CHAPTER

5

Series Solutions of Second Order Linear Equations

5.1

1. Apply the ratio test:

$$\lim_{n\to\infty}\frac{\left|(x-3)^{n+1}\right|}{|(x-3)^n|}=\lim_{n\to\infty}|x-3|=|x-3|\,.$$

Hence the series converges absolutely for $|x-3|<1$. The radius of convergence is $\rho=1$. The series diverges for $x=2$ and $x=4$, since the n-th term does not approach zero.

3. Applying the ratio test,

$$\lim_{n\to\infty}\frac{\left|n!\,x^{2n+2}\right|}{|(n+1)!\,x^{2n}|}=\lim_{n\to\infty}\frac{x^2}{n+1}=0\,.$$

The series converges absolutely for all values of x. Thus the radius of convergence is $\rho=\infty$.

4. Apply the ratio test:

$$\lim_{n\to\infty}\frac{\left|2^{n+1}x^{n+1}\right|}{|2^nx^n|}=\lim_{n\to\infty}2\,|x|=2\,|x|\,.$$

Hence the series converges absolutely for $2\,|x|<1$, or $|x|<1/2$. The radius of convergence is $\rho=1/2$. The series diverges for $x=\pm 1/2$, since the n-th term does not approach zero.

6. Applying the ratio test,

$$\lim_{n\to\infty}\frac{\left|n(x-x_0)^{n+1}\right|}{\left|(n+1)(x-x_0)^n\right|} = \lim_{n\to\infty}\frac{n}{n+1}\left|(x-x_0)\right| = \left|(x-x_0)\right|.$$

Hence the series converges absolutely for $|(x-x_0)|<1$. The radius of convergence is $\rho=1$. At $x=x_0+1$, we obtain the harmonic series, which is divergent. At the other endpoint, $x=x_0-1$, we obtain

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n},$$

which is conditionally convergent.

7. Apply the ratio test :

$$\lim_{n\to\infty}\frac{\left|3^{\,n}(n+1)^2(x+2)^{n+1}\right|}{\left|3^{\,n+1}n^2(x+2)^n\right|} = \lim_{n\to\infty}\frac{(n+1)^2}{3\,n^2}\left|(x+2)\right| = \frac{1}{3}\left|(x+2)\right|.$$

Hence the series converges absolutely for $\frac{1}{3}|x+2|<1$, or $|x+2|<3$. The radius of convergence is $\rho=3$. At $x=-5$ and $x=+1$, the series diverges, since the n-th term does not approach zero.

8. Applying the ratio test,

$$\lim_{n\to\infty}\frac{\left|n^n(n+1)!\,x^{n+1}\right|}{\left|(n+1)^{n+1}n!\,x^n\right|} = \lim_{n\to\infty}\frac{n^n}{(n+1)^n}\,|x| = \frac{1}{e}\,|x|,$$

since

$$\lim_{n\to\infty}\frac{n^n}{(n+1)^n} = \lim_{n\to\infty}(1+\frac{1}{n})^{-\,n} = e^{-1}.$$

Hence the series converges absolutely for $|x|<e$. The radius of convergence is $\rho=e$. At $x=\pm\, e$, the series diverges, since the n-th term does not approach zero. This follows from the fact that

$$\lim_{n\to\infty}\frac{n!\,e^n}{n^n\sqrt{2\pi n}} = 1.$$

10. We have $f(x)=e^x$, with $f^{(n)}(x)=e^x$, for $n=1,2,\ldots$. Therefore $f^{(n)}(0)=1$. Hence the Taylor expansion about $x_0=0$ is

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}.$$

Applying the ratio test,

$$\lim_{n\to\infty}\frac{\left|n!x^{n+1}\right|}{\left|(n+1)!\,x^n\right|} = \lim_{n\to\infty}\frac{1}{n+1}\,|x| = 0.$$

The radius of convergence is $\rho=\infty$.

11. We have $f(x) = x$, with $f'(x) = 1$ and $f^{(n)}(x) = 0$, for $n = 2, \ldots$. Clearly, $f(1) = 1$ and $f'(1) = 1$, with all other derivatives equal to zero. Hence the Taylor expansion about $x_0 = 1$ is

$$x = 1 + (x - 1).$$

Since the series has only a finite number of terms, it converges absolutely for all x.

14. We have $f(x) = 1/(1+x)$, $f'(x) = -1/(1+x)^2$, $f''(x) = 2/(1+x)^3, \ldots$ with $f^{(n)}(x) = (-1)^n n!/(1+x)^{n+1}$, for $n \geq 1$. It follows that $f^{(n)}(0) = (-1)^n n!$ for $n \geq 0$. Hence the Taylor expansion about $x_0 = 0$ is

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n.$$

Applying the ratio test,

$$\lim_{n \to \infty} \frac{|x^{n+1}|}{|x^n|} = \lim_{n \to \infty} |x| = |x|.$$

The series converges absolutely for $|x| < 1$, but diverges at $x = \pm 1$.

15. We have $f(x) = 1/(1-x)$, $f'(x) = 1/(1-x)^2$, $f''(x) = 2/(1-x)^3, \ldots$ with $f^{(n)}(x) = n!/(1-x)^{n+1}$, for $n \geq 1$. It follows that $f^{(n)}(0) = n!$, for $n \geq 0$. Hence the Taylor expansion about $x_0 = 0$ is

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Applying the ratio test,

$$\lim_{n \to \infty} \frac{|x^{n+1}|}{|x^n|} = \lim_{n \to \infty} |x| = |x|.$$

The series converges absolutely for $|x| < 1$, but diverges at $x = \pm 1$.

16. We have $f(x) = 1/(1-x)$, $f'(x) = 1/(1-x)^2$, $f''(x) = 2/(1-x)^3, \ldots$ with $f^{(n)}(x) = n!/(1-x)^{n+1}$, for $n \geq 1$. It follows that $f^{(n)}(2) = (-1)^{n+1} n!$ for $n \geq 0$. Hence the Taylor expansion about $x_0 = 2$ is

$$\frac{1}{1-x} = -\sum_{n=0}^{\infty} (-1)^n (x-2)^n.$$

Applying the ratio test,

$$\lim_{n \to \infty} \frac{|(x-2)^{n+1}|}{|(x-2)^n|} = \lim_{n \to \infty} |x-2| = |x-2|.$$

The series converges absolutely for $|x - 2| < 1$, but diverges at $x = 1$ and $x = 3$.

17. Applying the ratio test,

$$\lim_{n \to \infty} \frac{|(n+1)x^{n+1}|}{|n\,x^n|} = \lim_{n \to \infty} \frac{n+1}{n} |x| = |x|.$$

The series converges absolutely for $|x| < 1$. Term-by-term differentiation results in

$$y' = \sum_{n=1}^{\infty} n^2 x^{n-1} = 1 + 4x + 9x^2 + 16\,x^3 + \ldots$$

$$y'' = \sum_{n=2}^{\infty} n^2(n-1)\,x^{n-2} = 4 + 18x + 48x^2 + 100x^3 + \ldots$$

Shifting the indices, we can also write

$$y' = \sum_{n=0}^{\infty}(n+1)^2 x^n \quad \text{and} \quad y'' = \sum_{n=0}^{\infty}(n+2)^2(n+1)\,x^n.$$

20. Shifting the index in the second series, that is, setting $n = k+1$,

$$\sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{n=1}^{\infty} a_{n-1} x^n\,.$$

Hence

$$\begin{aligned}\sum_{k=0}^{\infty} a_{k+1} x^k + \sum_{k=0}^{\infty} a_k x^{k+1} &= \sum_{k=0}^{\infty} a_{k+1} x^k + \sum_{k=1}^{\infty} a_{k-1} x^k \\ &= a_1 + \sum_{k=1}^{\infty}(a_{k+1} + a_{k-1}) x^{k+1}.\end{aligned}$$

21. Shifting the index by 2, that is, setting $m = n-2$,

$$\begin{aligned}\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} &= \sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}\,x^m \\ &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n\,.\end{aligned}$$

22. Shift the index down by 2, that is, set $m = n+2$. It follows that

$$\sum_{n=0}^{\infty} a_n x^{n+2} = \sum_{m=2}^{\infty} a_{m-2}\,x^m = \sum_{n=2}^{\infty} a_{n-2}\,x^n\,.$$

24. Clearly,

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n.$$

Shifting the index in the first series, that is, setting $k = n-2$,

$$\begin{aligned}\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} &= \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}\,x^k \\ &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n.\end{aligned}$$

Hence

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\, x^n - \sum_{n=2}^{\infty} n(n-1)a_n\, x^n.$$

Note that when $n = 0$ and $n = 1$, the coefficients in the second series are zero. So

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - n(n-1)a_n\right] x^n.$$

26. Clearly,

$$\sum_{n=1}^{\infty} na_n\, x^{n-1} + x\sum_{n=0}^{\infty} a_n\, x^n = \sum_{n=1}^{\infty} na_n\, x^{n-1} + \sum_{n=0}^{\infty} a_n\, x^{n+1}.$$

Shifting the index in the first series, that is, setting $k = n-1$,

$$\sum_{n=1}^{\infty} na_n\, x^{n-1} = \sum_{k=0}^{\infty}(k+1)a_{k+1}x^k.$$

Shifting the index in the second series, that is, setting $k = n+1$,

$$\sum_{n=0}^{\infty} a_n\, x^{n+1} = \sum_{k=1}^{\infty} a_{k-1}x^k.$$

Combining the series, and starting the summation at $n = 1$,

$$\sum_{n=1}^{\infty} na_n\, x^{n-1} + x\sum_{n=0}^{\infty} a_n\, x^n = a_1 + \sum_{n=1}^{\infty}\left[(n+1)a_{n+1} + a_{n-1}\right] x^n.$$

27. We note that

$$x\sum_{n=2}^{\infty} n(n-1)a_n\, x^{n-2} + \sum_{n=0}^{\infty} a_n\, x^n = \sum_{n=2}^{\infty} n(n-1)a_n\, x^{n-1} + \sum_{n=0}^{\infty} a_n\, x^n.$$

