ , we find that  $-2e^{\pi/2} c_2 = 2$ . Hence we have  $c_2 = -e^{-\pi/2}$ . Therefore the specific solution is  $y(t) = -e^{t-\pi/2} \sin 2t$ . The solution oscillates with an exponentially growing amplitude.



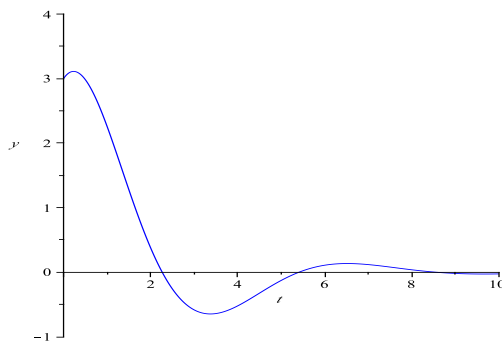
20. The characteristic equation is  $r^2 + 1 = 0$ , with roots  $r = \pm i$ . Hence the general solution is  $y = c_1 \cos t + c_2 \sin t$ . Its derivative is  $y' = -c_1 \sin t + c_2 \cos t$ . Based on the first condition,  $y(\pi/3) = 2$ , we require that  $c_1 + \sqrt{3}c_2 = 4$ . In order to satisfy the condition  $y'(\pi/3) = -4$ , we find that  $-\sqrt{3}c_1 + c_2 = -8$ . Solving these for the constants,  $c_1 = 1 + 2\sqrt{3}$  and  $c_2 = \sqrt{3} - 2$ . Hence the specific solution is a steady oscillation, given by  $y(t) = (1 + 2\sqrt{3}) \cos t + (\sqrt{3} - 2) \sin t$ .



21. From Problem 15, the general solution is  $y = c_1 e^{-t/2} \cos t + c_2 e^{-t/2} \sin t$ . Invoking the first initial condition,  $y(0) = 3$ , which implies that  $c_1 = 3$ . Substituting, it follows that  $y = 3e^{-t/2} \cos t + c_2 e^{-t/2} \sin t$ , and so the first derivative is

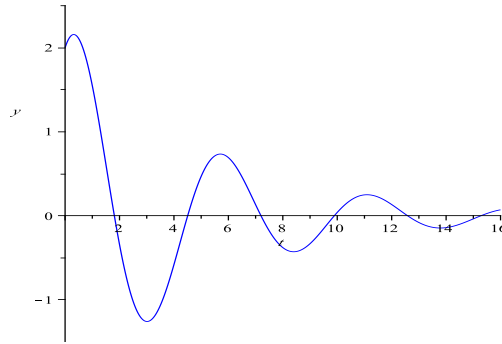
$$y' = -\frac{3}{2}e^{-t/2} \cos t - 3e^{-t/2} \sin t + c_2 e^{-t/2} \cos t - \frac{c_2}{2}e^{-t/2} \sin t.$$

Invoking the initial condition,  $y'(0) = 1$ , we find that  $-3/2 + c_2 = 1$ , and so  $c_2 = 5/2$ . Hence the specific solution is  $y(t) = 3e^{-t/2} \cos t + (5/2)e^{-t/2} \sin t$ . It oscillates with an exponentially decreasing amplitude.



24.(a) The characteristic equation is  $5r^2 + 2r + 7 = 0$ , with roots  $r = -(1 \pm i\sqrt{34})/5$ . The solution is  $u = c_1 e^{-t/5} \cos \frac{\sqrt{34}t}{5} + c_2 e^{-t/5} \sin \frac{\sqrt{34}t}{5}$ . Invoking the given initial conditions, we obtain the equations for the coefficients:  $c_1 = 2$ ,  $-2 + \sqrt{34}c_2 = 5$ . That is,  $c_1 = 2$ ,  $c_2 = 7/\sqrt{34}$ . Hence the specific solution is

$$u(t) = 2e^{-t/5} \cos \frac{\sqrt{34}t}{5} + \frac{7}{\sqrt{34}}e^{-t/5} \sin \frac{\sqrt{34}t}{5}.$$



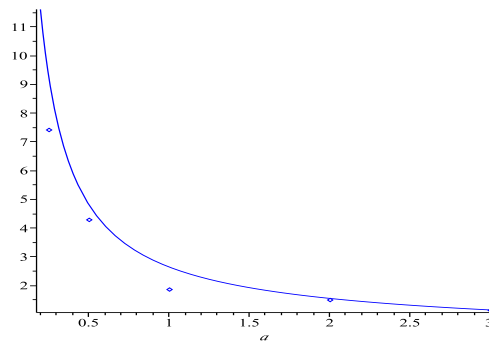
(b) Based on the graph of  $u(t)$ ,  $T$  is in the interval  $14 < t < 16$ . A numerical solution on that interval yields  $T \approx 14.5115$ .

26.(a) The characteristic equation is  $r^2 + 2ar + (a^2 + 1) = 0$ , with roots  $r = -a \pm i$ . Hence the general solution is  $y(t) = c_1 e^{-at} \cos t + c_2 e^{-at} \sin t$ . Based on the initial conditions, we find that  $c_1 = 1$  and  $c_2 = a$ . Therefore the specific solution is given by  $y(t) = e^{-at} \cos t + a e^{-at} \sin t = \sqrt{1 + a^2} e^{-at} \cos(t - \phi)$ , in which  $\phi = \arctan(a)$ .

(b) For estimation, note that  $|y(t)| \leq \sqrt{1 + a^2} e^{-at}$ . Now consider the inequality  $\sqrt{1 + a^2} e^{-at} \leq 1/10$ . The inequality holds for  $t \geq (1/a) \ln(10\sqrt{1 + a^2})$ . Therefore  $T \leq (1/a) \ln(10\sqrt{1 + a^2})$ . Setting  $a = 1$ , the numerical value is  $T \approx 1.8763$ .

(c) Similarly,  $T_{1/4} \approx 7.4284$ ,  $T_{1/2} \approx 4.3003$ ,  $T_2 \approx 1.5116$ .

(d)



Note that the estimates  $T_a$  approach the graph of  $(1/a) \ln(10\sqrt{1 + a^2})$  as  $a$  gets large.

27. Direct calculation gives the result. On the other hand, it was shown in Problem 3.2.37 that  $W(fg, fh) = f^2 W(g, h)$ . Hence  $W(e^{\lambda t} \cos \mu t, e^{\lambda t} \sin \mu t) = e^{2\lambda t} W(\cos \mu t, \sin \mu t) = e^{2\lambda t} [\cos \mu t (\sin \mu t)' - (\cos \mu t)' \sin \mu t] = \mu e^{2\lambda t}$ .

28.(a) Clearly,  $y_1$  and  $y_2$  are solutions. Also,  $W(\cos t, \sin t) = \cos^2 t + \sin^2 t = 1$ .

(b)  $y' = i e^{it}$ ,  $y'' = i^2 e^{it} = -e^{it}$ . Evidently,  $y$  is a solution and so  $y = c_1 y_1 + c_2 y_2$ .

(c) Setting  $t = 0$ ,  $1 = c_1 \cos 0 + c_2 \sin 0$ , and  $c_1 = 1$ .

(d) Differentiating,  $i e^{it} = c_2 \cos t$ . Setting  $t = 0$ ,  $i = c_2 \cos 0$  and hence  $c_2 = i$ . Therefore  $e^{it} = \cos t + i \sin t$ .

29. Euler's formula is  $e^{it} = \cos t + i \sin t$ . It follows that  $e^{-it} = \cos t - i \sin t$ . Adding these equation,  $e^{it} + e^{-it} = 2 \cos t$ . Subtracting the two equations results in  $e^{it} - e^{-it} = 2i \sin t$ .

30. Let  $r_1 = \lambda_1 + i\mu_1$ , and  $r_2 = \lambda_2 + i\mu_2$ . Then

$$\begin{aligned} e^{(r_1+r_2)t} &= e^{(\lambda_1+\lambda_2)t+i(\mu_1+\mu_2)t} = e^{(\lambda_1+\lambda_2)t} [\cos(\mu_1 + \mu_2)t + i \sin(\mu_1 + \mu_2)t] = \\ &= e^{(\lambda_1+\lambda_2)t} [(\cos \mu_1 t + i \sin \mu_1 t)(\cos \mu_2 t + i \sin \mu_2 t)] = \\ &= e^{\lambda_1 t} (\cos \mu_1 t + i \sin \mu_1 t) \cdot e^{\lambda_2 t} (\cos \mu_2 t + i \sin \mu_2 t) = e^{r_1 t} e^{r_2 t}. \end{aligned}$$

Hence  $e^{(r_1+r_2)t} = e^{r_1 t} e^{r_2 t}$ .

32. Clearly,  $u' = \lambda e^{\lambda t} \cos \mu t - \mu e^{\lambda t} \sin \mu t = e^{\lambda t} (\lambda \cos \mu t - \mu \sin \mu t)$  and then  $u'' = \lambda e^{\lambda t} (\lambda \cos \mu t - \mu \sin \mu t) + e^{\lambda t} (-\lambda \mu \sin \mu t - \mu^2 \cos \mu t)$ . Plugging these into the differential equation, dividing by  $e^{\lambda t} \neq 0$  and arranging the sine and cosine terms we obtain that the identity to prove is

$$(a(\lambda^2 - \mu^2) + b\lambda + c) \cos \mu t + (-2\lambda\mu a - b\mu) \sin \mu t = 0.$$

We know that  $\lambda \pm i\mu$  solves the characteristic equation  $ar^2 + br + c = 0$ , so  $a(\lambda - i\mu)^2 + b(\lambda - i\mu) + c = a(\lambda^2 - \mu^2) + b\lambda + c + i(-2\lambda\mu a - \mu b) = 0$ . If this complex number is zero, then both the real and imaginary parts of it are zero, but those are the coefficients of  $\cos \mu t$  and  $\sin \mu t$  in the above identity, which proves that  $au'' + bu' + cu = 0$ . The solution for  $v$  is analogous.

35. The equation transforms into  $y'' + y = 0$ . The characteristic roots are  $r = \pm i$ . The solution is  $y = c_1 \cos(x) + c_2 \sin(x) = c_1 \cos(\ln t) + c_2 \sin(\ln t)$ .

37. The equation transforms into  $y'' + 2y' + 1.25y = 0$ . The characteristic roots are  $r = -1 \pm i/2$ . The solution is

$$y = c_1 e^{-x} \cos(x/2) + c_2 e^{-x} \sin(x/2) = c_1 \frac{\cos(\frac{1}{2} \ln t)}{t} + c_2 \frac{\sin(\frac{1}{2} \ln t)}{t}.$$

38. The equation transforms into  $y'' - 5y' - 6y = 0$ . The characteristic roots are  $r = -1, 6$ . The solution is  $y = c_1 e^{-x} + c_2 e^{6x} = c_1 e^{-\ln t} + c_2 e^{6 \ln t} = c_1/t + c_2 t^6$ .

39. The equation transforms into  $y'' - 5y' + 6y = 0$ . The characteristic roots are  $r = 2, 3$ . The solution is  $y = c_1 e^{2x} + c_2 e^{3x} = c_1 e^{2 \ln t} + c_2 e^{3 \ln t} = c_1 t^2 + c_2 t^3$ .

41. The equation transforms into  $y'' + 2y' - 3y = 0$ . The characteristic roots are  $r = 1, -3$ . The solution is  $y = c_1 e^x + c_2 e^{-3x} = c_1 e^{\ln t} + c_2 e^{-3 \ln t} = c_1 t + c_2/t^3$ .

42. The equation transforms into  $y'' + 6y' + 10y = 0$ . The characteristic roots are  $r = -3 \pm i$ . The solution is

$$y = c_1 e^{-3x} \cos(x) + c_2 e^{-3x} \sin(x) = c_1 \frac{1}{t^3} \cos(\ln t) + c_2 \frac{1}{t^3} \sin(\ln t).$$

43.(a) By the chain rule,  $y'(x) = (dy/dx)x'$ . In general,  $dz/dt = (dz/dx)(dx/dt)$ . Setting  $z = (dy/dt)$ , we have

$$\frac{d^2 y}{dt^2} = \frac{dz}{dx} \frac{dx}{dt} = \frac{d}{dx} \left[ \frac{dy}{dx} \frac{dx}{dt} \right] \frac{dx}{dt} = \left[ \frac{d^2 y}{dx^2} \frac{dx}{dt} \right] \frac{dx}{dt} + \frac{dy}{dx} \frac{d}{dx} \left[ \frac{dx}{dt} \right] \frac{dx}{dt}.$$

However,

$$\frac{d}{dx} \left[ \frac{dx}{dt} \right] \frac{dx}{dt} = \left[ \frac{d^2 x}{dt^2} \right] \frac{dt}{dx} \cdot \frac{dx}{dt} = \frac{d^2 x}{dt^2}.$$

Hence

$$\frac{d^2 y}{dt^2} = \frac{d^2 y}{dx^2} \left[ \frac{dx}{dt} \right]^2 + \frac{dy}{dx} \frac{d^2 x}{dt^2}.$$

(b) Substituting the results in part (a) into the general differential equation,  $y'' + p(t)y' + q(t)y = 0$ , we find that

$$\frac{d^2 y}{dx^2} \left[ \frac{dx}{dt} \right]^2 + \frac{dy}{dx} \frac{d^2 x}{dt^2} + p(t) \frac{dy}{dx} \frac{dx}{dt} + q(t)y = 0.$$

Collecting the terms,

$$\left[ \frac{dx}{dt} \right]^2 \frac{d^2 y}{dx^2} + \left[ \frac{d^2 x}{dt^2} + p(t) \frac{dx}{dt} \right] \frac{dy}{dx} + q(t)y = 0.$$

(c) Assuming  $(dx/dt)^2 = kq(t)$ , and  $q(t) > 0$ , we find that  $dx/dt = \sqrt{kq(t)}$ , which can be integrated. That is,  $x = u(t) = \int \sqrt{kq(t)} dt = \int \sqrt{q(t)} dt$ , since  $k = 1$ .