Shifting the index in the first series, that is, setting $k = n-1$,

$$\sum_{n=2}^{\infty} n(n-1)a_n\, x^{n-1} = \sum_{k=1}^{\infty} k(k+1)a_{k+1}x^k = \sum_{k=0}^{\infty} k(k+1)a_{k+1}x^k,$$

since the coefficient of the term associated with $k = 0$ is zero. Combining the series,

$$x\sum_{n=2}^{\infty} n(n-1)a_n\, x^{n-2} + \sum_{n=0}^{\infty} a_n\, x^n = \sum_{n=0}^{\infty}\left[n(n+1)a_{n+1} + a_n\right] x^n.$$

5.2

1.(a,b,d) Let $y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$. Then

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n.$$

Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

or

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - a_n\right] x^n = 0\,.$$

Equating all the coefficients to zero,

$$(n+2)(n+1)a_{n+2} - a_n = 0\,, \qquad n = 0, 1, 2, \ldots\,.$$

We obtain the recurrence relation

$$a_{n+2} = \frac{a_n}{(n+1)(n+2)}\,, \qquad n = 0, 1, 2, \ldots\,.$$

The subscripts differ by two, so for $k = 1, 2, \ldots$

$$a_{2k} = \frac{a_{2k-2}}{(2k-1)2k} = \frac{a_{2k-4}}{(2k-3)(2k-2)(2k-1)2k} = \ldots = \frac{a_0}{(2k)!}$$

and

$$a_{2k+1} = \frac{a_{2k-1}}{2k(2k+1)} = \frac{a_{2k-3}}{(2k-2)(2k-1)2k(2k+1)} = \ldots = \frac{a_1}{(2k+1)!}\,.$$

Hence

$$y = a_0 \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} + a_1 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}\,.$$

The linearly independent solutions are

$$y_1 = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots = \cosh x$$

$$y_2 = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots = \sinh x\,.$$

(c) The Wronskian at 0 is 1.

4.(a,b,d) Let $y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$. Then

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n.$$

Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n + k^2x^2\sum_{n=0}^{\infty}a_nx^n = 0\,.$$

Rewriting the second summation,

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n + \sum_{n=2}^{\infty}k^2a_{n-2}\;x^n = 0\,,$$

that is,

$$2a_2 + 3\cdot 2\,a_3x + \sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2} + k^2a_{n-2}\right]x^n = 0\,.$$

Setting the coefficients equal to zero, we have $a_2 = 0$, $a_3 = 0$, and

$$(n+2)(n+1)a_{n+2} + k^2a_{n-2} = 0\,, \quad \text{ for } \; n = 2,3,4,\ldots\,.$$

The recurrence relation can be written as

$$a_{n+2} = -\frac{k^2a_{n-2}}{(n+2)(n+1)}\,, \quad n = 2,3,4,\ldots\,.$$

The indices differ by four, so a_4 , a_8 , a_{12} , ... are defined by

$$a_4 = -\frac{k^2a_0}{4\cdot 3}\,,\;\; a_8 = -\frac{k^2a_4}{8\cdot 7}\,,\, a_{12} = -\frac{k^2a_8}{12\cdot 11}\,,\;\ldots\,.$$

Similarly, a_5 , a_9 , a_{13} , ... are defined by

$$a_5 = -\frac{k^2a_1}{5\cdot 4}\,,\;\; a_9 = -\frac{k^2a_5}{9\cdot 8}\,,\, a_{13} = -\frac{k^2a_9}{13\cdot 12}\,,\;\ldots\,.$$

The remaining coefficients are zero. Therefore the general solution is

$$\begin{aligned} y \,&= a_0\left[1 - \frac{k^2}{4\cdot 3}x^4 + \frac{k^4}{8\cdot 7\cdot 4\cdot 3}x^8 - \frac{k^6}{12\cdot 11\cdot 8\cdot 7\cdot 4\cdot 3}x^{12} + \ldots\right] + \\ &+ a_1\left[x - \frac{k^2}{5\cdot 4}x^5 + \frac{k^4}{9\cdot 8\cdot 5\cdot 4}x^9 - \frac{k^6}{13\cdot 12\cdot 9\cdot 8\cdot 4\cdot 4}x^{13} + \ldots\right]. \end{aligned}$$

Note that for the even coefficients,

$$a_{4m} = -\frac{k^2a_{4m-4}}{(4m-1)4m}\,, \quad m = 1,2,3,\ldots$$

and for the odd coefficients,

$$a_{4m+1} = -\frac{k^2a_{4m-3}}{4m(4m+1)}\,, \quad m = 1,2,3,\ldots\,.$$

Hence the linearly independent solutions are

$$y_1(x) = 1 + \sum_{m=0}^{\infty}\frac{(-1)^{m+1}(k^2x^4)^{m+1}}{3\cdot 4\cdot 7\cdot 8\,\ldots\,(4m+3)(4m+4)}$$

$$y_2(x) = x\left[1 + \sum_{m=0}^{\infty}\frac{(-1)^{m+1}(k^2x^4)^{m+1}}{4\cdot 5\cdot 8\cdot 9\,\ldots\,(4m+4)(4m+5)}\right].$$

(c) The Wronskian at 0 is 1.

6.(a,b) Let $y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n .$$

Substitution into the ODE results in

$$(2 + x^2) \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n - x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 4 \sum_{n=0}^{\infty} a_n x^n = 0\,.$$

Before proceeding, write

$$x^2 \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n = \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

and

$$x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=1}^{\infty} n\, a_n x^n .$$

It follows that

$$4a_0 + 4a_2 + (3a_1 + 12a_3)x +$$
$$+ \sum_{n=2}^{\infty} \left[2(n+2)(n+1) a_{n+2} + n(n-1) a_n - n\, a_n + 4a_n\right] x^n = 0\,.$$

Equating the coefficients to zero, we find that $a_2 = -a_0$, $a_3 = -a_1/4$, and

$$a_{n+2} = -\frac{n^2 - 2n + 4}{2(n+2)(n+1)}\, a_n\,, \quad n = 0, 1, 2, \ldots\,.$$

The indices differ by two, so for $k = 0, 1, 2, \ldots$

$$a_{2k+2} = -\frac{(2k)^2 - 4k + 4}{2(2k+2)(2k+1)}\, a_{2k}$$

and

$$a_{2k+3} = -\frac{(2k+1)^2 - 4k + 2}{2(2k+3)(2k+2)}\, a_{2k+1} .$$

Hence the linearly independent solutions are

$$y_1(x) = 1 - x^2 + \frac{x^4}{6} - \frac{x^6}{30} + \ldots$$
$$y_2(x) = x - \frac{x^3}{4} + \frac{7x^5}{160} - \frac{19x^7}{1920} + \ldots\,.$$

(c) The Wronskian at 0 is 1.

7.(a,b,d) Let $y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n .$$

Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n + x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 2 \sum_{n=0}^{\infty} a_n x^n = 0 .$$

First write

$$x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=1}^{\infty} n\, a_n x^n .$$

We then obtain

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} + n\, a_n + 2a_n \right] x^n = 0 .$$

It follows that $a_2 = -a_0$ and $a_{n+2} = -a_n/(n+1)$, $n = 0, 1, 2, \ldots$. Note that the indices differ by two, so for $k = 1, 2, \ldots$

$$a_{2k} = -\frac{a_{2k-2}}{2k-1} = \frac{a_{2k-4}}{(2k-3)(2k-1)} = \ldots = \frac{(-1)^k a_0}{1 \cdot 3 \cdot 5 \ldots (2k-1)}$$

and

$$a_{2k+1} = -\frac{a_{2k-1}}{2k} = \frac{a_{2k-3}}{(2k-2)2k} = \ldots = \frac{(-1)^k a_1}{2 \cdot 4 \cdot 6 \ldots (2k)} .$$

Hence the linearly independent solutions are

$$y_1(x) = 1 - \frac{x^2}{1} + \frac{x^4}{1 \cdot 3} - \frac{x^6}{1 \cdot 3 \cdot 5} + \ldots = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{1 \cdot 3 \cdot 5 \ldots (2n-1)}$$

$$y_2(x) = x - \frac{x^3}{2} + \frac{x^5}{2 \cdot 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + \ldots = x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2 \cdot 4 \cdot 6 \ldots (2n)} .$$

(c) The Wronskian at 0 is 1.

9.(a,b,d) Let $y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n .$$

Substitution into the ODE results in

$$(1+x^2)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n - 4x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + 6\sum_{n=0}^{\infty}a_n x^n = 0\,.$$

Before proceeding, write

$$x^2\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n = \sum_{n=2}^{\infty}n(n-1)a_n x^n$$

and

$$x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = \sum_{n=1}^{\infty}n\,a_n x^n.$$

It follows that

$$6a_0 + 2a_2 + (2a_1 + 6a_3)x +$$

$$+\sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2} + n(n-1)a_n - 4n\,a_n + 6a_n\right]x^n = 0\,.$$

Setting the coefficients equal to zero, we obtain $a_2 = -3a_0$, $a_3 = -a_1/3$, and

$$a_{n+2} = -\frac{(n-2)(n-3)}{(n+1)(n+2)}\,a_n\,, \quad n = 0, 1, 2, \ldots\,.$$

Observe that for $n = 2$ and $n = 3$, we obtain $a_4 = a_5 = 0$. Since the indices differ by two, we also have $a_n = 0$ for $n \geq 4$. Therefore the general solution is a polynomial

$$y = a_0 + a_1 x - 3a_0 x^2 - a_1 x^3/3\,.$$

Hence the linearly independent solutions are

$$y_1(x) = 1 - 3x^2 \quad \text{and} \quad y_2(x) = x - x^3/3\,.$$

(c) The Wronskian is $(x^2+1)^2$. At $x = 0$ it is 1.