(d) Let  $k = 1$ . It follows that  $d^2 x/dt^2 + p(t)dx/dt = du/dt + p(t)u(t) = q'/2\sqrt{q} + p\sqrt{q}$ . Hence

$$\left[ \frac{d^2 x}{dt^2} + p(t) \frac{dx}{dt} \right] / \left[ \frac{dx}{dt} \right]^2 = \frac{q'(t) + 2p(t)q(t)}{2[q(t)]^{3/2}}.$$

As long as  $dx/dt \neq 0$ , the differential equation can be expressed as

$$\frac{d^2 y}{dx^2} + \left[ \frac{q'(t) + 2p(t)q(t)}{2[q(t)]^{3/2}} \right] \frac{dy}{dx} + y = 0.$$

For the case  $q(t) < 0$ , write  $q(t) = -[-q(t)]$ , and set  $(dx/dt)^2 = -q(t)$ .

45.  $p(t) = 3t$  and  $q(t) = t^2$ . We have  $x = \int t dt = t^2/2$ . Furthermore,

$$\frac{q'(t) + 2p(t)q(t)}{2[q(t)]^{3/2}} = \frac{1 + 3t^2}{t^2}.$$

The ratio is not constant, and therefore the equation cannot be transformed.

46.  $p(t) = t - 1/t$  and  $q(t) = t^2$ . We have  $x = \int t dt = t^2/2$ . Furthermore,

$$\frac{q'(t) + 2p(t)q(t)}{2[q(t)]^{3/2}} = 1.$$

The ratio is constant, and therefore the equation can be transformed. From Problem 43, the transformed equation is

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0.$$

Based on the methods in this section, the characteristic equation is  $r^2 + r + 1 = 0$ , with roots  $r = (-1 \pm i\sqrt{3})/2$ . The general solution is  $y(x) = c_1 e^{-x/2} \cos \sqrt{3}x/2 + c_2 e^{-x/2} \sin \sqrt{3}x/2$ . Since  $x = t^2/2$ , the solution in the original variable  $t$  is

$$y(t) = e^{-t^2/4} \left[ c_1 \cos(\sqrt{3} t^2/4) + c_2 \sin(\sqrt{3} t^2/4) \right].$$

## 3.4

2. The characteristic equation is  $9r^2 + 6r + 1 = 0$ , with the double root  $r = -1/3$ . The general solution is  $y(t) = c_1 e^{-t/3} + c_2 t e^{-t/3}$ .

3. The characteristic equation is  $4r^2 - 4r - 3 = 0$ , with roots  $r = -1/2, 3/2$ . The general solution is  $y(t) = c_1 e^{-t/2} + c_2 e^{3t/2}$ .

4. The characteristic equation is  $4r^2 + 12r + 9 = 0$ , with double root  $r = -3/2$ . The general solution is  $y(t) = (c_1 + c_2 t)e^{-3t/2}$ .

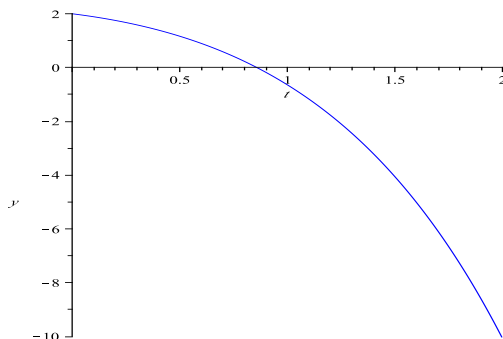
6. The characteristic equation is  $r^2 - 6r + 9 = 0$ , with the double root  $r = 3$ . The general solution is  $y(t) = c_1 e^{3t} + c_2 t e^{3t}$ .

7. The characteristic equation is  $4r^2 + 17r + 4 = 0$ , with roots  $r = -1/4, -4$ . The general solution is  $y(t) = c_1 e^{-t/4} + c_2 e^{-4t}$ .

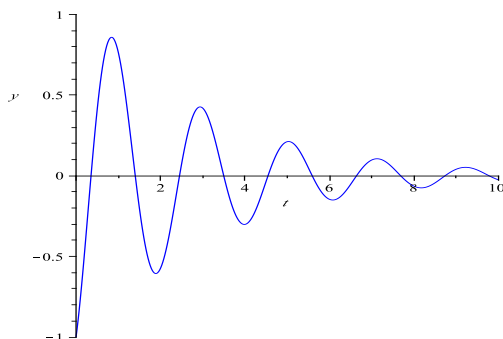
8. The characteristic equation is  $16r^2 + 24r + 9 = 0$ , with double root  $r = -3/4$ . The general solution is  $y(t) = c_1 e^{-3t/4} + c_2 t e^{-3t/4}$ .

10. The characteristic equation is  $2r^2 + 2r + 1 = 0$ . We obtain the complex roots  $r = (-1 \pm i)/2$ . The general solution is  $y(t) = c_1 e^{-t/2} \cos(t/2) + c_2 e^{-t/2} \sin(t/2)$ .

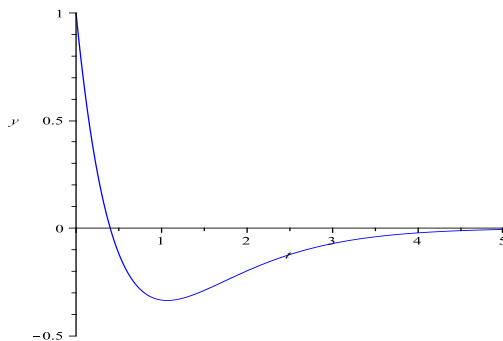
11. The characteristic equation is  $9r^2 - 12r + 4 = 0$ , with the double root  $r = 2/3$ . The general solution is  $y(t) = c_1 e^{2t/3} + c_2 t e^{2t/3}$ . Invoking the first initial condition, it follows that  $c_1 = 2$ . Now  $y'(t) = (4/3 + c_2)e^{2t/3} + 2c_2 t e^{2t/3}/3$ . Invoking the second initial condition,  $4/3 + c_2 = -1$ , or  $c_2 = -7/3$ . Hence we obtain the solution  $y(t) = 2e^{2t/3} - (7/3)te^{2t/3}$ . Since the second term dominates for large  $t$ ,  $y(t) \rightarrow -\infty$ .



13. The characteristic equation is  $9r^2 + 6r + 82 = 0$ . We obtain the complex roots  $r = -1/3 \pm 3i$ . The general solution is  $y(t) = c_1 e^{-t/3} \cos 3t + c_2 e^{-t/3} \sin 3t$ . Based on the first initial condition,  $c_1 = -1$ . Invoking the second initial condition, we conclude that  $1/3 + 3c_2 = 2$ , or  $c_2 = 5/9$ . Hence  $y(t) = -e^{-t/3} \cos 3t + (5/9)e^{-t/3} \sin 3t$ . The solution oscillates with an exponentially decreasing amplitude.



15.(a) The characteristic equation is  $4r^2 + 12r + 9 = 0$ , with double root  $r = -3/2$ . The general solution is  $y(t) = c_1 e^{-3t/2} + c_2 t e^{-3t/2}$ . Invoking the first initial condition, it follows that  $c_1 = 1$ . Now  $y'(t) = (-3/2 + c_2)e^{-3t/2} - (3/2)c_2 t e^{-3t/2}$ . The second initial condition requires that  $-3/2 + c_2 = -4$ , or  $c_2 = -5/2$ . Hence the specific solution is  $y(t) = e^{-3t/2} - (5/2)t e^{-3t/2}$ .



(b) The solution crosses the  $x$ -axis at  $t = 2/5$ .

(c) The solution has a minimum at the point  $(16/15, -5e^{-8/5}/3)$ .

(d) Given that  $y'(0) = b$ , we have  $-3/2 + c_2 = b$ , or  $c_2 = b + 3/2$ . Hence the solution is  $y(t) = e^{-3t/2} + (b + 3/2)t e^{-3t/2}$ . Since the second term dominates, the long-term solution depends on the sign of the coefficient  $b + 3/2$ . The critical value is  $b = -3/2$ .

16. The characteristic roots are  $r_1 = r_2 = 1/2$ . Hence the general solution is given by  $y(t) = c_1 e^{t/2} + c_2 t e^{t/2}$ . Invoking the initial conditions, we require that  $c_1 = 2$ , and that  $1 + c_2 = b$ . The specific solution is  $y(t) = 2e^{t/2} + (b - 1)t e^{t/2}$ . Since the second term dominates, the long-term solution depends on the sign of the coefficient  $b - 1$ . The critical value is  $b = 1$ .

18.(a) The characteristic roots are  $r_1 = r_2 = -2/3$ . Therefore the general solution is given by  $y(t) = c_1 e^{-2t/3} + c_2 t e^{-2t/3}$ . Invoking the initial conditions, we require that  $c_1 = a$ , and that  $-2a/3 + c_2 = -1$ . After solving for the coefficients, the specific solution is  $y(t) = a e^{-2t/3} + (2a/3 - 1)t e^{-2t/3}$ .

(b) Since the second term dominates, the long-term solution depends on the sign of the coefficient  $2a/3 - 1$ . The critical value is  $a = 3/2$ .

20.(a) The characteristic equation is  $r^2 + 2a r + a^2 = (r + a)^2 = 0$ .

(b) With  $p(t) = 2a$ , Abel's Formula becomes  $W(y_1, y_2) = c e^{-\int 2a dt} = c e^{-2at}$ .

(c)  $y_1(t) = e^{-at}$  is a solution. From part (b), with  $c = 1$ ,  $e^{-at} y_2'(t) + a e^{-at} y_2(t) = e^{-2at}$ , which can be written as  $(e^{at} y_2(t))' = 1$ , resulting in  $e^{at} y_2(t) = t$ .

22.(a) If the characteristic equation  $ar^2 + br + c$  has equal roots  $r_1$ , then  $ar_1^2 + br_1 + c = a(r - r_1)^2 = 0$ . Then clearly  $L[e^{rt}] = (ar^2 + br + c)e^{rt} = a(r - r_1)^2 e^{rt}$ . This gives immediately that  $L[e^{r_1 t}] = 0$ .

(b) Differentiating the identity in part (a) with respect to  $r$  we get  $(2ar + b)e^{rt} + (ar^2 + br + c)te^{rt} = 2a(r - r_1)e^{rt} + a(r - r_1)^2 te^{rt}$ . Again, this gives  $L[te^{r_1 t}] = 0$ .

23. Set  $y_2(t) = t^2 v(t)$ . Substitution into the differential equation results in

$$t^2(t^2 v'' + 4tv' + 2v) - 4t(t^2 v' + 2tv) + 6t^2 v = 0.$$

After collecting terms, we end up with  $t^4 v'' = 0$ . Hence  $v(t) = c_1 + c_2 t$ , and thus  $y_2(t) = c_1 t^2 + c_2 t^3$ . Setting  $c_1 = 0$  and  $c_2 = 1$ , we obtain  $y_2(t) = t^3$ .

24. Set  $y_2(t) = t v(t)$ . Substitution into the differential equation results in

$$t^2(tv'' + 2v') + 2t(tv' + v) - 2tv = 0.$$

After collecting terms, we end up with  $t^3 v'' + 4t^2 v' = 0$ . This equation is linear in the variable  $w = v'$ . It follows that  $v'(t) = ct^{-4}$ , and  $v(t) = c_1 t^{-3} + c_2$ . Thus



$y_2(t) = c_1 t^{-2} + c_2 t$ . Setting  $c_1 = 1$  and  $c_2 = 0$ , we obtain  $y_2(t) = t^{-2}$ .

26. Set  $y_2(t) = tv(t)$ . Substitution into the differential equation results in  $v'' - v' = 0$ . This equation is linear in the variable  $w = v'$ . It follows that  $v'(t) = c_1 e^t$ , and  $v(t) = c_1 e^t + c_2$ . Thus  $y_2(t) = c_1 t e^t + c_2 t$ . Setting  $c_1 = 1$  and  $c_2 = 0$ , we obtain  $y_2(t) = t e^t$ .

28. Set  $y_2(x) = e^x v(x)$ . Substitution into the differential equation results in  $v'' + (x-2)/(x-1)v' = 0$ . This equation is linear in the variable  $w = v'$ . An integrating factor is  $\mu = e^{\int (x-2)/(x-1) dx} = e^x/(x-1)$ . Rewrite the equation as  $[e^x v'/(x-1)]' = 0$ , from which it follows that  $v'(x) = c(x-1)e^{-x}$ . Hence  $v(x) = c_1 x e^{-x} + c_2$  and  $y_2(x) = c_1 x + c_2 e^x$ . Setting  $c_1 = 1$  and  $c_2 = 0$ , we obtain  $y_2(x) = x$ .

29. Set  $y_2(x) = y_1(x)v(x)$ , in which  $y_1(x) = x^{1/4}e^{2\sqrt{x}}$ . It can be verified that  $y_1$  is a solution of the differential equation, that is,  $x^2 y_1'' - (x - 0.1875)y_1 = 0$ . Substitution of the given form of  $y_2$  results in the differential equation  $2x^{9/4}v'' + (4x^{7/4} + x^{5/4})v' = 0$ . This equation is linear in the variable  $w = v'$ . An integrating factor is  $\mu = e^{\int [2x^{-1/2} + 1/(2x)] dx} = \sqrt{x} e^{4\sqrt{x}}$ . Rewrite the equation as  $[\sqrt{x} e^{4\sqrt{x}} v']' = 0$ , from which it follows that  $v'(x) = c e^{-4\sqrt{x}}/\sqrt{x}$ . Integrating,  $v(x) = c_1 e^{-4\sqrt{x}} + c_2$  and as a result,  $y_2(x) = c_1 x^{1/4} e^{-2\sqrt{x}} + c_2 x^{1/4} e^{2\sqrt{x}}$ . Setting  $c_1 = 1$  and  $c_2 = 0$ , we obtain  $y_2(x) = x^{1/4} e^{-2\sqrt{x}}$ .