10.(a,b,d) Let $y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$. Then

$$y'' = \sum_{n=2}^{\infty}n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n.$$

Substitution into the ODE results in

$$(4-x^2)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n + 2\sum_{n=0}^{\infty}a_n x^n = 0\,.$$

First write

$$x^2\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n = \sum_{n=2}^{\infty}n(n-1)a_n x^n.$$

It follows that

$$2a_0 + 8a_2 + (2a_1 + 24a_3)x +$$

$$+\sum_{n=2}^{\infty}\left[4(n+2)(n+1)a_{n+2}-n(n-1)a_n+2a_n\right]x^n=0\,.$$

We obtain $a_2=-a_0/4$, $a_3=-a_1/12$ and

$$4(n+2)a_{n+2}=(n-2)a_n\,,\quad n=0,1,2,\ldots\,.$$

Note that for $n=2$, $a_4=0$. Since the indices differ by two, we also have $a_{2k}=0$ for $k=2,3,\ldots$. On the other hand, for $k=1,2,\ldots,$

$$a_{2k+1}=\frac{(2k-3)a_{2k-1}}{4(2k+1)}=\frac{(2k-5)(2k-3)a_{2k-3}}{4^2(2k-1)(2k+1)}=\ldots=\frac{-a_1}{4^k(2k-1)(2k+1)}\,.$$

Therefore the general solution is

$$y=a_0+a_1x-a_0\frac{x^2}{4}-a_1\sum_{n=1}^{\infty}\frac{x^{2n+1}}{4^n(2n-1)(2n+1)}\,.$$

Hence the linearly independent solutions are $y_1(x)=1-x^2/4$ and

$$y_2(x)=x-\frac{x^3}{12}-\frac{x^5}{240}-\frac{x^7}{2240}-\ldots=x-\sum_{n=1}^{\infty}\frac{x^{2n+1}}{4^n(2n-1)(2n+1)}\,.$$

(c) The Wronskian at 0 is 1.

11.(a,b,d) Let $y=a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots$. Then

$$y'=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

and

$$y''=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n.$$

Substitution into the ODE results in

$$(3-x^2)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n-3x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n-\sum_{n=0}^{\infty}a_nx^n=0\,.$$

Before proceeding, write

$$x^2\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n=\sum_{n=2}^{\infty}n(n-1)a_nx^n$$

and

$$x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n=\sum_{n=1}^{\infty}n\,a_nx^n.$$

It follows that

$$6a_2-a_0+(-4a_1+18a_3)x+$$

$$+\sum_{n=2}^{\infty}\left[3(n+2)(n+1)a_{n+2}-n(n-1)a_n-3n\,a_n-a_n\right]x^n=0.$$

We obtain $a_2 = a_0/6$, $2a_3 = a_1/9$, and

$$3(n+2)a_{n+2} = (n+1)a_n\,, \quad n = 0, 1, 2, \ldots\,.$$

The indices differ by two, so for $k = 1, 2, \ldots$

$$a_{2k} = \frac{(2k-1)a_{2k-2}}{3(2k)} = \frac{(2k-3)(2k-1)a_{2k-4}}{3^2(2k-2)(2k)} = \ldots = \frac{3\cdot 5\,\ldots(2k-1)\,a_0}{3^k\cdot 2\cdot 4\ldots(2k)}$$

and

$$a_{2k+1} = \frac{(2k)a_{2k-1}}{3(2k+1)} = \frac{(2k-2)(2k)a_{2k-3}}{3^2(2k-1)(2k+1)} = \ldots = \frac{2\cdot 4\cdot 6\,\ldots(2k)\,a_1}{3^k\cdot 3\cdot 5\,\ldots(2k+1)}\,.$$

Hence the linearly independent solutions are

$$y_1(x) = 1 + \frac{x^2}{6} + \frac{x^4}{24} + \frac{5x^6}{432} + \ldots = 1 + \sum_{n=1}^{\infty} \frac{3\cdot 5\,\ldots(2n-1)\,x^{2n}}{3^n\cdot 2\cdot 4\,\ldots(2n)}$$

$$y_2(x) = x + \frac{2x^3}{9} + \frac{8x^5}{135} + \frac{16x^7}{945} + \ldots = x + \sum_{n=1}^{\infty} \frac{2\cdot 4\cdot 6\,\ldots(2n)\,x^{2n+1}}{3^n\cdot 3\cdot 5\,\ldots(2n+1)}\,.$$

(c) The Wronskian at 0 is 1.

12.(a,b,d) Let $y = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots$. Then

$$y' = \sum_{n=1}^{\infty} na_nx^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n.$$

Substitution into the ODE results in

$$(1-x)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n + x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty}a_nx^n = 0\,.$$

Before proceeding, write

$$x\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n = \sum_{n=1}^{\infty}(n+1)n\,a_{n+1}x^n$$

and

$$x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = \sum_{n=1}^{\infty}n\,a_nx^n.$$

It follows that

$$2a_2 - a_0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} - (n+1)n\,a_{n+1} + n\,a_n - a_n\right]x^n = 0.$$

We obtain $a_2 = a_0/2$ and

$$(n+2)(n+1)a_{n+2} - (n+1)n\,a_{n+1} + (n-1)a_n = 0$$

for $n = 0, 1, 2, \ldots$. Writing out the individual equations,

$$\begin{aligned} 3 \cdot 2\, a_3 - 2 \cdot 1\, a_2 &= 0 \\ 4 \cdot 3\, a_4 - 3 \cdot 2\, a_3 + a_2 &= 0 \\ 5 \cdot 4\, a_5 - 4 \cdot 3\, a_4 + 2\, a_3 &= 0 \\ 6 \cdot 5\, a_6 - 5 \cdot 4\, a_5 + 3\, a_4 &= 0 \\ &\vdots \end{aligned}$$

The coefficients are calculated successively as $a_3 = a_0/(2 \cdot 3)$, $a_4 = a_3/2 - a_2/12 = a_0/24$, $a_5 = 3a_4/5 - a_3/10 = a_0/120$, ... We can now see that for $n \geq 2$, a_n is proportional to a_0. In fact, for $n \geq 2$, $a_n = a_0/(n!)$. Therefore the general solution is

$$y = a_0 + a_1 x + \frac{a_0 x^2}{2!} + \frac{a_0 x^3}{3!} + \frac{a_0 x^4}{4!} + \ldots .$$

Hence the linearly independent solutions are $y_2(x) = x$ and

$$y_1(x) = 1 + \sum_{n=2}^{\infty} \frac{x^n}{n!} = e^x - x.$$

(c) The Wronskian is $e^x(1 - x)$. At $x = 0$ it is 1.

13.(a,b,d) Let $y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n .$$

Substitution into the ODE results in

$$2 \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n + x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 3 \sum_{n=0}^{\infty} a_n x^n = 0 \, .$$

First write

$$x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=1}^{\infty} n\, a_n x^n .$$

We then obtain

$$4a_2 + 3a_0 + \sum_{n=1}^{\infty} \left[2(n+2)(n+1) a_{n+2} + n\, a_n + 3a_n \right] x^n = 0 \, .$$

It follows that $a_2 = -3a_0/4$ and

$$2(n+2)(n+1) a_{n+2} + (n+3) a_n = 0$$

for $n = 0, 1, 2, \ldots$. The indices differ by two, so for $k = 1, 2, \ldots$

$$\begin{aligned} a_{2k} &= -\frac{(2k+1)a_{2k-2}}{2(2k-1)(2k)} = \frac{(2k-1)(2k+1)a_{2k-4}}{2^2(2k-3)(2k-2)(2k-1)(2k)} = \ldots \\ &= \frac{(-1)^k 3\cdot 5 \ldots (2k+1)}{2^k\,(2k)!}\,a_0\,. \end{aligned}$$

and

$$\begin{aligned} a_{2k+1} &= -\frac{(2k+2)a_{2k-1}}{2(2k)(2k+1)} = \frac{(2k)(2k+2)a_{2k-3}}{2^2(2k-2)(2k-1)(2k)(2k+1)} = \ldots \\ &= \frac{(-1)^k\, 4\cdot 6 \ldots (2k)(2k+2)}{2^k\,(2k+1)!}\,a_1\,. \end{aligned}$$

Hence the linearly independent solutions are

$$y_1(x) = 1 - \frac{3}{4}x^2 + \frac{5}{32}x^4 - \frac{7}{384}x^6 + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n 3\cdot 5 \ldots (2n+1)}{2^n\,(2n)!}x^{2n}$$

$$y_2(x) = x - \frac{1}{3}x^3 + \frac{1}{20}x^5 - \frac{1}{210}x^7 + \ldots = x + \sum_{n=1}^{\infty} \frac{(-1)^n 4\cdot 6 \ldots (2n+2)}{2^n\,(2n+1)!}x^{2n+1}.$$

(c) The Wronskian at 0 is 1.

15.(a) From Problem 2, we have

$$y_1(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n\,n!} \quad \text{and} \quad y_2(x) = \sum_{n=0}^{\infty} \frac{2^n\,n!\,x^{2n+1}}{(2n+1)!}\,.$$

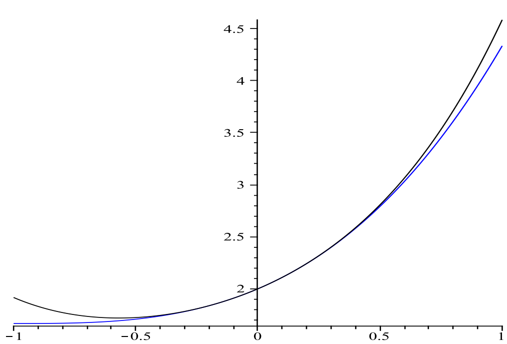
Since $a_0 = y(0)$ and $a_1 = y'(0)$, we have $y(x) = 2\,y_1(x) + y_2(x)$. That is,

$$y(x) = 2 + x + x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{15}x^5 + \frac{1}{24}x^6 + \ldots\,.$$

The four- and five-term polynomial approximations are

$$p_4 = 2 + x + x^2 + x^3/3, \text{ and } p_5 = 2 + x + x^2 + x^3/3 + x^4/4\,.$$

(b)



(c) The four-term approximation p_4 appears to be reasonably accurate (within 10%) on the interval $|x| < 0.7$.

17.(a) From Problem 7, the linearly independent solutions are

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{1 \cdot 3 \cdot 5 \ldots (2n-1)} \text{ and } y_2(x) = x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2 \cdot 4 \cdot 6 \ldots (2n)}.$$

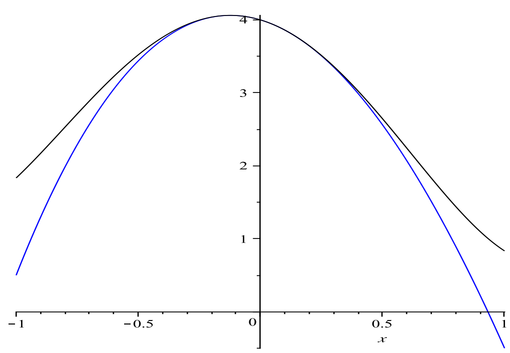
Since $a_0 = y(0)$ and $a_1 = y'(0)$, we have $y(x) = 4\,y_1(x) - y_2(x)$. That is,

$$y(x) = 4 - x - 4x^2 + \frac{1}{2}x^3 + \frac{4}{3}x^4 - \frac{1}{8}x^5 - \frac{4}{15}x^6 + \ldots.$$

The four- and five-term polynomial approximations are

$$p_4 = 4 - x - 4x^2 + \frac{1}{2}x^3, \text{ and } p_5 = 4 - x - 4x^2 + \frac{1}{2}x^3 + \frac{4}{3}x^4.$$

(b)



(c) The four-term approximation p_4 appears to be reasonably accurate (within 10%) on the interval $|x| < 0.5$.

18.(a) From Problem 12, we have

$$y_1(x) = 1 + \sum_{n=2}^{\infty} \frac{x^n}{n!} \quad \text{and} \quad y_2(x) = x.$$

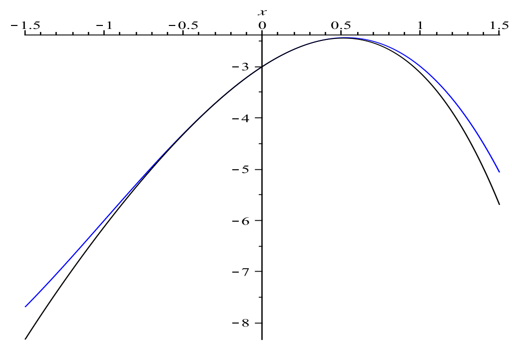
Since $a_0 = y(0)$ and $a_1 = y'(0)$, we have $y(x) = -3\,y_1(x) + 2\,y_2(x)$. That is,

$$y(x) = -3 + 2x - \frac{3}{2}x^2 - \frac{1}{2}x^3 - \frac{1}{8}x^4 - \frac{1}{40}x^5 - \frac{1}{240}x^6 + \ldots.$$

The four- and five-term polynomial approximations are

$$p_4 = -3 + 2x - \frac{3}{2}x^2 - \frac{1}{2}x^3, \text{ and } p_5 = -3 + 2x - \frac{3}{2}x^2 - \frac{1}{2}x^3 - \frac{1}{8}x^4.$$

(b)



(c) The four-term approximation p_4 appears to be reasonably accurate (within 10%) on the interval $|x| < 0.9$.

20. Two linearly independent solutions of Airy's equation (about $x_0 = 0$) are

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{2 \cdot 3 \ldots (3n-1)(3n)}$$

$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{x^{3n+1}}{3 \cdot 4 \ldots (3n)(3n+1)}\,.$$

Applying the ratio test to the terms of $y_1(x)$,

$$\lim_{n\to\infty} \frac{\left|2 \cdot 3 \ldots (3n-1)(3n)\, x^{3n+3}\right|}{\left|2 \cdot 3 \ldots (3n+2)(3n+3)\, x^{3n}\right|} = \lim_{n\to\infty} \frac{1}{(3n+1)(3n+2)(3n+3)}\, |x|^3 = 0\,.$$

Similarly, applying the ratio test to the terms of $y_2(x)$,

$$\lim_{n\to\infty} \frac{\left|3 \cdot 4 \ldots (3n)(3n+1)\, x^{3n+4}\right|}{\left|3 \cdot 4 \ldots (3n+3)(3n+4)\, x^{3n+1}\right|} = \lim_{n\to\infty} \frac{1}{(3n+2)(3n+3)(3n+4)}\, |x|^3 = 0\,.$$

Hence both series converge absolutely for all x.

21. Let $y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n.$$

Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n - 2x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \lambda \sum_{n=0}^{\infty} a_n x^n = 0\,.$$

First write

$$x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n=\sum_{n=1}^{\infty}n\,a_nx^n.$$

We then obtain

$$2a_2+\lambda\,a_0+\sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2}-2n\,a_n+\lambda\,a_n\right]x^n=0\,.$$

Setting the coefficients equal to zero, it follows that

$$a_{n+2}=\frac{(2n-\lambda)}{(n+1)(n+2)}\,a_n$$

for $n=0,1,2,\ldots$. Note that the indices differ by two, so for $k=1,2,\ldots$

$$\begin{aligned}a_{2k}&=\frac{(4k-4-\lambda)a_{2k-2}}{(2k-1)2k}=\frac{(4k-8-\lambda)(4k-4-\lambda)a_{2k-4}}{(2k-3)(2k-2)(2k-1)2k}\\&=(-1)^k\frac{\lambda\ldots(\lambda-4k+8)(\lambda-4k+4)}{(2k)!}\,a_0\,.\end{aligned}$$

and

$$\begin{aligned}a_{2k+1}&=\frac{(4k-2-\lambda)a_{2k-1}}{2k(2k+1)}=\frac{(4k-6-\lambda)(4k-2-\lambda)a_{2k-3}}{(2k-2)(2k-1)2k(2k+1)}\\&=(-1)^k\frac{(\lambda-2)\ldots(\lambda-4k+6)(\lambda-4k+2)}{(2k+1)!}\,a_1.\end{aligned}$$

Hence the linearly independent solutions of the Hermite equation (about $x_0=0$) are

$$y_1(x)=1-\frac{\lambda}{2!}x^2+\frac{\lambda(\lambda-4)}{4!}x^4-\frac{\lambda(\lambda-4)(\lambda-8)}{6\,!}x^6+\ldots$$

$$y_2(x)=x-\frac{\lambda-2}{3\,!}x^3+\frac{(\lambda-2)(\lambda-6)}{5\,!}x^5-\frac{(\lambda-2)(\lambda-6)(\lambda-10)}{7\,!}x^7+\ldots.$$

(b) Based on the recurrence relation

$$a_{n+2}=\frac{(2n-\lambda)}{(n+1)(n+2)}\,a_n\,,$$

the series solution will terminate as long as λ is a nonnegative even integer. If $\lambda=2m$, then one or the other of the solutions in part (b) will contain at most $m/2+1$ terms. In particular, we obtain the polynomial solutions corresponding to $\lambda=0,2,4,6,8,10$:

$\lambda=0$	$y_1(x)=1$
$\lambda=2$	$y_2(x)=x$
$\lambda=4$	$y_1(x)=1-2x^2$
$\lambda=6$	$y_2(x)=x-2x^3/3$
$\lambda=8$	$y_1(x)=1-4x^2+4x^4/3$
$\lambda=10$	$y_2(x)=x-4x^3/3+4x^5/15$

(c) Observe that if $\lambda = 2n$, and $a_0 = a_1 = 1$, then

$$a_{2k} = (-1)^k \frac{2n \ldots (2n-4k+8)(2n-4k+4)}{(2k)!}$$

and

$$a_{2k+1} = (-1)^k \frac{(2n-2) \ldots (2n-4k+6)(2n-4k+2)}{(2k+1)!}.$$

for $k = 1, 2, \ldots [n/2]$. It follows that the coefficient of x^n, in y_1 and y_2, is

$$a_n = \begin{cases} (-1)^k \frac{4^k\, k!}{(2k)!} & \text{for } n = 2k \\ (-1)^k \frac{4^k\, k!}{(2k+1)!} & \text{for } n = 2k+1 \end{cases}$$

Then by definition,

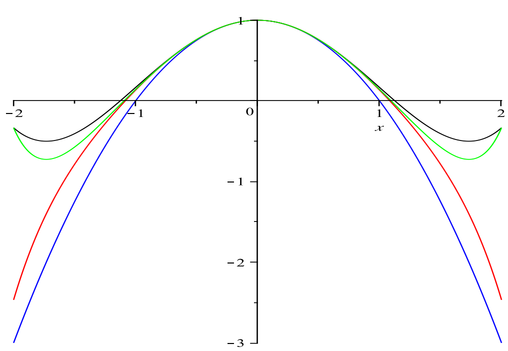
$$H_n(x) = \begin{cases} (-1)^k\ 2^n \frac{(2k)!}{4^k\, k!} y_1(x) = (-1)^k\ \frac{(2k)!}{k!} y_1(x) & \text{for } n = 2k \\ (-1)^k\ 2^n \frac{(2k+1)!}{4^k\, k!} y_2(x) = (-1)^k\ \frac{2\ (2k+1)!}{k!} y_2(x) & \text{for } n = 2k+1 \end{cases}$$

Therefore the first six Hermite polynomials are

$$\begin{aligned} H_0(x) &= 1 \\ H_1(x) &= 2x \\ H_2(x) &= 4x^2 - 2 \\ H_3(x) &= 8x^8 - 12x \\ H_4(x) &= 16x^4 - 48x^2 + 12 \\ H_5(x) &= 32x^5 - 160x^3 + 120x \end{aligned}$$

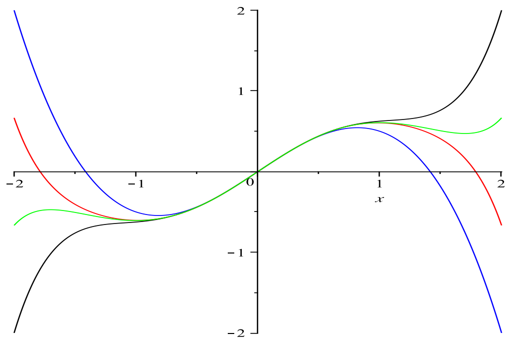
24. The series solution is given by

$$y(x) = 1 - x^2 + \frac{x^4}{6} - \frac{x^6}{30} + \frac{x^8}{120} + \ldots.$$



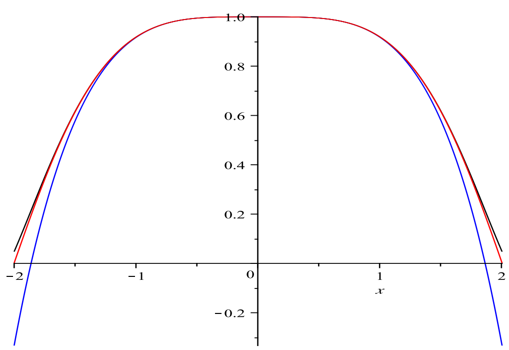
25. The series solution is given by

$$y(x) = x - \frac{x^3}{2} + \frac{x^5}{2 \cdot 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + \frac{x^9}{2 \cdot 4 \cdot 6 \cdot 8} - \ldots.$$