31. Direct substitution verifies that  $y_1(t) = e^{-\delta x^2/2}$  is a solution of the differential equation. Now set  $y_2(x) = y_1(x)v(x)$ . Substitution of  $y_2$  into the equation results in  $v'' - \delta x v' = 0$ . This equation is linear in the variable  $w = v'$ . An integrating factor is  $\mu = e^{-\delta x^2/2}$ . Rewrite the equation as  $[e^{-\delta x^2/2} v']' = 0$ , from which it follows that  $v'(x) = c_1 e^{\delta x^2/2}$ . Integrating, we obtain

$$v(x) = c_1 \int_0^x e^{\delta u^2/2} du + v(0).$$

Hence

$$y_2(x) = c_1 e^{-\delta x^2/2} \int_0^x e^{\delta u^2/2} du + c_2 e^{-\delta x^2/2}.$$

Setting  $c_2 = 0$ , we obtain a second independent solution.

33. After writing the differential equation in standard form, we have  $p(t) = 3/t$ . Based on Abel's identity,  $W(y_1, y_2) = c_1 e^{-\int 3/t dt} = c_1 t^{-3}$ . As shown in Problem 32, two solutions of a second order linear equation satisfy  $(y_2/y_1)' = W(y_1, y_2)/y_1^2$ . In the given problem,  $y_1(t) = t^{-1}$ . Hence  $(t y_2)' = c_1 t^{-1}$ . Integrating both sides of the equation,  $y_2(t) = c_1 t^{-1} \ln t + c_2 t^{-1}$ . Setting  $c_1 = 1$  and  $c_2 = 0$  we obtain  $y_2(t) = t^{-1} \ln t$ .

35. After writing the differential equation in standard form, we have  $p(x) = -x/(x-1)$ . Based on Abel's identity,  $W(y_1, y_2) = c e^{\int x/(x-1) dx} = c e^x(x-1)$ . Two solutions of a second order linear equation satisfy  $(y_2/y_1)' = W(y_1, y_2)/y_1^2$ . In the given problem,  $y_1(x) = e^x$ . Hence  $(e^{-x} y_2)' = c e^{-x}(x-1)$ . Integrating both

sides of the equation,  $y_2(x) = c_1x + c_2e^x$ . Setting  $c_1 = 1$  and  $c_2 = 0$ , we obtain  $y_2(x) = x$ .

36. Write the differential equation in standard form to find  $p(x) = 1/x$ . Based on Abel's identity,  $W(y_1, y_2) = ce^{-\int 1/x dx} = cx^{-1}$ . Two solutions of a second order linear differential equation satisfy  $(y_2/y_1)' = W(y_1, y_2)/y_1^2$ . In the given problem,  $y_1(x) = x^{-1/2} \sin x$ . Hence

$$\left(\frac{\sqrt{x}}{\sin x} y_2\right)' = c \frac{1}{\sin^2 x}.$$

Integrating both sides of the equation,  $y_2(x) = c_1x^{-1/2} \cos x + c_2x^{-1/2} \sin x$ . Setting  $c_1 = 1$  and  $c_2 = 0$ , we obtain  $y_2(x) = x^{-1/2} \cos x$ .

38.(a) The characteristic equation is  $ar^2 + c = 0$ . If  $a, c > 0$ , then the roots are  $r = \pm i\sqrt{c/a}$ . The general solution is

$$y(t) = c_1 \cos \sqrt{\frac{c}{a}} t + c_2 \sin \sqrt{\frac{c}{a}} t,$$

which is bounded.

(b) The characteristic equation is  $ar^2 + br = 0$ . The roots are  $r = 0, -b/a$ , and hence the general solution is  $y(t) = c_1 + c_2e^{-bt/a}$ . Clearly,  $y(t) \rightarrow c_1$ . With the given initial conditions,  $c_1 = y_0 + (a/b)y'_0$ .

39. Note that  $2 \cos t \sin t = \sin 2t$ . Then  $1 - k \cos t \sin t = 1 - (k/2) \sin 2t$ . Now if  $0 < k < 2$ , then  $(k/2) \sin 2t < |\sin 2t|$  and  $-(k/2) \sin 2t > -|\sin 2t|$ . Hence

$$1 - k \cos t \sin t = 1 - \frac{k}{2} \sin 2t > 1 - |\sin 2t| \geq 0.$$

40. The equation transforms into  $y'' - 4y' + 4y = 0$ . We obtain a double root  $r = 2$ . The solution is  $y = c_1e^{2x} + c_2xe^{2x} = c_1e^{2 \ln t} + c_2 \ln t e^{2 \ln t} = c_1t^2 + c_2t^2 \ln t$ .

42. The equation transforms into  $y'' - 7y'/2 + 5y/2 = 0$ . The characteristic roots are  $r = 1, 5/2$ , so the solution is  $y = c_1e^x + c_2e^{5x/2} = c_1e^{\ln t} + c_2e^{5 \ln t/2} = c_1t + c_2t^{5/2}$ .

43. The equation transforms into  $y'' + 2y' + y = 0$ . We get a double root  $r = -1$ . The solution is  $y = c_1e^{-x} + c_2xe^{-x} = c_1e^{-\ln t} + c_2 \ln t e^{-\ln t} = c_1t^{-1} + c_2t^{-1} \ln t$ .

44. The equation transforms into  $y'' - 3y' + 9y/4 = 0$ . We obtain the double root  $r = 3/2$ . The solution is  $y = c_1e^{3x/2} + c_2xe^{3x/2} = c_1e^{3 \ln t/2} + c_2 \ln t e^{3 \ln t/2} = c_1t^{3/2} + c_2t^{3/2} \ln t$ .

## 3.5

2. The characteristic equation for the homogeneous problem is  $r^2 + 2r + 5 = 0$ , with complex roots  $r = -1 \pm 2i$ . Hence  $y_c(t) = c_1 e^{-t} \cos 2t + c_2 e^{-t} \sin 2t$ . Since the function  $g(t) = 3 \sin 2t$  is not proportional to the solutions of the homogeneous equation, set  $Y = A \cos 2t + B \sin 2t$ . Substitution into the given differential equation, and comparing the coefficients, results in the system of equations  $B - 4A = 3$  and  $A + 4B = 0$ . Hence  $Y = -(12/17) \cos 2t + (3/17) \sin 2t$ . The general solution is  $y(t) = y_c(t) + Y$ .

3. The characteristic equation for the homogeneous problem is  $r^2 - r - 2 = 0$ , with roots  $r = -1, 2$ . Hence  $y_c(t) = c_1 e^{-t} + c_2 e^{2t}$ . Set  $Y = At^2 + Bt + C$ . Substitution into the given differential equation, and comparing the coefficients, results in the system of equations  $-2A = 4$ ,  $-2A - 2B = -2$  and  $2A - B - 2C = 0$ . Hence  $Y = -2t^2 + 3t - 7/2$ . The general solution is  $y(t) = y_c(t) + Y$ .

4. The characteristic equation for the homogeneous problem is  $r^2 + r - 6 = 0$ , with roots  $r = -3, 2$ . Hence  $y_c(t) = c_1 e^{-3t} + c_2 e^{2t}$ . Set  $Y = Ae^{3t} + Be^{-2t}$ . Substitution into the given differential equation, and comparing the coefficients, results in the system of equations  $6A = 12$  and  $-4B = 12$ . Hence  $Y = 2e^{3t} - 3e^{-2t}$ . The general solution is  $y(t) = y_c(t) + Y$ .

5. The characteristic equation for the homogeneous problem is  $r^2 - 2r - 3 = 0$ , with roots  $r = -1, 3$ . Hence  $y_c(t) = c_1 e^{-t} + c_2 e^{3t}$ . Note that the assignment  $Y = Ate^{-t}$  is not sufficient to match the coefficients. Try  $Y = Ate^{-t} + Bt^2 e^{-t}$ . Substitution into the differential equation, and comparing the coefficients, results in the system of equations  $-4A + 2B = 0$  and  $-8B = -3$ . This implies that  $Y = (3/16)te^{-t} + (3/8)t^2 e^{-t}$ . The general solution is  $y(t) = y_c(t) + Y$ .

7. The characteristic equation for the homogeneous problem is  $r^2 + 9 = 0$ , with complex roots  $r = \pm 3i$ . Hence  $y_c(t) = c_1 \cos 3t + c_2 \sin 3t$ . To simplify the analysis, set  $g_1(t) = 6$  and  $g_2(t) = t^2 e^{3t}$ . By inspection, we have  $Y_1 = 2/3$ . Based on the form of  $g_2$ , set  $Y_2 = Ae^{3t} + Bte^{3t} + Ct^2 e^{3t}$ . Substitution into the differential equation, and comparing the coefficients, results in the system of equations  $18A + 6B + 2C = 0$ ,  $18B + 12C = 0$ , and  $18C = 1$ . Hence

$$Y_2 = \frac{1}{162}e^{3t} - \frac{1}{27}te^{3t} + \frac{1}{18}t^2 e^{3t}.$$

The general solution is  $y(t) = y_c(t) + Y_1 + Y_2$ .

9. The characteristic equation for the homogeneous problem is  $2r^2 + 3r + 1 = 0$ , with roots  $r = -1, -1/2$ . Hence  $y_c(t) = c_1 e^{-t} + c_2 e^{-t/2}$ . To simplify the analysis, set  $g_1(t) = t^2$  and  $g_2(t) = 3 \sin t$ . Based on the form of  $g_1$ , set  $Y_1 = A + Bt + Ct^2$ . Substitution into the differential equation, and comparing the coefficients, results in the system of equations  $A + 3B + 4C = 0$ ,  $B + 6C = 0$ , and  $C = 1$ . Hence we obtain  $Y_1 = 14 - 6t + t^2$ . On the other hand, set  $Y_2 = D \cos t + E \sin t$ . After substitution into the ODE, we find that  $D = -9/10$  and  $E = -3/10$ . The general solution is  $y(t) = y_c(t) + Y_1 + Y_2$ .

11. The characteristic equation for the homogeneous problem is  $r^2 + \omega_0^2 = 0$ , with complex roots  $r = \pm \omega_0 i$ . Hence  $y_c(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$ . Since  $\omega \neq \omega_0$ , set  $Y = A \cos \omega t + B \sin \omega t$ . Substitution into the ODE and comparing the coefficients results in the system of equations  $(\omega_0^2 - \omega^2)A = 1$  and  $(\omega_0^2 - \omega^2)B = 0$ . Hence

$$Y = \frac{1}{\omega_0^2 - \omega^2} \cos \omega t.$$

The general solution is  $y(t) = y_c(t) + Y$ .

12. From Problem 11,  $y_c(t)$  is known. Since  $\cos \omega_0 t$  is a solution of the homogeneous problem, set  $Y = At \cos \omega_0 t + Bt \sin \omega_0 t$ . Substitution into the given ODE and comparing the coefficients results in  $A = 0$  and  $B = 1/2\omega_0$ . Hence the general solution is  $y(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + t \sin \omega_0 t / (2\omega_0)$ .

14. The characteristic equation for the homogeneous problem is  $r^2 - r - 2 = 0$ , with roots  $r = -1, 2$ . Hence  $y_c(t) = c_1 e^{-t} + c_2 e^{2t}$ . Based on the form of the right hand side, that is,  $\cosh(2t) = (e^{2t} + e^{-2t})/2$ , set  $Y = At e^{2t} + B e^{-2t}$ . Substitution into the given ODE and comparing the coefficients results in  $A = 1/6$  and  $B = 1/8$ . Hence the general solution is  $y(t) = c_1 e^{-t} + c_2 e^{2t} + t e^{2t}/6 + e^{-2t}/8$ .

16. The characteristic equation for the homogeneous problem is  $r^2 + 4 = 0$ , with roots  $r = \pm 2i$ . Hence  $y_c(t) = c_1 \cos 2t + c_2 \sin 2t$ . Set  $Y_1 = A + Bt + Ct^2$ . Comparing the coefficients of the respective terms, we find that  $A = -1/8$ ,  $B = 0$ ,  $C = 1/4$ . Now set  $Y_2 = D e^t$ , and obtain  $D = 3/5$ . Hence the general solution is  $y(t) = c_1 \cos 2t + c_2 \sin 2t - 1/8 + t^2/4 + 3e^t/5$ . Invoking the initial conditions, we require that  $19/40 + c_1 = 0$  and  $3/5 + 2c_2 = 2$ . Hence  $c_1 = -19/40$  and  $c_2 = 7/10$ .

17. The characteristic equation for the homogeneous problem is  $r^2 - 2r + 1 = 0$ , with a double root  $r = 1$ . Hence  $y_c(t) = c_1 e^t + c_2 t e^t$ . Consider  $g_1(t) = t e^t$ . Note that  $g_1$  is a solution of the homogeneous problem. Set  $Y_1 = At^2 e^t + Bt^3 e^t$  (the first term is not sufficient for a match). Upon substitution, we obtain  $Y_1 = t^3 e^t/6$ . By inspection,  $Y_2 = 4$ . Hence the general solution is  $y(t) = c_1 e^t + c_2 t e^t + t^3 e^t/6 + 4$ . Invoking the initial conditions, we require that  $c_1 + 4 = 1$  and  $c_1 + c_2 = 1$ . Hence  $c_1 = -3$  and  $c_2 = 4$ .

19. The characteristic equation for the homogeneous problem is  $r^2 + 4 = 0$ , with roots  $r = \pm 2i$ . Hence  $y_c(t) = c_1 \cos 2t + c_2 \sin 2t$ . Since the function  $\sin 2t$  is a solution of the homogeneous problem, set  $Y = At \cos 2t + Bt \sin 2t$ . Upon substitution, we obtain  $Y = -3t \cos 2t/4$ . Hence the general solution is  $y(t) = c_1 \cos 2t + c_2 \sin 2t - 3t \cos 2t/4$ . Invoking the initial conditions, we require that  $c_1 = 2$  and  $2c_2 - (3/4) = -1$ . Hence  $c_1 = 2$  and  $c_2 = -1/8$ .