27. The series solution is given by

$$y(x) = 1 - \frac{x^4}{12} + \frac{x^8}{672} - \frac{x^{12}}{88704} + \ldots .$$



28. Let $y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n .$$

Substitution into the ODE results in

$$(1-x) \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n + x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - 2 \sum_{n=0}^{\infty} a_n x^n = 0 .$$

After appropriately shifting the indices, it follows that

$$2a_2 - 2a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} - (n+1)n\, a_{n+1} + n\, a_n - 2\, a_n \right] x^n = 0.$$

We find that $a_2 = a_0$ and

$$(n+2)(n+1) a_{n+2} - (n+1)n\, a_{n+1} + (n-2) a_n = 0$$

for $n = 1, 2, \ldots$. Writing out the individual equations,

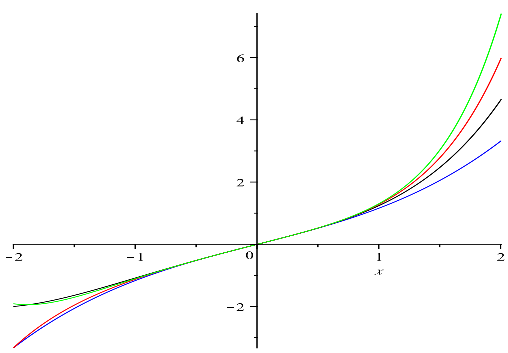
$$\begin{aligned} 3\cdot 2\, a_3 - 2\cdot 1\, a_2 - a_1 &= 0 \\ 4\cdot 3\, a_4 - 3\cdot 2\, a_3 &= 0 \\ 5\cdot 4\, a_5 - 4\cdot 3\, a_4 + a_3 &= 0 \\ 6\cdot 5\, a_6 - 5\cdot 4\, a_5 + 2\, a_4 &= 0 \\ &\vdots \end{aligned}$$

Since $a_0 = 0$ and $a_1 = 1$, the remaining coefficients satisfy the equations

$$\begin{aligned} 3\cdot 2\, a_3 - 1 &= 0 \\ 4\cdot 3\, a_4 - 3\cdot 2\, a_3 &= 0 \\ 5\cdot 4\, a_5 - 4\cdot 3\, a_4 + a_3 &= 0 \\ 6\cdot 5\, a_6 - 5\cdot 4\, a_5 + 2\, a_4 &= 0 \\ &\vdots \end{aligned}$$

That is, $a_3 = 1/6$, $a_4 = 1/12$, $a_5 = 1/24$, $a_6 = 1/45, \ldots$. Hence the series solution of the initial value problem is

$$y(x) = x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{24}x^5 + \frac{1}{45}x^6 + \frac{13}{1008}x^7 + \ldots .$$



5.3

2. Let $y = \phi(x)$ be a solution of the initial value problem. First note that

$$y'' = -(\sin x)y' - (\cos x)y .$$

Differentiating twice,

$$\begin{aligned} y''' &= -(\sin x)y'' - 2(\cos x)y' + (\sin x)y \\ y^{(4)} &= -(\sin x)y''' - 3(\cos x)y'' + 3(\sin x)y' + (\cos x)y . \end{aligned}$$

Given that $\phi(0) = 0$ and $\phi'(0) = 1$, the first equation gives $\phi''(0) = 0$ and the last two equations give $\phi'''(0) = -2$ and $\phi^{(4)}(0) = 0$.

4. Let $y = \phi(x)$ be a solution of the initial value problem. First note that

$$y'' = -x^2 y' - (\sin x)y\,.$$

Differentiating twice,

$$y''' = -x^2 y'' - (2x + \sin x)y' - (\cos x)y$$
$$y^{(4)} = -x^2 y''' - (4x + \sin x)y'' - (2 + 2\cos x)y' + (\sin x)y\,.$$

Given that $\phi(0) = a_0$ and $\phi'(0) = a_1$, the first equation gives $\phi''(0) = 0$ and the last two equations give $\phi'''(0) = -a_0$ and $\phi^{(4)}(0) = -4a_1$.

5. Clearly, $p(x) = 4$ and $q(x) = 6x$ are analytic for all x. Hence the series solutions converge everywhere.

8. The only root of $P(x) = x$ is zero. Hence $\rho_{min} = 1$.

12. The Taylor series expansion of e^x, about $x_0 = 0$, is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}\,.$$

Let $y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$. Substituting into the ODE,

$$\left[\sum_{n=0}^{\infty} \frac{x^n}{n!}\right]\left[\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n\right] + x\sum_{n=0}^{\infty} a_n x^n = 0\,.$$

First note that

$$x\sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_{n-1}x^n = a_0\,x + a_1 x^2 + a_2 x^3 + \ldots + a_{n-1}x^n + \ldots\,.$$

The coefficient of x^n in the product of the two series is

$$c_n = 2a_2\frac{1}{n!} + 6a_3\frac{1}{(n-1)!} + 12a_4\frac{1}{(n-2)!} + \ldots$$
$$\ldots + (n+1)n\,a_{n+1} + (n+2)(n+1)a_{n+2}\,.$$

Expanding the individual series, it follows that

$$2a_2 + (2a_2 + 6a_3)x + (a_2 + 6a_3 + 12a_4)x^2 + (a_2 + 6a_3 + 12a_4 + 20a_5)x^3 + \ldots$$
$$\ldots + a_0\,x + a_1 x^2 + a_2 x^3 + \ldots = 0\,.$$

Setting the coefficients equal to zero, we obtain the system $2a_2 = 0$, $2a_2 + 6a_3 + a_0 = 0$, $a_2 + 6a_3 + 12a_4 + a_1 = 0$, $a_2 + 6a_3 + 12a_4 + 20a_5 + a_2 = 0, \ldots$. Hence the general solution is

$$y(x) = a_0 + a_1 x - a_0\frac{x^3}{6} + (a_0 - a_1)\frac{x^4}{12} + (2a_1 - a_0)\frac{x^5}{40} + (\frac{4}{3}a_0 - 2a_1)\frac{x^6}{120} + \ldots\,.$$

We find that two linearly independent solutions ($W(y_1, y_2)(0) = 1$) are

$$y_1(x) = 1 - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{40} + \ldots$$

$$y_2(x) = x - \frac{x^4}{12} + \frac{x^5}{20} - \frac{x^6}{60} + \ldots$$

Since $p(x) = 0$ and $q(x) = xe^{-x}$ converge everywhere, $\rho = \infty$.

13. The Taylor series expansion of $\cos x$, about $x_0 = 0$, is

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

Let $y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$. Substituting into the ODE,

$$\left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}\right] \left[\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n\right] + \sum_{n=1}^{\infty} n a_n x^n - 2 \sum_{n=0}^{\infty} a_n x^n = 0.$$

The coefficient of x^n in the product of the two series is

$$c_n = 2a_2 b_n + 6a_3 b_{n-1} + 12 a_4 b_{n-2} + \ldots + (n+1) n a_{n+1} b_1 + (n+2)(n+1) a_{n+2} b_0,$$

in which $\cos x = b_0 + b_1 x + b_2 x^2 + \ldots + b_n x^n + \ldots$. It follows that

$$2a_2 - 2a_0 + \sum_{n=1}^{\infty} c_n x^n + \sum_{n=1}^{\infty} (n-2) a_n x^n = 0.$$

Expanding the product of the series, it follows that

$$2a_2 - 2a_0 + 6a_3 x + (-a_2 + 12 a_4) x^2 + (-3a_3 + 20 a_5) x^3 + \ldots$$
$$\ldots - a_1 x + a_3 x^3 + 2 a_4 x^4 + \ldots = 0.$$

Setting the coefficients equal to zero, $a_2 - a_0 = 0$, $6a_3 - a_1 = 0$, $-a_2 + 12 a_4 = 0$, $-3a_3 + 20 a_5 + a_3 = 0$, $\ldots$. Hence the general solution is

$$y(x) = a_0 + a_1 x + a_0 x^2 + a_1 \frac{x^3}{6} + a_0 \frac{x^4}{12} + a_1 \frac{x^5}{60} + a_0 \frac{x^6}{120} + a_1 \frac{x^7}{560} + \ldots.$$

We find that two linearly independent solutions ($W(y_1, y_2)(0) = 1$) are

$$y_1(x) = 1 + x^2 + \frac{x^4}{12} + \frac{x^6}{120} + \ldots$$

$$y_2(x) = x + \frac{x^3}{6} + \frac{x^5}{60} + \frac{x^7}{560} + \ldots$$

The nearest zero of $P(x) = \cos x$ is at $x = \pm\pi/2$. Hence $\rho_{min} = \pi/2$.

14. The Taylor series expansion of $\ln(1+x)$, about $x_0 = 0$, is

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}.$$

Let $y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$. Substituting into the ODE,

$$\left[\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}\right] \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n$$

$$+\left[\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n}\right]\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - x\sum_{n=0}^{\infty}a_n x^n = 0\,.$$

The first product is the series

$$2a_2 + (-2a_2 + 6a_3)x + (a_2 - 6a_3 + 12a_4)x^2 + (-a_2 + 6a_3 - 12a_4 + 20a_5)x^3 + \ldots$$

The second product is the series

$$a_1 x + (2a_2 - a_1/2)x^2 + (3a_3 - a_2 + a_1/3)x^3 + (4a_4 - 3a_3/2 + 2a_2/3 - a_1/4)x^3 + \ldots$$

Combining the series and equating the coefficients to zero, we obtain

$$\begin{aligned} 2a_2 &= 0 \\ -2a_2 + 6a_3 + a_1 - a_0 &= 0 \\ 12a_4 - 6a_3 + 3a_2 - 3a_1/2 &= 0 \\ 20a_5 - 12a_4 + 9a_3 - 3a_2 + a_1/3 &= 0 \\ &\vdots \end{aligned}$$

Hence the general solution is

$$y(x) = a_0 + a_1 x + (a_0 - a_1)\frac{x^3}{6} + (2a_0 + a_1)\frac{x^4}{24} + a_1\frac{7x^5}{120} + (\frac{5}{3}a_1 - a_0)\frac{x^6}{120} + \ldots$$

We find that two linearly independent solutions ($W(y_1, y_2)(0) = 1$) are

$$\begin{aligned} y_1(x) &= 1 + \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^6}{120} + \ldots \\ y_2(x) &= x - \frac{x^3}{6} + \frac{x^4}{24} + \frac{7x^5}{120} + \ldots \end{aligned}$$

The coefficient $p(x) = e^x \ln(1 + x)$ is analytic at $x_0 = 0$, but its power series has a radius of convergence $\rho = 1$.

15. If $y_1 = x$ and $y_2 = x^2$ are solutions, then substituting y_2 into the ODE results in

$$2\,P(x) + 2x\,Q(x) + x^2 R(x) = 0\,.$$

Setting $x = 0$, we find that $P(0) = 0$. Similarly, substituting y_1 into the ODE results in $Q(0) = 0$. Therefore $P(x)/Q(x)$ and $R(x)/P(x)$ may not be analytic. If they were, Theorem 3.2.1 would guarantee that y_1 and y_2 were the only two solutions. But note that an arbitrary value of $y(0)$ cannot be a linear combination of $y_1(0)$ and $y_2(0)$. Hence $x_0 = 0$ must be a singular point.

16. Let $y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$. Substituting into the ODE,

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}\,x^n - \sum_{n=0}^{\infty}a_n x^n = 0\,.$$

That is,

$$\sum_{n=0}^{\infty}\left[(n+1)a_{n+1} - a_n\right]x^n = 0\,.$$

Setting the coefficients equal to zero, we obtain

$$a_{n+1} = \frac{a_n}{n+1}$$

for $n = 0, 1, 2, \ldots$. It is easy to see that $a_n = a_0/(n\,!)$. Therefore the general solution is

$$y(x) = a_0\left[1 + x + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \ldots\right] = a_0 e^x.$$

The coefficient $a_0 = y(0)$, which can be arbitrary.

17. Let $y = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots$. Substituting into the ODE,

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}\,x^n - x\sum_{n=0}^{\infty}a_nx^n = 0\,.$$

That is,

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}\,x^n - \sum_{n=1}^{\infty}a_{n-1}x^n = 0\,.$$

Combining the series, we have

$$a_1 + \sum_{n=1}^{\infty}\left[(n+1)a_{n+1} - a_{n-1}\right]x^n = 0\,.$$

Setting the coefficient equal to zero, $a_1 = 0$ and $a_{n+1} = a_{n-1}/(n+1)$ for $n = 1, 2, \ldots$. Note that the indices differ by two, so for $k = 1, 2, \ldots$

$$a_{2k} = \frac{a_{2k-2}}{(2k)} = \frac{a_{2k-4}}{(2k-2)(2k)} = \ldots = \frac{a_0}{2\cdot 4\,\ldots(2k)}$$

and

$$a_{2k+1} = 0\,.$$

Hence the general solution is

$$y(x) = a_0\left[1 + \frac{x^2}{2} + \frac{x^4}{2^2\,2\,!} + \frac{x^6}{2^3\,3\,!} + \ldots + \frac{x^{2n}}{2^n\,n\,!} + \ldots\right] = a_0 e^{x^2/2}.$$

The coefficient $a_0 = y(0)$, which can be arbitrary.

19. Let $y = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots$. Substituting into the ODE,

$$(1-x)\sum_{n=0}^{\infty}(n+1)a_{n+1}\,x^n - \sum_{n=0}^{\infty}a_nx^n = 0\,.$$

That is,

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}\,x^n - \sum_{n=1}^{\infty}n\,a_n\,x^n - \sum_{n=0}^{\infty}a_nx^n = 0\,.$$

Combining the series, we have

$$a_1 - a_0 + \sum_{n=1}^{\infty}\left[(n+1)a_{n+1} - n\,a_n - a_n\right]x^n = 0\,.$$

Setting the coefficients equal to zero, $a_1 = a_0$ and $a_{n+1} = a_n$ for $n = 0, 1, 2, \ldots$. Hence the general solution is

$$y(x) = a_0\left[1 + x + x^2 + x^3 + \ldots + x^n + \ldots\right] = a_0 \frac{1}{1-x}\,.$$

The coefficient $a_0 = y(0)$, which can be arbitrary.

21. Let $y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$. Substituting into the ODE,

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}\,x^n + x\sum_{n=0}^{\infty} a_n x^n = 1 + x\,.$$

That is,

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}\,x^n + \sum_{n=1}^{\infty} a_{n-1} x^n = 1 + x\,.$$

Combining the series, and the nonhomogeneous terms, we have

$$(a_1 - 1) + (2a_2 + a_0 - 1)x + \sum_{n=2}^{\infty}\left[(n+1)a_{n+1} + a_{n-1}\right]x^n = 0\,.$$

Setting the coefficients equal to zero, we obtain $a_1 = 1$, $2a_2 + a_0 - 1 = 0$, and

$$a_n = -\frac{a_{n-2}}{n}\,, \qquad n = 3, 4, \ldots\,.$$

The indices differ by two, so for $k = 2, 3, \ldots$

$$a_{2k} = -\frac{a_{2k-2}}{(2k)} = \frac{a_{2k-4}}{(2k-2)(2k)} = \ldots = \frac{(-1)^{k-1}\,a_2}{4\cdot 6\ldots(2k)} = \frac{(-1)^k(a_0-1)}{2\cdot 4\cdot 6\ldots(2k)}\,,$$

and for $k = 1, 2, \ldots$

$$a_{2k+1} = -\frac{a_{2k-1}}{(2k+1)} = \frac{a_{2k-3}}{(2k-1)(2k+1)} = \ldots = \frac{(-1)^k}{3\cdot 5\ldots(2k+1)}\,.$$

Hence the general solution is

$$y(x) = a_0 + x + \frac{1-a_0}{2}x^2 - \frac{x^3}{3} + a_0\frac{x^4}{2^2\,2\,!} + \frac{x^5}{3\cdot 5} - a_0\frac{x^6}{2^3\,3\,!} - \ldots$$

Collecting the terms containing a_0,

$$y(x) = a_0\left[1 - \frac{x^2}{2} + \frac{x^4}{2^2\,2\,!} - \frac{x^6}{2^3\,3\,!} + \ldots\right]$$
$$+\left[x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{2^2\,2\,!} + \frac{x^5}{3\cdot 5} + \frac{x^6}{2^3\,3\,!} - \frac{x^7}{3\cdot 5\cdot 7} + \ldots\right].$$

Upon inspection, we find that

$$y(x) = a_0 e^{-x^2/2} + \left[x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{2^2\,2\,!} + \frac{x^5}{3\cdot 5} + \frac{x^6}{2^3\,3\,!} - \frac{x^7}{3\cdot 5\cdot 7} + \ldots\right].$$

Note that the given ODE is first order linear, with integrating factor $\mu(x) = e^{x^2/2}$. The general solution is given by

$$y(x) = e^{-x^2/2}\int_0^x e^{u^2/2}du + (y(0)-1)e^{-x^2/2} + 1\,.$$

23. If $\alpha = 0$, then $y_1(x) = 1$. If $\alpha = 2n$, then $a_{2m} = 0$ for $m \geq n+1$. As a result,

$$y_1(x) = 1$$
$$+ \sum_{m=1}^{n} (-1)^m \frac{2^m n(n-1)\ldots(n-m+1)(2n+1)(2n+3)\ldots(2n+2m-1)}{(2m)!} x^{2m}.$$

$\alpha = 0$	1
$\alpha = 2$	$1 - 3x^2$
$\alpha = 4$	$1 - 10x^2 + \frac{35}{3}x^4$

If $\alpha = 2n+1$, then $a_{2m+1} = 0$ for $m \geq n+1$. As a result,

$$y_2(x) = x$$
$$+ \sum_{m=1}^{n} (-1)^m \frac{2^m n(n-1)\ldots(n-m+1)(2n+3)(2n+5)\ldots(2n+2m+1)}{(2m+1)!} x^{2m+1}.$$

$\alpha = 1$	x
$\alpha = 3$	$x - \frac{5}{3}x^3$
$\alpha = 5$	$x - \frac{14}{3}x^3 + \frac{21}{5}x^5$

24.(a) Based on Problem 23,

$\alpha = 0$	1	$y_1(1) = 1$
$\alpha = 2$	$1 - 3x^2$	$y_1(1) = -2$
$\alpha = 4$	$1 - 10x^2 + \frac{35}{3}x^4$	$y_1(1) = \frac{8}{3}$

Normalizing the polynomials, we obtain

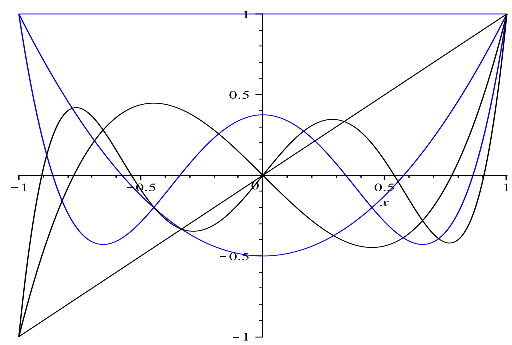
$$P_0(x) = 1$$
$$P_2(x) = -\frac{1}{2} + \frac{3}{2}x^2$$
$$P_4(x) = \frac{3}{8} - \frac{15}{4}x^2 + \frac{35}{8}x^4$$

$\alpha = 1$	x	$y_2(1) = 1$
$\alpha = 3$	$x - \frac{5}{3}x^3$	$y_2(1) = -\frac{2}{3}$
$\alpha = 5$	$x - \frac{14}{3}x^3 + \frac{21}{5}x^5$	$y_2(1) = \frac{8}{15}$

Similarly,

$$P_1(x) = x$$
$$P_3(x) = -\frac{3}{2}x + \frac{5}{2}x^3$$
$$P_5(x) = \frac{15}{8}x - \frac{35}{4}x^3 + \frac{63}{8}x^5$$

(b)



(c) $P_0(x)$ has no roots. $P_1(x)$ has one root at $x = 0$. The zeros of $P_2(x)$ are at $x = \pm 1/\sqrt{3}$. The zeros of $P_3(x)$ are $x = 0, \pm\sqrt{3/5}$. The roots of $P_4(x)$ are given by $x^2 = (15 + 2\sqrt{30})/35$, $(15 - 2\sqrt{30})/35$. The roots of $P_5(x)$ are given by $x = 0$ and $x^2 = (35 + 2\sqrt{70})/63$, $(35 - 2\sqrt{70})/63$.