20. The characteristic equation for the homogeneous problem is  $r^2 + 2r + 5 = 0$ , with complex roots  $r = -1 \pm 2i$ . Hence  $y_c(t) = c_1 e^{-t} \cos 2t + c_2 e^{-t} \sin 2t$ . Based on the form of  $g(t)$ , set  $Y = At e^{-t} \cos 2t + Bt e^{-t} \sin 2t$ . After comparing coefficients, we obtain  $Y = t e^{-t} \sin 2t$ . Hence the general solution is  $y(t) = c_1 e^{-t} \cos 2t + c_2 e^{-t} \sin 2t + t e^{-t} \sin 2t$ . Invoking the initial conditions, we require

that  $c_1 = 1$  and  $-c_1 + 2c_2 = 0$ . Hence  $c_1 = 1$  and  $c_2 = 1/2$ .

22.(a) The characteristic equation for the homogeneous problem is  $r^2 + 1 = 0$ , with complex roots  $r = \pm i$ . Hence  $y_c(t) = c_1 \cos t + c_2 \sin t$ . Let  $g_1(t) = t \sin t$  and  $g_2(t) = t$ . By inspection, it is easy to see that  $Y_2(t) = t$ . Based on the form of  $g_1(t)$ , set  $Y_1(t) = At \cos t + Bt \sin t + Ct^2 \cos t + Dt^2 \sin t$ .

(b) Substitution into the equation and comparing the coefficients results in  $A = 0$ ,  $B = 1/4$ ,  $C = -1/4$ , and  $D = 0$ . Hence  $Y(t) = t + t \sin t / 4 - t^2 \cos t / 4$ .

23.(a) The characteristic equation for the homogeneous problem is  $r^2 - 5r + 6 = 0$ , with roots  $r = 2, 3$ . Hence  $y_c(t) = c_1 e^{2t} + c_2 e^{3t}$ . Consider  $g_1(t) = e^{2t}(3t + 4) \sin t$ , and  $g_2(t) = e^t \cos 2t$ . Based on the form of these functions on the right hand side of the ODE, set  $Y_2(t) = e^t(A_1 \cos 2t + A_2 \sin 2t)$  and  $Y_1(t) = (B_1 + B_2 t)e^{2t} \sin t + (C_1 + C_2 t)e^{2t} \cos t$ .

(b) Substitution into the equation and comparing the coefficients results in

$$Y(t) = -\frac{1}{20}(e^t \cos 2t + 3e^t \sin 2t) + \frac{3}{2}te^{2t}(\cos t - \sin t) + e^{2t}\left(\frac{1}{2} \cos t - 5 \sin t\right).$$

25.(a) We obtain the double characteristic root  $r = 2$ . Hence  $y_c(t) = c_1 e^{2t} + c_2 t e^{2t}$ . Consider the functions  $g_1(t) = 2t^2$ ,  $g_2(t) = 4te^{2t}$ , and  $g_3(t) = t \sin 2t$ . The corresponding forms of the respective parts of the particular solution are  $Y_1(t) = A_0 + A_1 t + A_2 t^2$ ,  $Y_2(t) = e^{2t}(B_2 t^2 + B_3 t^3)$ , and  $Y_3(t) = t(C_1 \cos 2t + C_2 \sin 2t) + (D_1 \cos 2t + D_2 \sin 2t)$ .

(b) Substitution into the equation and comparing the coefficients results in

$$Y(t) = \frac{1}{4}(3 + 4t + 2t^2) + \frac{2}{3}t^3 e^{2t} + \frac{1}{8}t \cos 2t + \frac{1}{16}(\cos 2t - \sin 2t).$$

26.(a) The homogeneous solution is  $y_c(t) = c_1 \cos 2t + c_2 \sin 2t$ . Since  $\cos 2t$  and  $\sin 2t$  are both solutions of the homogeneous equation, set

$$Y(t) = t(A_0 + A_1 t + A_2 t^2) \cos 2t + t(B_0 + B_1 t + B_2 t^2) \sin 2t.$$

(b) Substitution into the equation and comparing the coefficients results in

$$Y(t) = \left(\frac{13}{32}t - \frac{1}{12}t^3\right) \cos 2t + \frac{1}{16}(28t + 13t^2) \sin 2t.$$

27.(a) The homogeneous solution is  $y_c(t) = c_1 e^{-t} + c_2 t e^{-2t}$ . None of the functions on the right hand side are solutions of the homogenous equation. In order to include all possible combinations of the derivatives, consider

$$Y(t) = e^t(A_0 + A_1 t + A_2 t^2) \cos 2t + e^t(B_0 + B_1 t + B_2 t^2) \sin 2t + e^{-t}(C_1 \cos t + C_2 \sin t) + D e^t.$$

(b) Substitution into the differential equation and comparing the coefficients results in

$$Y(t) = e^t(A_0 + A_1t + A_2t^2) \cos 2t + e^t(B_0 + B_1t + B_2t^2) \sin 2t \\ + e^{-t}\left(-\frac{3}{2} \cos t + \frac{3}{2} \sin t\right) + 2e^t/3,$$

in which  $A_0 = -4105/35152$ ,  $A_1 = 73/676$ ,  $A_2 = -5/52$ ,  $B_0 = -1233/35152$ ,  $B_1 = 10/169$ ,  $B_2 = 1/52$ .

28.(a) The homogeneous solution is  $y_c(t) = c_1e^{-t} \cos 2t + c_2e^{-t} \sin 2t$ . None of the terms on the right hand side are solutions of the homogenous equation. In order to include the appropriate combinations of derivatives, consider

$$Y(t) = e^{-t}(A_1t + A_2t^2) \cos 2t + e^{-t}(B_1t + B_2t^2) \sin 2t + \\ + e^{-2t}(C_0 + C_1t) \cos 2t + e^{-2t}(D_0 + D_1t) \sin 2t.$$

(b) Substitution into the differential equation and comparing the coefficients results in

$$Y(t) = \frac{3}{16}te^{-t} \cos 2t + \frac{3}{8}t^2e^{-t} \sin 2t \\ - \frac{1}{25}e^{-2t}(7 + 10t) \cos 2t + \frac{1}{25}e^{-2t}(1 + 5t) \sin 2t.$$

30. The homogeneous solution is  $y_c(t) = c_1 \cos \lambda t + c_2 \sin \lambda t$ . Since the differential operator does not contain a first derivative (and  $\lambda \neq m\pi$ ), we can set

$$Y(t) = \sum_{m=1}^N C_m \sin m\pi t.$$

Substitution into the differential equation yields

$$- \sum_{m=1}^N m^2\pi^2 C_m \sin m\pi t + \lambda^2 \sum_{m=1}^N C_m \sin m\pi t = \sum_{m=1}^N a_m \sin m\pi t.$$

Equating coefficients of the individual terms, we obtain

$$C_m = \frac{a_m}{\lambda^2 - m^2\pi^2}, \quad m = 1, 2 \dots N.$$

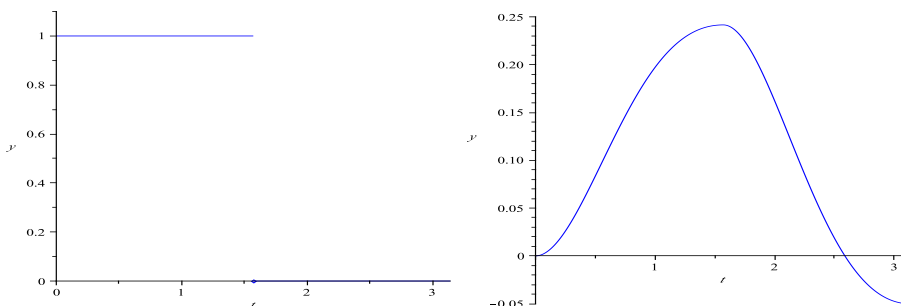
32. The homogeneous solution is  $y_c(t) = c_1e^{-t} \cos 2t + c_2e^{-t} \sin 2t$ . The input function is independent of the homogeneous solutions, on any interval. Since the right hand side is piecewise constant, it follows by inspection that

$$Y(t) = \begin{cases} 1/5, & 0 \leq t \leq \pi/2 \\ 0, & t > \pi/2 \end{cases}.$$

For  $0 \leq t \leq \pi/2$ , the general solution is  $y(t) = c_1 e^{-t} \cos 2t + c_2 e^{-t} \sin 2t + 1/5$ . Invoking the initial conditions  $y(0) = y'(0) = 0$ , we require that  $c_1 = -1/5$ , and that  $c_2 = -1/10$ . Hence

$$y(t) = \frac{1}{5} - \frac{1}{10}(2e^{-t} \cos 2t + e^{-t} \sin 2t)$$

on the interval  $0 \leq t \leq \pi/2$ . We now have the values  $y(\pi/2) = (1 + e^{-\pi/2})/5$ , and  $y'(\pi/2) = 0$ . For  $t > \pi/2$ , the general solution is  $y(t) = d_1 e^{-t} \cos 2t + d_2 e^{-t} \sin 2t$ . It follows that  $y(\pi/2) = -e^{-\pi/2} d_1$  and  $y'(\pi/2) = e^{-\pi/2} d_1 - 2e^{-\pi/2} d_2$ . Since the solution is continuously differentiable, we require that  $-e^{-\pi/2} d_1 = (1 + e^{-\pi/2})/5$  and  $e^{-\pi/2} d_1 - 2e^{-\pi/2} d_2 = 0$ . Solving for the coefficients,  $d_1 = 2d_2 = -(e^{\pi/2} + 1)/5$ .



34. Since  $a, b, c > 0$ , the roots of the characteristic equation have negative real parts. That is,  $r = \alpha \pm \beta i$ , where  $\alpha < 0$ . Hence the homogeneous solution is

$$y_c(t) = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t.$$

If  $g(t) = d$ , then the general solution is

$$y(t) = d/c + c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t.$$

Since  $\alpha < 0$ ,  $y(t) \rightarrow d/c$  as  $t \rightarrow \infty$ . If  $c = 0$ , then the characteristic roots are  $r = 0$  and  $r = -b/a$ . The ODE becomes  $ay'' + by' = d$ . Integrating both sides, we find that  $ay' + by = dt + c_1$ . The general solution can be expressed as

$$y(t) = dt/b + c_1 + c_2 e^{-bt/a}.$$

In this case, the solution grows without bound. If  $b = 0$ , also, then the differential equation can be written as  $y'' = d/a$ , which has general solution  $y(t) = dt^2/2a + c_1 + c_2$ . Hence the assertion is true only if the coefficients are positive.

35. (a) Since  $D$  is a linear operator,  $D^2 y + bDy + cy = D^2 y - (r_1 + r_2)Dy + r_1 r_2 y = D^2 y - r_2 Dy - r_1 Dy + r_1 r_2 y = D(Dy - r_2 y) - r_1(Dy - r_2 y) = (D - r_1)(D - r_2)y$ .

(b) Let  $u = (D - r_2)y$ . Then the ODE (i) can be written as  $(D - r_1)u = g(t)$ , that is,  $u' - r_1 u = g(t)$ . The latter is a linear first order equation in  $u$ . Its general solution is

$$u(t) = e^{r_1 t} \int_{t_0}^t e^{-r_1 \tau} g(\tau) d\tau + c_1 e^{r_1 t}.$$

From above, we have  $y' - r_2y = u(t)$ . This equation is also a first order ODE. Hence the general solution of the original second order equation is

$$y(t) = e^{r_2t} \int_{t_0}^t e^{-r_2\tau} u(\tau) d\tau + c_2 e^{r_2t}.$$

Note that the solution  $y(t)$  contains two arbitrary constants.

37. Note that  $(2D^2 + 3D + 1)y = (2D + 1)(D + 1)y$ . Let  $u = (D + 1)y$ , and solve the ODE  $2u' + u = t^2 + 3 \sin t$ . This equation is a linear first order ODE, with solution

$$\begin{aligned} u(t) &= e^{-t/2} \int_{t_0}^t e^{\tau/2} \left[ \tau^2/2 + \frac{3}{2} \sin \tau \right] d\tau + c e^{-t/2} = \\ &= t^2 - 4t + 8 - \frac{6}{5} \cos t + \frac{3}{5} \sin t + c e^{-t/2}. \end{aligned}$$

Now consider the ODE  $y' + y = u(t)$ . The general solution of this first order ODE is

$$y(t) = e^{-t} \int_{t_0}^t e^{\tau} u(\tau) d\tau + c_2 e^{-t},$$

in which  $u(t)$  is given above. Substituting for  $u(t)$  and performing the integration,

$$y(t) = t^2 - 6t + 14 - \frac{9}{10} \cos t - \frac{3}{10} \sin t + c_1 e^{-t/2} + c_2 e^{-t}.$$

38. We have  $(D^2 + 2D + 1)y = (D + 1)(D + 1)y$ . Let  $u = (D + 1)y$ , and consider the ODE  $u' + u = 2e^{-t}$ . The general solution is  $u(t) = 2te^{-t} + ce^{-t}$ . We therefore have the first order equation  $u' + u = 2te^{-t} + ce^{-t}$ . The general solution of the latter differential equation is

$$y(t) = e^{-t} \int_{t_0}^t [2\tau + c_1] d\tau + c_2 e^{-t} = e^{-t}(t^2 + c_1t + c_2).$$

39. We have  $(D^2 + 2D)y = D(D + 2)y$ . Let  $u = (D + 2)y$ , and consider the equation  $u' = 3 + 4 \sin 2t$ . Direct integration results in  $u(t) = 3t - 2 \cos 2t + c$ . The problem is reduced to solving the ODE  $y' + 2y = 3t - 2 \cos 2t + c$ . The general solution of this first order differential equation is

$$\begin{aligned} y(t) &= e^{-2t} \int_{t_0}^t e^{2\tau} [3\tau - 2 \cos 2\tau + c] d\tau + c_2 e^{-2t} = \\ &= \frac{3}{2}t - \frac{1}{2}(\cos 2t + \sin 2t) + c_1 + c_2 e^{-2t}. \end{aligned}$$

## 3.6

1. The solution of the homogeneous equation is  $y_c(t) = c_1 e^{2t} + c_2 e^{3t}$ . The functions  $y_1(t) = e^{2t}$  and  $y_2(t) = e^{3t}$  form a fundamental set of solutions. The Wronskian



of these functions is  $W(y_1, y_2) = e^{5t}$ . Using the method of variation of parameters, the particular solution is given by  $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ , in which

$$u_1(t) = - \int \frac{e^{3t}(2e^t)}{W(t)} dt = 2e^{-t} \quad \text{and} \quad u_2(t) = \int \frac{e^{2t}(2e^t)}{W(t)} dt = -e^{-2t}.$$

Hence the particular solution is  $Y(t) = 2e^t - e^t = e^t$ .

3. The solution of the homogeneous equation is  $y_c(t) = c_1e^{-t} + c_2te^{-t}$ . The functions  $y_1(t) = e^{-t}$  and  $y_2(t) = te^{-t}$  form a fundamental set of solutions. The Wronskian of these functions is  $W(y_1, y_2) = e^{-2t}$ . Using the method of variation of parameters, the particular solution is given by  $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ , in which

$$u_1(t) = - \int \frac{te^{-t}(3e^{-t})}{W(t)} dt = -3t^2/2 \quad \text{and} \quad u_2(t) = \int \frac{e^{-t}(3e^{-t})}{W(t)} dt = 3t.$$

Hence the particular solution is  $Y(t) = -3t^2e^{-t}/2 + 3t^2e^{-t} = 3t^2e^{-t}/2$ .

4. The functions  $y_1(t) = e^{t/2}$  and  $y_2(t) = te^{t/2}$  form a fundamental set of solutions. The Wronskian of these functions is  $W(y_1, y_2) = e^t$ . First write the equation in standard form, so that  $g(t) = 4e^{t/2}$ . Using the method of variation of parameters, the particular solution is given by  $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ , in which

$$u_1(t) = - \int \frac{te^{t/2}(4e^{t/2})}{W(t)} dt = -2t^2 \quad \text{and} \quad u_2(t) = \int \frac{e^{t/2}(4e^{t/2})}{W(t)} dt = 4t.$$

Hence the particular solution is  $Y(t) = -2t^2e^{t/2} + 4t^2e^{t/2} = 2t^2e^{t/2}$ .

6. The solution of the homogeneous equation is  $y_c(t) = c_1 \cos 3t + c_2 \sin 3t$ . The two functions  $y_1(t) = \cos 3t$  and  $y_2(t) = \sin 3t$  form a fundamental set of solutions, with  $W(y_1, y_2) = 3$ . The particular solution is given by  $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ , in which

$$u_1(t) = - \int \frac{\sin 3t(9 \sec^2 3t)}{W(t)} dt = -\csc 3t$$

$$u_2(t) = \int \frac{\cos 3t(9 \sec^2 3t)}{W(t)} dt = \ln(\sec 3t + \tan 3t),$$

since  $0 < t < \pi/6$ . Hence  $Y(t) = -1 + (\sin 3t) \ln(\sec 3t + \tan 3t)$ . The general solution is given by

$$y(t) = c_1 \cos 3t + c_2 \sin 3t + (\sin 3t) \ln(\sec 3t + \tan 3t) - 1.$$

7. The functions  $y_1(t) = e^{-2t}$  and  $y_2(t) = te^{-2t}$  form a fundamental set of solutions. The Wronskian of these functions is  $W(y_1, y_2) = e^{-4t}$ . The particular solution is given by  $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ , in which

$$u_1(t) = - \int \frac{te^{-2t}(t^{-2}e^{-2t})}{W(t)} dt = -\ln t \quad \text{and} \quad u_2(t) = \int \frac{e^{-2t}(t^{-2}e^{-2t})}{W(t)} dt = -1/t.$$

Hence the particular solution is  $Y(t) = -e^{-2t} \ln t - e^{-2t}$ . Since the second term is a solution of the homogeneous equation, the general solution is given by

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t} - e^{-2t} \ln t.$$

8. The solution of the homogeneous equation is  $y_c(t) = c_1 \cos 2t + c_2 \sin 2t$ . The two functions  $y_1(t) = \cos 2t$  and  $y_2(t) = \sin 2t$  form a fundamental set of solutions, with  $W(y_1, y_2) = 2$ . The particular solution is given by  $Y(t) = u_1(t) y_1(t) + u_2(t) y_2(t)$ , in which

$$u_1(t) = - \int \frac{\sin 2t(3 \csc 2t)}{W(t)} dt = -3t/2$$

$$u_2(t) = \int \frac{\cos 2t(3 \csc 2t)}{W(t)} dt = \frac{3}{4} \ln(\sin 2t),$$

since  $0 < t < \pi/2$ . Hence  $Y(t) = -(3/2)t \cos 2t + (3/4)(\sin 2t) \ln(\sin 2t)$ . The general solution is given by

$$y(t) = c_1 \cos 2t + c_2 \sin 2t - \frac{3}{2}t \cos 2t + \frac{3}{4}(\sin 2t) \ln(\sin 2t).$$

9. The functions  $y_1(t) = \cos(t/2)$  and  $y_2(t) = \sin(t/2)$  form a fundamental set of solutions. The Wronskian of these functions is  $W(y_1, y_2) = 1/2$ . First write the ODE in standard form, so that  $g(t) = \sec(t/2)/2$ . The particular solution is given by  $Y(t) = u_1(t) y_1(t) + u_2(t) y_2(t)$ , in which

$$u_1(t) = - \int \frac{\cos(t/2) [\sec(t/2)]}{2W(t)} dt = 2 \ln(\cos(t/2))$$

$$u_2(t) = \int \frac{\sin(t/2) [\sec(t/2)]}{2W(t)} dt = t.$$

The particular solution is  $Y(t) = 2 \cos(t/2) \ln(\cos(t/2)) + t \sin(t/2)$ . The general solution is given by

$$y(t) = c_1 \cos(t/2) + c_2 \sin(t/2) + 2 \cos(t/2) \ln(\cos(t/2)) + t \sin(t/2).$$

10. The solution of the homogeneous equation is  $y_c(t) = c_1 e^t + c_2 t e^t$ . The functions  $y_1(t) = e^t$  and  $y_2(t) = t e^t$  form a fundamental set of solutions, with  $W(y_1, y_2) = e^{2t}$ . The particular solution is given by  $Y(t) = u_1(t) y_1(t) + u_2(t) y_2(t)$ , in which

$$u_1(t) = - \int \frac{t e^t (e^t)}{W(t)(1+t^2)} dt = -\frac{1}{2} \ln(1+t^2)$$

$$u_2(t) = \int \frac{e^t (e^t)}{W(t)(1+t^2)} dt = \arctan t.$$

The particular solution is  $Y(t) = -(1/2)e^t \ln(1+t^2) + t e^t \arctan(t)$ . Hence the general solution is given by

$$y(t) = c_1 e^t + c_2 t e^t - \frac{1}{2} e^t \ln(1+t^2) + t e^t \arctan(t).$$

12. The functions  $y_1(t) = \cos 2t$  and  $y_2(t) = \sin 2t$  form a fundamental set of solutions, with  $W(y_1, y_2) = 2$ . The particular solution is given by  $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ , in which

$$u_1(t) = -\frac{1}{2} \int_{t_0}^t g(s) \sin 2s \, ds \quad \text{and} \quad u_2(t) = \frac{1}{2} \int_{t_0}^t g(s) \cos 2s \, ds.$$

Hence the particular solution is

$$Y(t) = -\frac{1}{2} \cos 2t \int_{t_0}^t g(s) \sin 2s \, ds + \frac{1}{2} \sin 2t \int_{t_0}^t g(s) \cos 2s \, ds.$$

Note that  $\sin 2t \cos 2s - \cos 2t \sin 2s = \sin(2t - 2s)$ . It follows that

$$Y(t) = \frac{1}{2} \int_{t_0}^t g(s) \sin(2t - 2s) \, ds.$$

The general solution of the differential equation is given by

$$y(t) = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{2} \int_{t_0}^t g(s) \sin(2t - 2s) \, ds.$$

13. Note first that  $p(t) = 0$ ,  $q(t) = -2/t^2$  and  $g(t) = (3t^2 - 1)/t^2$ . The functions  $y_1(t)$  and  $y_2(t)$  are solutions of the homogeneous equation, verified by substitution. The Wronskian of these two functions is  $W(y_1, y_2) = -3$ . Using the method of variation of parameters, the particular solution is  $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ , in which

$$u_1(t) = - \int \frac{t^{-1}(3t^2 - 1)}{t^2 W(t)} \, dt = t^{-2}/6 + \ln t$$

$$u_2(t) = \int \frac{t^2(3t^2 - 1)}{t^2 W(t)} \, dt = -t^3/3 + t/3.$$

Therefore  $Y(t) = 1/6 + t^2 \ln t - t^2/3 + 1/3$ .

15. Observe that  $g(t) = t e^{2t}$ . The functions  $y_1(t)$  and  $y_2(t)$  are a fundamental set of solutions. The Wronskian of these two functions is  $W(y_1, y_2) = t e^t$ . Using the method of variation of parameters, the particular solution is  $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ , in which

$$u_1(t) = - \int \frac{e^t(t e^{2t})}{W(t)} \, dt = -e^{2t}/2 \quad \text{and} \quad u_2(t) = \int \frac{(1+t)(t e^{2t})}{W(t)} \, dt = t e^t.$$

Therefore  $Y(t) = -(1+t)e^{2t}/2 + t e^{2t} = -e^{2t}/2 + t e^{2t}/2$ .

16. Observe that  $g(t) = 2(1-t)e^{-t}$ . Direct substitution of  $y_1(t) = e^t$  and  $y_2(t) = t$  verifies that they are solutions of the homogeneous equation. The Wronskian of the two solutions is  $W(y_1, y_2) = (1-t)e^t$ . Using the method of variation of parameters, the particular solution is  $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ , in which

$$u_1(t) = - \int \frac{2t(1-t)e^{-t}}{W(t)} \, dt = t e^{-2t} + e^{-2t}/2$$

$$u_2(t) = \int \frac{2(1-t)}{W(t)} dt = -2e^{-t}.$$

Therefore  $Y(t) = te^{-t} + e^{-t}/2 - 2te^{-t} = -te^{-t} + e^{-t}/2$ .

17. Note that  $g(x) = \ln x$ . The functions  $y_1(x) = x^2$  and  $y_2(x) = x^2 \ln x$  are solutions of the homogeneous equation, as verified by substitution. The Wronskian of the solutions is  $W(y_1, y_2) = x^3$ . Using the method of variation of parameters, the particular solution is  $Y(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$ , in which

$$u_1(x) = - \int \frac{x^2 \ln x (\ln x)}{W(x)} dx = -(\ln x)^3/3$$

$$u_2(x) = \int \frac{x^2 (\ln x)}{W(x)} dx = (\ln x)^2/2.$$

Therefore  $Y(x) = -x^2(\ln x)^3/3 + x^2(\ln x)^3/2 = x^2(\ln x)^3/6$ .

19. First write the equation in standard form. Note that the forcing function becomes  $g(x)/(1-x)$ . The functions  $y_1(x) = e^x$  and  $y_2(x) = x$  are a fundamental set of solutions, as verified by substitution. The Wronskian of the solutions is  $W(y_1, y_2) = (1-x)e^x$ . Using the method of variation of parameters, the particular solution is  $Y(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$ , in which

$$u_1(x) = - \int_{x_0}^x \frac{\tau(g(\tau))}{(1-\tau)W(\tau)} d\tau \quad \text{and} \quad u_2(x) = \int_{x_0}^x \frac{e^\tau(g(\tau))}{(1-\tau)W(\tau)} d\tau.$$

Therefore

$$\begin{aligned} Y(x) &= -e^x \int_{x_0}^x \frac{\tau(g(\tau))}{(1-\tau)W(\tau)} d\tau + x \int_{x_0}^x \frac{e^\tau(g(\tau))}{(1-\tau)W(\tau)} d\tau = \\ &= \int_{x_0}^x \frac{(xe^\tau - e^x\tau)g(\tau)}{(1-\tau)^2 e^\tau} d\tau. \end{aligned}$$

20. First write the equation in standard form. The forcing function becomes  $g(x)/x^2$ . The functions  $y_1(x) = x^{-1/2} \sin x$  and  $y_2(x) = x^{-1/2} \cos x$  are a fundamental set of solutions. The Wronskian of the solutions is  $W(y_1, y_2) = -1/x$ . Using the method of variation of parameters, the particular solution is  $Y(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$ , in which

$$u_1(x) = \int_{x_0}^x \frac{\cos \tau(g(\tau))}{\tau\sqrt{\tau}} d\tau \quad \text{and} \quad u_2(x) = - \int_{x_0}^x \frac{\sin \tau(g(\tau))}{\tau\sqrt{\tau}} d\tau.$$

Therefore

$$\begin{aligned} Y(x) &= \frac{\sin x}{\sqrt{x}} \int_{x_0}^x \frac{\cos \tau(g(\tau))}{\tau\sqrt{\tau}} dt - \frac{\cos x}{\sqrt{x}} \int_{x_0}^x \frac{\sin \tau(g(\tau))}{\tau\sqrt{\tau}} d\tau = \\ &= \frac{1}{\sqrt{x}} \int_{x_0}^x \frac{\sin(x-\tau)g(\tau)}{\tau\sqrt{\tau}} d\tau. \end{aligned}$$

21. Let  $y_1(t)$  and  $y_2(t)$  be a fundamental set of solutions, and  $W(t) = W(y_1, y_2)$  be the corresponding Wronskian. Any solution,  $u(t)$ , of the homogeneous equation is

a linear combination  $u(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t)$ . Invoking the initial conditions, we require that

$$\begin{aligned} y_0 &= \alpha_1 y_1(t_0) + \alpha_2 y_2(t_0) \\ y'_0 &= \alpha_1 y'_1(t_0) + \alpha_2 y'_2(t_0) \end{aligned}$$

Note that this system of equations has a unique solution, since  $W(t_0) \neq 0$ . Now consider the nonhomogeneous problem,  $L[v] = g(t)$ , with homogeneous initial conditions. Using the method of variation of parameters, the particular solution is given by

$$Y(t) = -y_1(t) \int_{t_0}^t \frac{y_2(s)g(s)}{W(s)} ds + y_2(t) \int_{t_0}^t \frac{y_1(s)g(s)}{W(s)} ds.$$

The general solution of the IVP (iii) is

$$v(t) = \beta_1 y_1(t) + \beta_2 y_2(t) + Y(t) = \beta_1 y_1(t) + \beta_2 y_2(t) + y_1(t)u_1(t) + y_2(t)u_2(t)$$

in which  $u_1$  and  $u_2$  are defined above. Invoking the initial conditions, we require that

$$\begin{aligned} 0 &= \beta_1 y_1(t_0) + \beta_2 y_2(t_0) + Y(t_0) \\ 0 &= \beta_1 y'_1(t_0) + \beta_2 y'_2(t_0) + Y'(t_0) \end{aligned}$$

Based on the definition of  $u_1$  and  $u_2$ ,  $Y(t_0) = 0$ . Furthermore, since  $y_1 u'_1 + y_2 u'_2 = 0$ , it follows that  $Y'(t_0) = 0$ . Hence the only solution of the above system of equations is the trivial solution. Therefore  $v(t) = Y(t)$ . Now consider the function  $y = u + v$ . Then  $L[y] = L[u + v] = L[u] + L[v] = g(t)$ . That is,  $y(t)$  is a solution of the nonhomogeneous problem. Further,  $y(t_0) = u(t_0) + v(t_0) = y_0$ , and similarly,  $y'(t_0) = y'_0$ . By the uniqueness theorems,  $y(t)$  is the unique solution of the initial value problem.