25. Observe that

$$P_n(-1) = \frac{(-1)^n}{2^n} \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)!}{k!(n-k)!(n-2k)!} = (-1)^n P_n(1).$$

But $P_n(1) = 1$ for all nonnegative integers n.

27. We have

$$(x^2 - 1)^n = \sum_{k=0}^{n} \frac{(-1)^{n-k}\, n!}{k!(n-k)!} x^{2k},$$

which is a polynomial of degree $2n$. Differentiating n times,

$$\frac{d^n}{dx^n}(x^2 - 1)^n = \sum_{k=\mu}^{n} \frac{(-1)^{n-k}\, n!}{k!(n-k)!}(2k)(2k-1)\ldots(2k-n+1)x^{2k-n},$$

in which the lower index is $\mu = [n/2] + 1$. Note that if $n = 2m+1$, then $\mu = m + 1$. Now shift the index, by setting $k = n - j$. Hence

$$\begin{aligned}\frac{d^n}{dx^n}(x^2 - 1)^n &= \sum_{j=0}^{[n/2]} \frac{(-1)^j\, n!}{(n-j)!j!}(2n-2j)(2n-2j-1)\ldots(n-2j+1)x^{n-2j} \\ &= n! \sum_{j=0}^{[n/2]} \frac{(-1)^j (2n-2j)!}{(n-j)!j!(n-2j)!} x^{n-2j}.\end{aligned}$$

Based on Problem 25,

$$\frac{d^n}{dx^n}(x^2 - 1)^n = n!\, 2^n P_n(x).$$

29. Since the $n+1$ polynomials $P_0,\ P_1,\ \ldots,\ P_n$ are linearly independent, and the degree of P_k is k, any polynomial f of degree n can be expressed as a linear combination

$$f(x) = \sum_{k=0}^{n} a_k P_k(x)\,.$$

Multiplying both sides by P_m and integrating,

$$\int_{-1}^{1} f(x)P_m(x)dx = \sum_{k=0}^{n} a_k \int_{-1}^{1} P_k(x)P_m(x)dx\,.$$

Based on Problem 28,

$$\int_{-1}^{1} P_k(x)P_m(x)dx = \frac{2}{2m+1}\delta_{km}.$$

Hence

$$\int_{-1}^{1} f(x)P_m(x)dx = \frac{2}{2m+1}\,a_m\,.$$

5.4

1. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where $F(r) = r(r-1) + 4r + 2 = r^2 + 3r + 2$. The roots are $r = -2\,,\ -1$. Hence the general solution, for $x \neq 0$, is $y = c_1\, x^{-2} + c_2\, x^{-1}$.

3. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where $F(r) = r(r-1) - 3r + 4 = r^2 - 4r + 4$. The root is $r = 2$, with multiplicity two. Hence the general solution, for $x \neq 0$, is $y = (c_1 + c_2\ \ln|x|)\, x^2$.

5. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where $F(r) = r(r-1) - r + 1 = r^2 - 2r + 1$. The root is $r = 1$, with multiplicity two. Hence the general solution, for $x \neq 0$, is $y = (c_1 + c_2\ \ln|x|)\, x$.

6. Substitution of $y = (x-1)^r$ results in the quadratic equation $F(r) = 0$, where $F(r) = r^2 + 7r + 12$. The roots are $r = -3\,,\ -4$. Hence the general solution, for $x \neq 1$, is $y = c_1\,(x-1)^{-3} + c_2(x-1)^{-4}$.

7. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where $F(r) = r^2 + 5r - 1$. The roots are $r = -(5 \pm \sqrt{29})/2$. Hence the general solution, for $x \neq 0$, is $y = c_1\,|x|^{-(5+\sqrt{29})/2} + c_2\,|x|^{-(5-\sqrt{29})/2}$.

8. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where $F(r) = r^2 - 3r + 3$. The roots are complex, with $r = (3 \pm i\sqrt{3})/2$. Hence the general solution, for $x \neq 0$, is

$$y = c_1\,|x|^{3/2}\cos(\frac{\sqrt{3}}{2}\,\ln|x|) + c_2\,|x|^{3/2}\sin(\frac{\sqrt{3}}{2}\,\ln|x|).$$

10. Substitution of $y = (x-2)^r$ results in the quadratic equation $F(r) = 0$, where $F(r) = r^2 + 4r + 8$. The roots are complex, with $r = -2 \pm 2i$. Hence the general solution, for $x \neq 2$, is $y = c_1\,(x-2)^{-2}\cos(2\,\ln|x-2|) + c_2(x-2)^{-2}\sin(2\,\ln|x-2|)$.

11. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where $F(r) = r^2 + r + 4$. The roots are complex, with $r = -(1 \pm i\sqrt{15})/2$. Hence the general solution, for $x \neq 0$, is

$$y = c_1\,|x|^{-1/2}\cos(\frac{\sqrt{15}}{2}\,\ln|x|) + c_2\,|x|^{-1/2}\sin(\frac{\sqrt{15}}{2}\,\ln|x|).$$

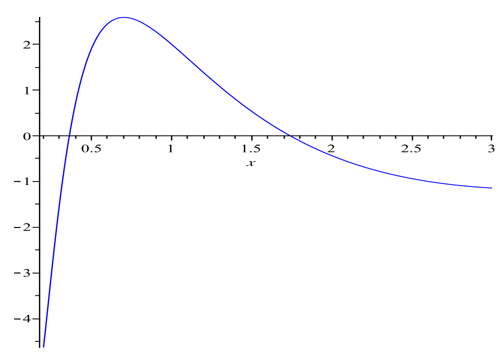
12. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where $F(r) = r^2 - 5r + 4$. The roots are $r = 1$, 4. Hence the general solution is $y = c_1\,x + c_2\,x^4$.

14. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where $F(r) = 4r^2 + 4r + 17$. The roots are complex, with $r = -1/2 \pm 2i$. Hence the general solution, for $x > 0$, is $y = c_1\,x^{-1/2}\cos(2\,\ln\,x) + c_2\,x^{-1/2}\sin(2\,\ln\,x)$. Invoking the initial conditions, we obtain the system of equations

$$c_1 = 2, \qquad -\frac{1}{2}c_1 + 2c_2 = -3.$$

Hence the solution of the initial value problem is

$$y(x) = 2\,x^{-1/2}\cos(2\,\ln\,x) - x^{-1/2}\sin(2\,\ln\,x).$$



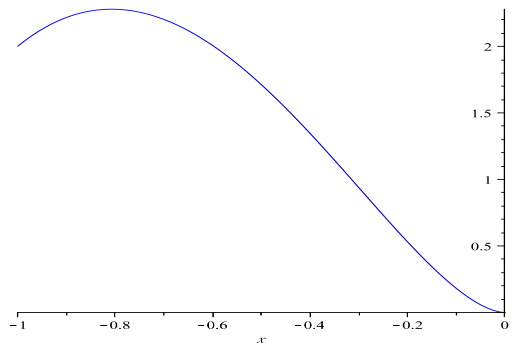
As $x \to 0^+$, the solution decreases without bound.

15. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where $F(r) = r^2 - 4r + 4$. The root is $r = 2$, with multiplicity two. Hence the general solution, for $x < 0$, is $y = (c_1 + c_2\,\ln\,|x|)\,x^2$. Invoking the initial conditions, we obtain the system of equations

$$c_1 = 2, \qquad -2c_1 - c_2 = 3.$$

Hence the solution of the initial value problem is

$$y(x) = (2 - 7\,\ln\,|x|)\,x^2.$$



We find that $y(x) \to 0$ as $x \to 0^-$.

18. We see that $P(x) = 0$ when $x = 0$ and 1. Since the three coefficients have no factors in common, both of these points are singular points. Near $x = 0$,

$$\lim_{x\to 0} x\,p(x) = \lim_{x\to 0} x\frac{2x}{x^2(1-x)^2} = 2\,.$$

$$\lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2\frac{4}{x^2(1-x)^2} = 4\,.$$

The singular point $x = 0$ is regular. Considering $x = 1$,

$$\lim_{x\to 1}(x-1)p(x) = \lim_{x\to 1}(x-1)\frac{2x}{x^2(1-x)^2}\,.$$

The latter limit does not exist. Hence $x = 1$ is an irregular singular point.

19. $P(x) = 0$ when $x = 0$ and 1. Since the three coefficients have no common factors, both of these points are singular points. Near $x = 0$,

$$\lim_{x\to 0} x\,p(x) = \lim_{x\to 0} x\frac{x-2}{x^2(1-x)}\,.$$

The limit does not exist, and so $x = 0$ is an irregular singular point. Considering $x = 1$,

$$\lim_{x\to 1}(x-1)p(x) = \lim_{x\to 1}(x-1)\frac{x-2}{x^2(1-x)} = 1\,.$$

$$\lim_{x\to 1}(x-1)^2 q(x) = \lim_{x\to 1}(x-1)^2\frac{-3x}{x^2(1-x)} = 0\,.$$

Hence $x = 1$ is a regular singular point.

20. $P(x) = 0$ when $x = 0$ and ± 1. Since the three coefficients have no common factors, both of these points are singular points. Near $x = 0$,

$$\lim_{x\to 0} x\,p(x) = \lim_{x\to 0} x\frac{2}{x^3(1-x^2)}\,.$$

The limit does not exist, and so $x = 0$ is an irregular singular point. Near $x = -1$,

$$\lim_{x\to -1}(x+1)p(x) = \lim_{x\to -1}(x+1)\frac{2}{x^3(1-x^2)} = -1\,.$$

$$\lim_{x\to -1}(x+1)^2 q(x) = \lim_{x\to -1}(x+1)^2 \frac{2}{x^3(1-x^2)} = 0\,.$$

Hence $x=-1$ is a regular singular point. At $x=1$,

$$\lim_{x\to 1}(x-1)p(x) = \lim_{x\to 1}(x-1)\frac{2}{x^3(1-x^2)} = -1\,.$$

$$\lim_{x\to 1}(x-1)^2 q(x) = \lim_{x\to 1}(x-1)^2 \frac{2}{x^3(1-x^2)} = 0\,.$$

Hence $x=1$ is a regular singular point.