23.(a) A fundamental set of solutions is  $y_1(t) = \cos t$  and  $y_2(t) = \sin t$ . The Wronskian  $W(t) = y_1 y'_2 - y'_1 y_2 = 1$ . By the result in Problem 22,

$$\begin{aligned} Y(t) &= \int_{t_0}^t \frac{\cos(s) \sin(t) - \cos(t) \sin(s)}{W(s)} g(s) ds \\ &= \int_{t_0}^t [\cos(s) \sin(t) - \cos(t) \sin(s)] g(s) ds. \end{aligned}$$

Finally, we have  $\cos(s) \sin(t) - \cos(t) \sin(s) = \sin(t - s)$ .

(b) Using Problem 21 and part (a), the solution is

$$y(t) = y_0 \cos t + y'_0 \sin t + \int_0^t \sin(t - s) g(s) ds.$$

24. A fundamental set of solutions is  $y_1(t) = e^{at}$  and  $y_2(t) = e^{bt}$ . The Wronskian  $W(t) = y_1 y'_2 - y'_1 y_2 = (b - a)e^{(a+b)t}$ . By the result in Problem 22,

$$Y(t) = \int_{t_0}^t \frac{e^{as} e^{bt} - e^{at} e^{bs}}{W(s)} g(s) ds = \frac{1}{b - a} \int_{t_0}^t \frac{e^{as} e^{bt} - e^{at} e^{bs}}{e^{(a+b)s}} g(s) ds.$$

Hence the particular solution is

$$Y(t) = \frac{1}{b-a} \int_{t_0}^t [e^{b(t-s)} - e^{a(t-s)}] g(s) ds.$$

26. A fundamental set of solutions is  $y_1(t) = e^{at}$  and  $y_2(t) = te^{at}$ . The Wronskian  $W(t) = y_1 y_2' - y_1' y_2 = e^{2at}$ . By the result in Problem 22,

$$Y(t) = \int_{t_0}^t \frac{te^{as+at} - se^{at+as}}{W(s)} g(s) ds = \int_{t_0}^t \frac{(t-s)e^{as+at}}{e^{2as}} g(s) ds.$$

Hence the particular solution is

$$Y(t) = \int_{t_0}^t (t-s)e^{a(t-s)} g(s) ds.$$

27. The form of the kernel depends on the characteristic roots. If the roots are real and distinct,

$$K(t-s) = \frac{e^{b(t-s)} - e^{a(t-s)}}{b-a}.$$

If the roots are real and identical,

$$K(t-s) = (t-s)e^{a(t-s)}.$$

If the roots are complex conjugates,

$$K(t-s) = \frac{e^{\lambda(t-s)} \sin \mu(t-s)}{\mu}.$$

28. Let  $y(t) = v(t)y_1(t)$ , in which  $y_1(t)$  is a solution of the homogeneous equation. Substitution into the given ODE results in

$$v''y_1 + 2v'y_1' + vy_1'' + p(t)[v'y_1 + vy_1'] + q(t)vy_1 = g(t).$$

By assumption,  $y_1'' + p(t)y_1' + q(t)y_1 = 0$ , hence  $v(t)$  must be a solution of the ODE

$$v''y_1 + [2y_1' + p(t)y_1]v' = g(t).$$

Setting  $w = v'$ , we also have  $w'y_1 + [2y_1' + p(t)y_1]w = g(t)$ .

30. First write the equation as  $y'' + 7t^{-1}y + 5t^{-2}y = t^{-1}$ . As shown in Problem 28, the function  $y(t) = t^{-1}v(t)$  is a solution of the given ODE as long as  $v$  is a solution of

$$t^{-1}v'' + [-2t^{-2} + 7t^{-2}]v' = t^{-1},$$

that is,  $v'' + 5t^{-1}v' = 1$ . This ODE is linear and first order in  $v'$ . The integrating factor is  $\mu = t^5$ . The solution is  $v' = t/6 + ct^{-5}$ . Direct integration now results in  $v(t) = t^2/12 + c_1t^{-4} + c_2$ . Hence  $y(t) = t/12 + c_1t^{-5} + c_2t^{-1}$ .

31. Write the equation as  $y'' - t^{-1}(1+t)y + t^{-1}y = te^{2t}$ . As shown in Problem 28, the function  $y(t) = (1+t)v(t)$  is a solution of the given ODE as long as  $v$  is a solution of

$$(1+t)v'' + [2 - t^{-1}(1+t)^2]v' = te^{2t},$$

that is,

$$v'' - \frac{1+t^2}{t(t+1)}v' = \frac{t}{t+1}e^{2t}.$$

This equation is first order linear in  $v'$ , with integrating factor  $\mu = t^{-1}(1+t)^2e^{-t}$ . The solution is  $v' = (t^2e^{2t} + c_1te^t)/(1+t)^2$ . Integrating, we obtain  $v(t) = e^{2t}/2 - e^{2t}/(t+1) + c_1e^t/(t+1) + c_2$ . Hence the solution of the original ODE is  $y(t) = (t-1)e^{2t}/2 + c_1e^t + c_2(t+1)$ .

32. Write the equation as  $y'' + t(1-t)^{-1}y - (1-t)^{-1}y = 2(1-t)e^{-t}$ . The function  $y(t) = e^tv(t)$  is a solution to the given ODE as long as  $v$  is a solution of

$$e^tv'' + [2e^t + t(1-t)^{-1}e^t]v' = 2(1-t)e^{-t},$$

that is,  $v'' + [(2-t)/(1-t)]v' = 2(1-t)e^{-2t}$ . This equation is first order linear in  $v'$ , with integrating factor  $\mu = e^t/(t-1)$ . The solution is

$$v' = (t-1)(2e^{-2t} + c_1e^{-t}).$$

Integrating, we obtain  $v(t) = (1/2 - t)e^{-2t} - c_1te^{-t} + c_2$ . Hence the solution of the original ODE is  $y(t) = (1/2 - t)e^{-t} - c_1t + c_2e^t$ .

## 3.7

1.  $R \cos \delta = 3$  and  $R \sin \delta = 4$ , so  $R = \sqrt{25} = 5$  and  $\delta = \arctan(4/3)$ . We obtain that  $u = 5 \cos(2t - \arctan(4/3))$ .

3.  $R \cos \delta = 4$  and  $R \sin \delta = -2$ , so  $R = \sqrt{20} = 2\sqrt{5}$  and  $\delta = -\arctan(1/2)$ . We obtain that  $u = 2\sqrt{5} \cos(3t + \arctan(1/2))$ .

4.  $R \cos \delta = -2$  and  $R \sin \delta = -3$ , so  $R = \sqrt{13}$  and  $\delta = \pi + \arctan(3/2)$ . We obtain that  $u = \sqrt{13} \cos(\pi t - \pi - \arctan(3/2))$ .

5. The spring constant is  $k = 2/(1/2) = 4$  lb/ft. Mass  $m = 2/32 = 1/16$  lb-s<sup>2</sup>/ft. Since there is no damping, the equation of motion is  $u''/16 + 4u = 0$ , that is,  $u'' + 64u = 0$ . The initial conditions are  $u(0) = 1/4$  ft,  $u'(0) = 0$  ft/s. The general solution is  $u(t) = A \cos 8t + B \sin 8t$ . Invoking the initial conditions, we have  $u(t) = \cos 8t/4$ .  $R = 1/4$  ft,  $\delta = 0$  rad,  $\omega_0 = 8$  rad/s, and  $T = \pi/4$  s.

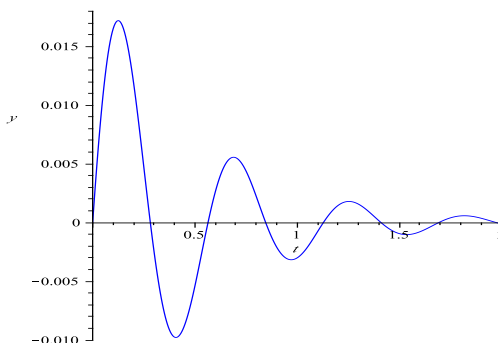
7. The spring constant is  $k = 3/(1/4) = 12$  lb/ft. Mass  $m = 3/32$  lb-s<sup>2</sup>/ft. Since there is no damping, the equation of motion is  $3u''/32 + 12u = 0$ , that is,  $u'' + 128u = 0$ . The initial conditions are  $u(0) = -1/12$  ft,  $u'(0) = 2$  ft/s. The general solution is  $u(t) = A \cos 8\sqrt{2}t + B \sin 8\sqrt{2}t$ . Invoking the initial conditions, we have

$$u(t) = -\frac{1}{12} \cos 8\sqrt{2}t + \frac{1}{4\sqrt{2}} \sin 8\sqrt{2}t.$$

$R = \sqrt{11/288}$  ft,  $\delta = \pi - \arctan(3/\sqrt{2})$  rad,  $\omega_0 = 8\sqrt{2}$  rad/s,  $T = \pi/(4\sqrt{2})$  s.

10. The spring constant is  $k = 16/(1/4) = 64$  lb/ft. Mass  $m = 1/2$  lb-s<sup>2</sup>/ft. The damping coefficient is  $\gamma = 2$  lb-s/ft. Hence the equation of motion is  $u''/2 + 2u' + 64u = 0$ , that is,  $u'' + 4u' + 128u = 0$ . The initial conditions are  $u(0) = 0$  ft,  $u'(0) = 1/4$  ft/s. The general solution is  $u(t) = A \cos 2\sqrt{31}t + B \sin 2\sqrt{31}t$ . Invoking the initial conditions, we have

$$u(t) = \frac{1}{8\sqrt{31}} e^{-2t} \sin 2\sqrt{31}t.$$



Solving  $u(t) = 0$ , on the interval  $[0.2, 0.4]$ , we obtain  $t = \pi/2\sqrt{31} = 0.2821$  s. Based on the graph, and the solution of  $u(t) = -0.01/12$  ft, we have  $|u(t)| \leq 0.01$  for  $t \geq \tau = 1.5927$ .

11. The spring constant is  $k = 3/(.1) = 30$  N/m. The damping coefficient is given as  $\gamma = 3/5$  N-s/m. Hence the equation of motion is  $2u'' + 3u'/5 + 30u = 0$ , that is,  $u'' + 0.3u' + 15u = 0$ . The initial conditions are  $u(0) = 0.05$  m and  $u'(0) = 0.01$  m/s. The general solution is  $u(t) = A \cos \mu t + B \sin \mu t$ , in which  $\mu = 3.87008$  rad/s. Invoking the initial conditions, we have  $u(t) = e^{-0.15t}(0.05 \cos \mu t + 0.00452 \sin \mu t)$ . Also,  $\mu/\omega_0 = 3.87008/\sqrt{15} \approx 0.99925$ .

13. The frequency of the undamped motion is  $\omega_0 = 1$ . The quasi frequency of the damped motion is  $\mu = \sqrt{4 - \gamma^2}/2$ . Setting  $\mu = 2\omega_0/3$ , we obtain  $\gamma = 2\sqrt{5}/3$ .

14. The spring constant is  $k = mg/L$ . The equation of motion for an undamped system is  $mu'' + mg/L = 0$ . Hence the natural frequency of the system is  $\omega_0 = \sqrt{g/L}$ . The period is  $T = 2\pi/\omega_0$ .

15. The general solution of the system is  $u(t) = A \cos \gamma(t - t_0) + B \sin \gamma(t - t_0)$ . Invoking the initial conditions, we have  $u(t) = u_0 \cos \gamma(t - t_0) + (u'_0/\gamma) \sin \gamma(t - t_0)$ . Clearly, the functions  $v = u_0 \cos \gamma(t - t_0)$  and  $w = (u'_0/\gamma) \sin \gamma(t - t_0)$  satisfy the given criteria.

16. Note that  $r \sin(\omega_0 t - \theta) = r \sin \omega_0 t \cos \theta - r \cos \omega_0 t \sin \theta$ . Comparing the given expressions, we have  $A = -r \sin \theta$  and  $B = r \cos \theta$ . That is,  $r = R = \sqrt{A^2 + B^2}$ , and  $\tan \theta = -A/B = -1/\tan \delta$ . The latter relation is also  $\tan \theta + \cot \delta = 1$ .

18. The system is critically damped, when  $R = 2\sqrt{L/C}$ . Here  $R = 1000$  ohms.



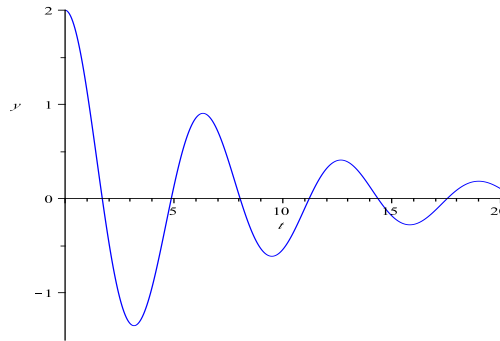
21.(a) Let  $u = Re^{-\gamma t/2m} \cos(\mu t - \delta)$ . Then attains a maximum when  $\mu t_k - \delta = 2k\pi$ . Hence  $T_d = t_{k+1} - t_k = 2\pi/\mu$ .

(b)  $u(t_k)/u(t_{k+1}) = e^{-\gamma t_k/2m} / e^{-\gamma t_{k+1}/2m} = e^{(\gamma t_{k+1} - \gamma t_k)/2m}$ . Hence  $u(t_k)/u(t_{k+1}) = e^{\gamma(2\pi/\mu)/2m} = e^{\gamma T_d/2m}$ .

(c)  $\Delta = \ln [u(t_k)/u(t_{k+1})] = \gamma(2\pi/\mu)/2m = \pi\gamma/\mu m$ .

22. The spring constant is  $k = 16/(1/4) = 64$  lb/ft. Mass  $m = 1/2$  lb-s<sup>2</sup>/ft. The damping coefficient is  $\gamma = 2$  lb-s/ft. The quasi frequency is  $\mu = 2\sqrt{31}$  rad/s. Hence  $\Delta = 2\pi/\sqrt{31} \approx 1.1285$ .

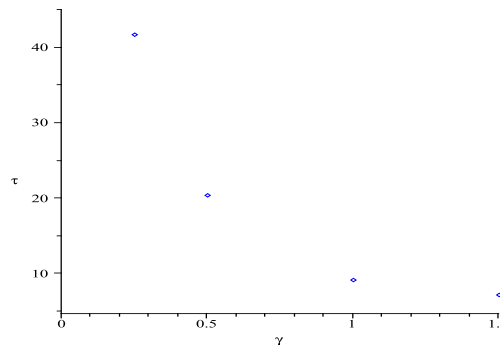
25.(a) The solution of the IVP is  $u(t) = e^{-t/8}(2 \cos 3\sqrt{7}t/8 + (2\sqrt{7}/21) \sin 3\sqrt{7}t/8)$ .



Using the plot, and numerical analysis,  $\tau \approx 41.715$ .

(b) For  $\gamma = 0.5$ ,  $\tau \approx 20.402$ ; for  $\gamma = 1.0$ ,  $\tau \approx 9.168$ ; for  $\gamma = 1.5$ ,  $\tau \approx 7.184$ .

(c)



(d) For  $\gamma = 1.6$ ,  $\tau \approx 7.218$ ; for  $\gamma = 1.7$ ,  $\tau \approx 6.767$ ; for  $\gamma = 1.8$ ,  $\tau \approx 5.473$ ; for  $\gamma = 1.9$ ,  $\tau \approx 6.460$ .  $\tau$  steadily decreases to about  $\tau_{min} \approx 4.873$ , corresponding to the critical value  $\gamma_0 \approx 1.73$ .

(e) We can rewrite the solution as  $u(t) = Re^{-\gamma t/2} \cos(\mu t - \delta)$ , where  $R = 4/\sqrt{4 - \gamma^2}$ .

Neglecting the cosine factor, we can approximate  $\tau$  by solving  $Re^{-\gamma\tau/2} = 1/100$ , thus finding

$$\tau \approx \frac{2}{\gamma} \ln(100R) = \frac{2}{\gamma} \ln \left( \frac{400}{\sqrt{4 - \gamma^2}} \right).$$

For  $\gamma = 0.25$ ,  $\tau \approx 42.4495$ ; for  $\gamma = 0.5$ ,  $\tau \approx 21.3223$ ; for  $\gamma = 1.0$ ,  $\tau \approx 10.8843$ ; for  $\gamma = 1.5$ ,  $\tau \approx 7.61554$ ; for  $\gamma = 1.6$ ,  $\tau \approx 7.26143$ ; for  $\gamma = 1.7$ ,  $\tau \approx 6.98739$ ; for  $\gamma = 1.8$ ,  $\tau \approx 6.80965$ ; for  $\gamma = 1.9$ ,  $\tau \approx 6.80239$ .

26.(a) The characteristic equation is  $mr^2 + \gamma r + k = 0$ . Since  $\gamma^2 < 4km$ , the roots are  $r_{1,2} = (-\gamma \pm i\sqrt{4mk - \gamma^2})/2m$ . The general solution is

$$u(t) = e^{-\gamma t/2m} \left[ A \cos \frac{\sqrt{4mk - \gamma^2}}{2m} t + B \sin \frac{\sqrt{4mk - \gamma^2}}{2m} t \right].$$

Invoking the initial conditions,  $A = u_0$  and  $B = (2mv_0 - \gamma u_0)/\sqrt{4mk - \gamma^2}$ .

(b) We can write  $u(t) = R e^{-\gamma t/2m} \cos(\mu t - \delta)$ , in which

$$R = \sqrt{u_0^2 + \frac{(2mv_0 - \gamma u_0)^2}{4mk - \gamma^2}} \quad \text{and} \quad \delta = \arctan \left[ \frac{(2mv_0 - \gamma u_0)}{u_0 \sqrt{4mk - \gamma^2}} \right].$$

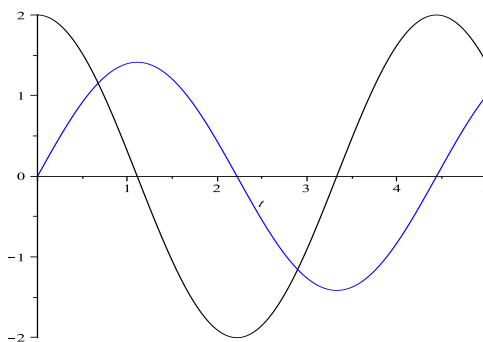
(c)

$$R = \sqrt{u_0^2 + \frac{(2mv_0 - \gamma u_0)^2}{4mk - \gamma^2}} = 2\sqrt{\frac{m(ku_0^2 + \gamma u_0 v_0 + mv_0^2)}{4mk - \gamma^2}} = \sqrt{\frac{a + b\gamma}{4mk - \gamma^2}}.$$

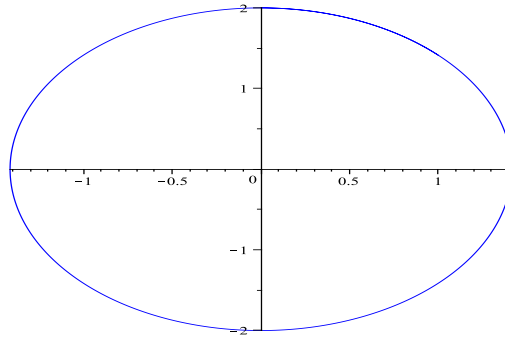
It is evident that  $R$  increases (monotonically) without bound as  $\gamma \rightarrow (2\sqrt{mk})^-$ .

28.(a) The general solution is  $u(t) = A \cos \sqrt{2}t + B \sin \sqrt{2}t$ . Invoking the initial conditions, we have  $u(t) = \sqrt{2} \sin \sqrt{2}t$ .

(b)



(c)



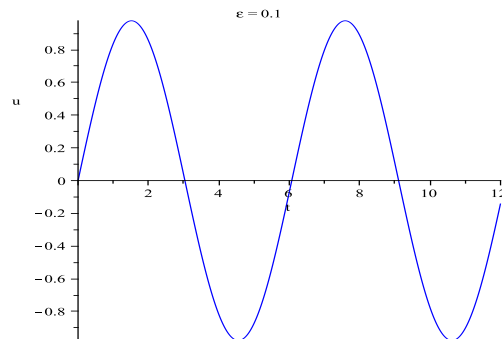
The condition  $u'(0) = 2$  implies that  $u(t)$  initially increases. Hence the phase point travels clockwise.

31. Based on Newton's second law, with the positive direction to the right,  $\sum F = mu''$ , where  $\sum F = -ku - \gamma u'$ . Hence the equation of motion is  $mu'' + \gamma u' + ku = 0$ . The only difference in this problem is that the equilibrium position is located at the unstretched configuration of the spring.

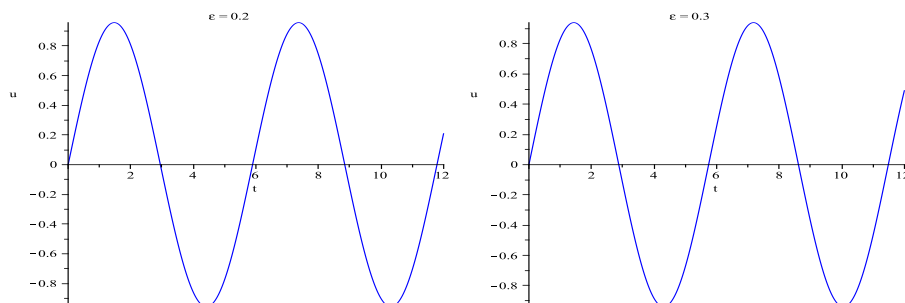
32.(a) The restoring force exerted by the spring is  $F_s = -(ku + \epsilon u^3)$ . The opposing viscous force is  $F_d = -\gamma u'$ . Based on Newton's second law, with the positive direction to the right,  $F_s + F_d = mu''$ . Hence the equation of motion is  $mu'' + \gamma u' + ku + \epsilon u^3 = 0$ .

(b) With the specified parameter values, the equation of motion is  $u'' + u = 0$ . The general solution of this ODE is  $u(t) = A \cos t + B \sin t$ . Invoking the initial conditions, the specific solution is  $u(t) = \sin t$ . Clearly, the amplitude is  $R = 1$ , and the period of the motion is  $T = 2\pi$ .

(c) Given  $\epsilon = 0.1$ , the equation of motion is  $u'' + u + 0.1u^3 = 0$ . A solution of the IVP can be generated numerically. We estimate  $A = 0.98$  and  $T = 6.07$ .

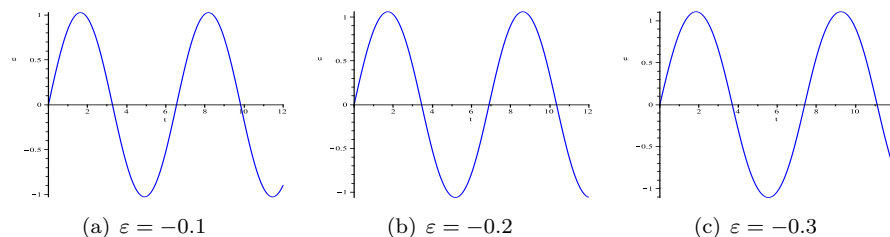


(d) For  $\epsilon = 0.2$ ,  $A = 0.96$  and  $T = 5.90$ . For  $\epsilon = 0.3$ ,  $A = 0.94$  and  $T = 5.74$ .



(e) The amplitude and period both seem to decrease.

(f) For  $\epsilon = -0.1$ ,  $A = 1.03$  and  $T = 6.55$ . For  $\epsilon = -0.2$ ,  $A = 1.06$  and  $T = 6.90$ . For  $\epsilon = -0.3$ ,  $A = 1.11$  and  $T = 7.41$ . The amplitude and period both seem to increase.



## 3.8

2. We have  $\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$ . Subtracting the two identities, we obtain  $\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2 \cos \alpha \sin \beta$ . Setting  $\alpha + \beta = 7t$  and  $\alpha - \beta = 6t$ , we get that  $\alpha = 6.5t$  and  $\beta = 0.5t$ . This implies that  $\sin 7t - \sin 6t = 2 \sin(t/2) \cos(13t/2)$ .

3. Consider the trigonometric identities  $\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$ . Adding the two identities, we get  $\cos(\alpha - \beta) + \cos(\alpha + \beta) = 2 \cos \alpha \cos \beta$ . Comparing the expressions, set  $\alpha + \beta = 2\pi t$  and  $\alpha - \beta = \pi t$ . This means  $\alpha = 3\pi t/2$  and  $\beta = \pi t/2$ . Upon substitution, we have  $\cos(\pi t) + \cos(2\pi t) = 2 \cos(3\pi t/2) \cos(\pi t/2)$ .

4. Adding the two identities  $\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$ , it follows that  $\sin(\alpha - \beta) + \sin(\alpha + \beta) = 2 \sin \alpha \cos \beta$ . Setting  $\alpha + \beta = 4t$  and  $\alpha - \beta = 3t$ , we have  $\alpha = 7t/2$  and  $\beta = t/2$ . Hence  $\sin 3t + \sin 4t = 2 \sin(7t/2) \cos(t/2)$ .

6. Using MKS units, the spring constant is  $k = 5(9.8)/0.1 = 490$  N/m, and the damping coefficient is  $\gamma = 2/0.04 = 50$  N-s/m. The equation of motion is

$$5u'' + 50u' + 490u = 10 \sin(t/2).$$

The initial conditions are  $u(0) = 0$  m and  $u'(0) = 0.03$  m/s.

8.(a) The homogeneous solution is  $u_c(t) = Ae^{-5t} \cos \sqrt{73}t + Be^{-5t} \sin \sqrt{73}t$ . Based on the method of undetermined coefficients, the particular solution is

$$U(t) = \frac{1}{153281} [-160 \cos(t/2) + 3128 \sin(t/2)].$$

Hence the general solution of the ODE is  $u(t) = u_c(t) + U(t)$ . Invoking the initial conditions, we find that

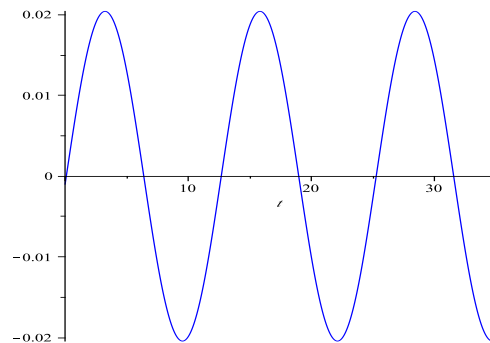
$$A = 160/153281 \text{ and } B = 383443\sqrt{73}/1118951300.$$

Hence the response is

$$u(t) = \frac{1}{153281} \left[ 160 e^{-5t} \cos \sqrt{73}t + \frac{383443\sqrt{73}}{7300} e^{-5t} \sin \sqrt{73}t \right] + U(t).$$

(b)  $u_c(t)$  is the transient part and  $U(t)$  is the steady state part of the response.

(c)



(d) The amplitude of the forced response is given by  $R = 2/\Delta$ , in which

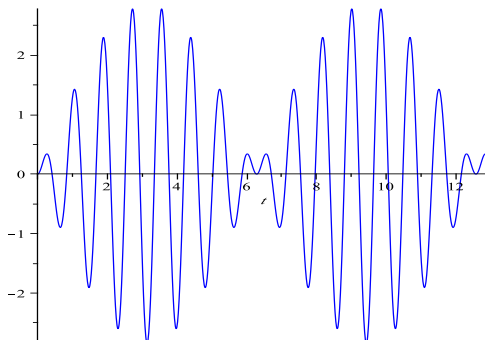
$$\Delta = \sqrt{25(98 - \omega^2)^2 + 2500\omega^2}.$$

The maximum amplitude is attained when  $\Delta$  is a minimum. Hence the amplitude is maximum at  $\omega = 4\sqrt{3}$  rad/s.

9. The spring constant is  $k = 12$  lb/ft and hence the equation of motion is

$$\frac{6}{32}u'' + 12u = 4 \cos 7t,$$

that is,  $u'' + 64u = (64/3) \cos 7t$ . The initial conditions are  $u(0) = 0$  ft,  $u'(0) = 0$  ft/s. The general solution is  $u(t) = A \cos 8t + B \sin 8t + (64/45) \cos 7t$ . Invoking the initial conditions, we have  $u(t) = -(64/45) \cos 8t + (64/45) \cos 7t = (128/45) \sin(t/2) \sin(15t/2)$ .



12. The equation of motion is  $2u'' + u' + 3u = 3 \cos 3t - 2 \sin 3t$ . Since the system is damped, the steady state response is equal to the particular solution. Using the method of undetermined coefficients, we obtain  $u_{ss}(t) = (\sin 3t - \cos 3t)/6$ . Further, we find that  $R = \sqrt{2}/6$  and  $\delta = \arctan(-1) = 3\pi/4$ . Hence we can write  $u_{ss}(t) = (\sqrt{2}/6) \cos(3t - 3\pi/4)$ .

13.(a,b) Plug in  $u(t) = R \cos(\omega t - \delta)$  into the equation  $mu'' + \gamma u' + ku = F_0 \cos \omega t$ , then use trigonometric identities and compare the coefficients of  $\cos \omega t$  and  $\sin \omega t$ . The result follows.

(c) The amplitude of the steady-state response is given by

$$R = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}}.$$

Since  $F_0$  is constant, the amplitude is maximum when the denominator of  $R$  is minimum. Let  $z = \omega^2$ , and consider the function  $f(z) = m^2(\omega_0^2 - z)^2 + \gamma^2 z$ . Note that  $f(z)$  is a quadratic, with minimum at  $z = \omega_0^2 - \gamma^2/2m^2$ . Hence the amplitude  $R$  attains a maximum at  $\omega_{max}^2 = \omega_0^2 - \gamma^2/2m^2$ . Furthermore, since  $\omega_0^2 = k/m$ ,

$$\omega_{max}^2 = \omega_0^2 \left[ 1 - \frac{\gamma^2}{2km} \right].$$

Substituting  $\omega^2 = \omega_{max}^2$  into the expression for the amplitude,

$$R = \frac{F_0}{\sqrt{\gamma^4/4m^2 + \gamma^2(\omega_0^2 - \gamma^2/2m^2)}} = \frac{F_0}{\sqrt{\omega_0^2 \gamma^2 - \gamma^4/4m^2}} = \frac{F_0}{\gamma \omega_0 \sqrt{1 - \gamma^2/4mk}}.$$

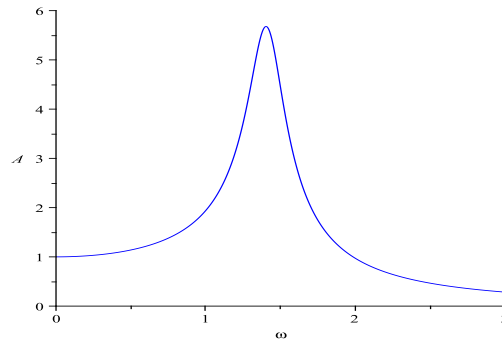
17.(a) The steady state part of the solution  $U(t) = A \cos \omega t + B \sin \omega t$  may be found by substituting this expression into the differential equation and solving for  $A$  and  $B$ . We find that

$$A = \frac{32(2 - \omega^2)}{64 - 63\omega^2 + 16\omega^4}, \quad B = \frac{8\omega}{64 - 63\omega^2 + 16\omega^4}.$$

(b) The amplitude is

$$A = \frac{8}{\sqrt{64 - 63\omega^2 + 16\omega^4}}.$$

(c)



(d) See Problem 13. The amplitude is maximum when the denominator of  $A$  is minimum. That is, when  $\omega = \omega_{max} = 3\sqrt{14}/8 \approx 1.4031$ . Hence  $A = 64/\sqrt{127}$ .

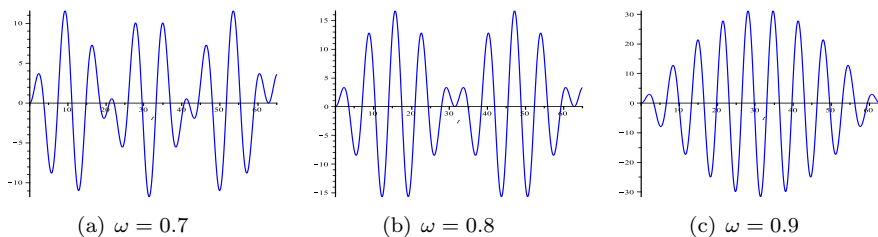
18.(a) The homogeneous solution is  $u_c(t) = A \cos t + B \sin t$ . Based on the method of undetermined coefficients, the particular solution is

$$U(t) = \frac{3}{1 - \omega^2} \cos \omega t.$$

Hence the general solution of the ODE is  $u(t) = u_c(t) + U(t)$ . Invoking the initial conditions, we find that  $A = 3/(\omega^2 - 1)$  and  $B = 0$ . Hence the response is

$$u(t) = \frac{3}{1 - \omega^2} [\cos \omega t - \cos t].$$

(b)



Note that

$$u(t) = \frac{6}{1 - \omega^2} \sin \left[ \frac{(1 - \omega)t}{2} \right] \sin \left[ \frac{(\omega + 1)t}{2} \right].$$

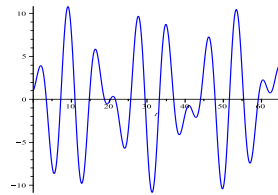
19.(a) The homogeneous solution is  $u_c(t) = A \cos t + B \sin t$ . Based on the method of undetermined coefficients, the particular solution is

$$U(t) = \frac{3}{1 - \omega^2} \cos \omega t.$$

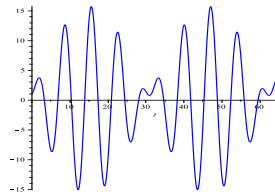
Hence the general solution is  $u(t) = u_c(t) + U(t)$ . Invoking the initial conditions, we find that  $A = (\omega^2 + 2)/(\omega^2 - 1)$  and  $B = 1$ . Hence the response is

$$u(t) = \frac{1}{1 - \omega^2} [3 \cos \omega t - (\omega^2 + 2) \cos t] + \sin t.$$

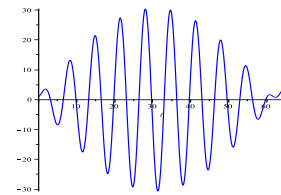
(b)



(a)  $\omega = 0.7$



(b)  $\omega = 0.8$

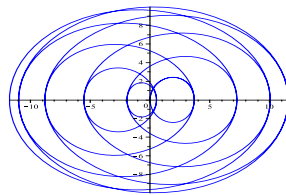


(c)  $\omega = 0.9$

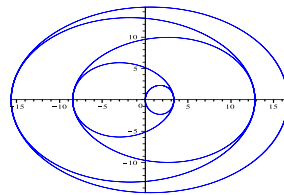
Note that

$$u(t) = \frac{6}{1 - \omega^2} \sin \left[ \frac{(1 - \omega)t}{2} \right] \sin \left[ \frac{(\omega + 1)t}{2} \right] + \cos t + \sin t.$$

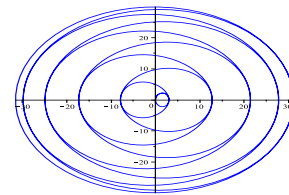
20.



(a)  $\omega = 0.7$



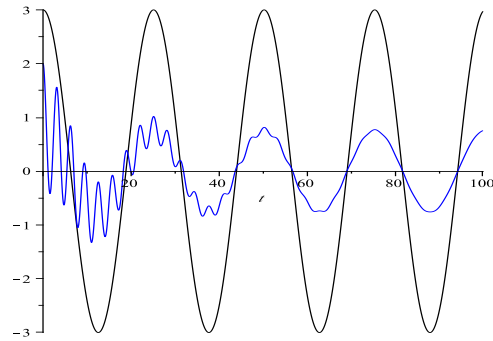
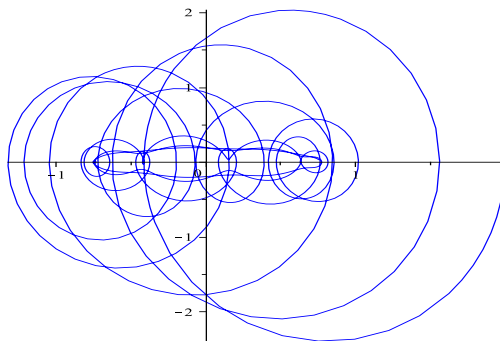
(b)  $\omega = 0.8$



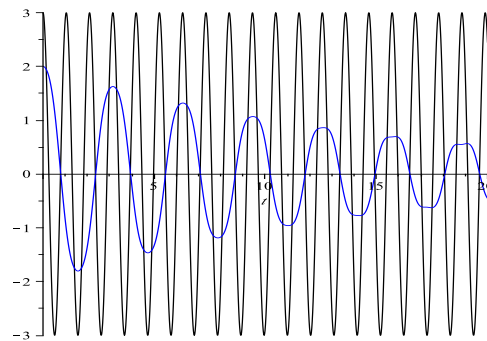
(c)  $\omega = 0.9$



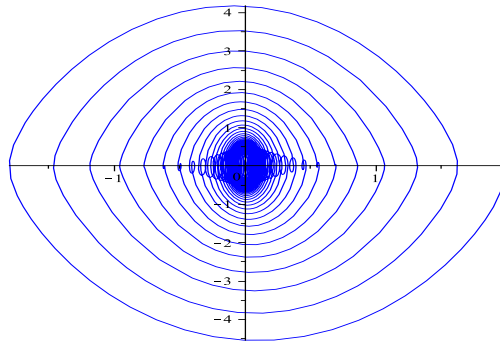
21.(a)

(b) Phase plot -  $u'$  vs  $u$  :

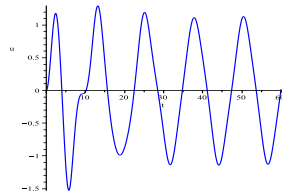
23.(a)



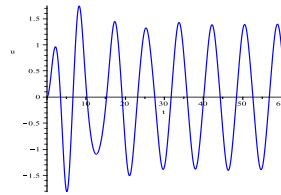
(b) Phase plot -  $u'$  vs  $u$  :



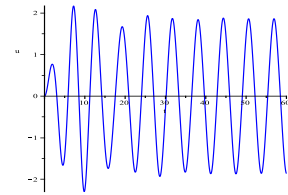
25.(a)



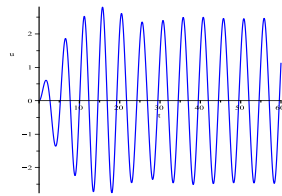
(a)  $\omega = 0.5$



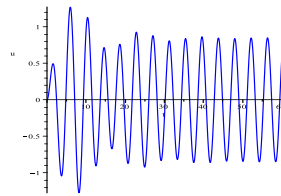
(b)  $\omega = 0.75$



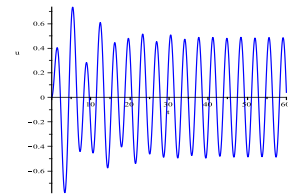
(c)  $\omega = 1$



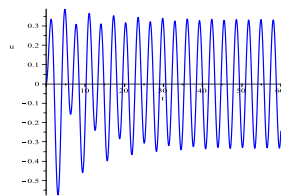
(d)  $\omega = 1.25$



(e)  $\omega = 1.5$

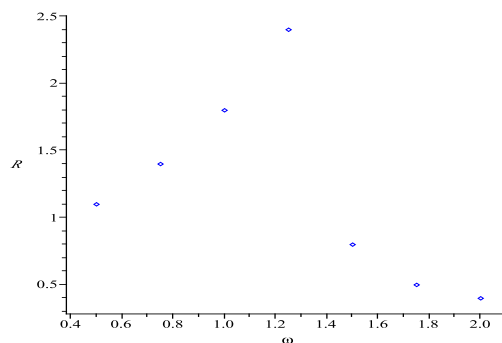


(f)  $\omega = 1.75$



(g)  $\omega = 2$

(b)  $R$  vs  $\omega$



(c) The amplitude for a similar system with a linear spring is given by

$$R = \frac{5}{\sqrt{25 - 49\omega^2 + 25\omega^4}}.$$