22. The only singular point is at $x=0$. We find that

$$\lim_{x\to 0} x\,p(x) = \lim_{x\to 0} x\frac{x}{x^2} = 1\,.$$

$$\lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2\,\frac{x^2-\nu^2}{x^2} = -\nu^2.$$

Hence $x=0$ is a regular singular point.

23. The only singular point is at $x=-3$. We find that

$$\lim_{x\to -3}(x+3)p(x) = \lim_{x\to -3}(x+3)\frac{-2x}{x+3} = 6\,.$$

$$\lim_{x\to -3}(x+3)^2 q(x) = \lim_{x\to -3}(x+3)^2\,\frac{1-x^2}{x+3} = 0.$$

Hence $x=-3$ is a regular singular point.

24. Dividing the ODE by $x(1-x^2)^3$, we find that

$$p(x) = \frac{1}{x(1-x^2)} \quad \text{and} \quad q(x) = \frac{2}{x(1+x)^2(1-x)^3}\,.$$

The singular points are at $x=0$ and ± 1. For $x=0$,

$$\lim_{x\to 0} x\,p(x) = \lim_{x\to 0} x\frac{1}{x(1-x^2)} = 1\,.$$

$$\lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2\,\frac{2}{x(1+x)^2(1-x)^3} = 0\,.$$

Hence $x=0$ is a regular singular point. For $x=-1$,

$$\lim_{x\to -1}(x+1)p(x) = \lim_{x\to -1}(x+1)\frac{1}{x(1-x^2)} = -\frac{1}{2}\,.$$

$$\lim_{x\to -1}(x+1)^2 q(x) = \lim_{x\to -1}(x+1)^2\,\frac{2}{x(1+x)^2(1-x)^3} = -\frac{1}{4}\,.$$

Hence $x=-1$ is a regular singular point. For $x=1$,

$$\lim_{x\to 1}(x-1)p(x) = \lim_{x\to 1}(x-1)\frac{1}{x(1-x^2)} = -\frac{1}{2}\,.$$

$$\lim_{x \to 1} (x-1)^2 q(x) = \lim_{x \to 1} (x-1)^2 \frac{2}{x(1+x)^2(1-x)^3}.$$

The latter limit does not exist. Hence $x = 1$ is an irregular singular point.

25. Dividing the ODE by $(x+2)^2(x-1)$, we find that

$$p(x) = \frac{3}{(x+2)^2} \quad \text{and} \quad q(x) = \frac{-2}{(x+2)(x-1)}.$$

The singular points are at $x = -2$ and 1. For $x = -2$,

$$\lim_{x \to -2} (x+2)p(x) = \lim_{x \to -2} (x+2) \frac{3}{(x+2)^2}.$$

The limit does not exist. Hence $x = -2$ is an irregular singular point. For $x = 1$,

$$\lim_{x \to 1} (x-1)p(x) = \lim_{x \to 1} (x-1) \frac{3}{(x+2)^2} = 0.$$

$$\lim_{x \to 1} (x-1)^2 q(x) = \lim_{x \to 1} (x-1)^2 \frac{-2}{(x+2)(x-1)} = 0.$$

Hence $x = 1$ is a regular singular point.

26. $P(x) = 0$ when $x = 0$ and 3. Since the three coefficients have no common factors, both of these points are singular points. Near $x = 0$,

$$\lim_{x \to 0} x\, p(x) = \lim_{x \to 0} x \frac{x+1}{x(3-x)} = \frac{1}{3}.$$

$$\lim_{x \to 0} x^2 q(x) = \lim_{x \to 0} x^2 \frac{-2}{x(3-x)} = 0.$$

Hence $x = 0$ is a regular singular point. For $x = 3$,

$$\lim_{x \to 3} (x-3)p(x) = \lim_{x \to 3} (x-3) \frac{x+1}{x(3-x)} = -\frac{4}{3}.$$

$$\lim_{x \to 3} (x-3)^2 q(x) = \lim_{x \to 3} (x-3)^2 \frac{-2}{x(3-x)} = 0.$$

Hence $x = 3$ is a regular singular point.

27. Dividing the ODE by $(x^2 + x - 2)$, we find that

$$p(x) = \frac{x+1}{(x+2)(x-1)} \quad \text{and} \quad q(x) = \frac{2}{(x+2)(x-1)}.$$

The singular points are at $x = -2$ and 1. For $x = -2$,

$$\lim_{x \to -2} (x+2)p(x) = \lim_{x \to -2} \frac{x+1}{x-1} = \frac{1}{3}.$$

$$\lim_{x \to -2} (x+2)^2 q(x) = \lim_{x \to -2} \frac{2(x+2)}{x-1} = 0.$$

Hence $x = -2$ is a regular singular point. For $x = 1$,

$$\lim_{x \to 1} (x-1)p(x) = \lim_{x \to 1} \frac{x+1}{x+2} = \frac{2}{3}\,.$$

$$\lim_{x \to 1} (x-1)^2 q(x) = \lim_{x \to 1} \frac{2(x-1)}{(x+2)} = 0\,.$$

Hence $x = 1$ is a regular singular point.

29. Note that $p(x) = \ln|x|$ and $q(x) = 3x$. Evidently, $p(x)$ is not analytic at $x_0 = 0$. Furthermore, the function $x\,p(x) = x\ln|x|$ does not have a Taylor series about $x_0 = 0$. Hence $x = 0$ is an irregular singular point.

30. $P(x) = 0$ when $x = 0$. Since the three coefficients have no common factors, $x = 0$ is a singular point. The Taylor series of $e^x - 1$, about $x = 0$, is

$$e^x - 1 = x + x^2/2 + x^3/6 + \ldots\,.$$

Hence the function $x\,p(x) = 2(e^x - 1)/x$ is analytic at $x = 0$. Similarly, the Taylor series of $e^{-x}\cos x$, about $x = 0$, is

$$e^{-x}\cos x = 1 - x + x^3/3 - x^4/6 + \ldots\,.$$

The function $x^2 q(x) = e^{-x}\cos x$ is also analytic at $x = 0$. Hence $x = 0$ is a regular singular point.

31. $P(x) = 0$ when $x = 0$. Since the three coefficients have no common factors, $x = 0$ is a singular point. The Taylor series of $\sin x$, about $x = 0$, is

$$\sin x = x - x^3/3! + x^5/5! - \ldots\,.$$

Hence the function $x\,p(x) = -3\sin x/x$ is analytic at $x = 0$. On the other hand, $q(x)$ is a rational function, with

$$\lim_{x \to 0} x^2 q(x) = \lim_{x \to 0} x^2 \frac{1 + x^2}{x^2} = 1\,.$$

Hence $x = 0$ is a regular singular point.

32. $P(x) = 0$ when $x = 0$. Since the three coefficients have no common factors, $x = 0$ is a singular point. We find that

$$\lim_{x \to 0} x\,p(x) = \lim_{x \to 0} x\frac{1}{x} = 1\,.$$

Although the function $R(x) = \cot x$ does not have a Taylor series about $x = 0$, note that $x^2 q(x) = x\cot x = 1 - x^2/3 - x^4/45 - 2x^6/945 - \ldots$. Hence $x = 0$ is a regular singular point. Furthermore, $q(x) = \cot x/x^2$ is undefined at $x = \pm n\pi$. Therefore the points $x = \pm n\pi$ are also singular points. First note that

$$\lim_{x \to \pm n\pi} (x \mp n\pi)p(x) = \lim_{x \to \pm n\pi} (x \mp n\pi)\frac{1}{x} = 0\,.$$

Furthermore, since $\cot\, x$ has period π ,

$$q(x) = \cot\, x/x = \cot(x \mp n\pi)/x = \cot(x \mp n\pi)\frac{1}{(x \mp n\pi) \pm n\pi}.$$

Therefore

$$(x \mp n\pi)^2 q(x) = (x \mp n\pi)\cot(x \mp n\pi)\left[\frac{(x \mp n\pi)}{(x \mp n\pi) \pm n\pi}\right].$$

From above,

$$(x \mp n\pi)\cot(x \mp n\pi) = 1 - (x \mp n\pi)^2/3 - (x \mp n\pi)^4/45 - \ldots .$$

Note that the function in brackets is analytic near $x = \pm n\pi$. It follows that the function $(x \mp n\pi)^2 q(x)$ is also analytic near $x = \pm n\pi$. Hence all the singular points are regular.

34. The singular points are located at $x = \pm n\pi$, $n = 0, 1, \ldots$. Dividing the ODE by $x\, \sin\, x$, we find that $x\, p(x) = 3\, \csc\, x$ and $x^2 q(x) = x^2 \csc\, x$. Evidently, $x\, p(x)$ is not even defined at $x = 0$. Hence $x = 0$ is an irregular singular point. On the other hand, the Taylor series of $x\, \csc\, x$, about $x = 0$, is

$$x\, \csc\, x = 1 + x^2/6 + 7x^4 360 + \ldots .$$

Noting that $\csc(x \mp n\pi) = (-1)^n \csc\, x$,

$$\begin{aligned}(x \mp n\pi)p(x) &= 3(-1)^n (x \mp n\pi)\csc(x \mp n\pi)/x \\ &= 3(-1)^n (x \mp n\pi)\csc(x \mp n\pi)\left[\frac{1}{(x \mp n\pi) \pm n\pi}\right].\end{aligned}$$

It is apparent that $(x \mp n\pi)p(x)$ is analytic at $x = \pm n\pi$. Similarly,

$$(x \mp n\pi)^2 q(x) = (x \mp n\pi)^2 \csc\, x = (-1)^n (x \mp n\pi)^2 \csc(x \mp n\pi),$$

which is also analytic at $x = \pm n\pi$. Hence all other singular points are regular.

36. Substitution of $y = x^r$ results in the quadratic equation $r^2 - r + \beta = 0$. The roots are

$$r = \frac{1 \pm \sqrt{1 - 4\beta}}{2}.$$

If $\beta > 1/4$, the roots are complex, with $r_{1,2} = (1 \pm i\sqrt{4\beta - 1}\,)/2$. Hence the general solution, for $x \neq 0$, is

$$y = c_1\, |x|^{1/2} \cos(\frac{1}{2}\sqrt{4\beta - 1}\ \ln|x|) + c_2\, |x|^{1/2} \sin(\frac{1}{2}\sqrt{4\beta - 1}\ \ln|x|).$$

Since the trigonometric factors are bounded, $y(x) \to 0$ as $x \to 0$. If $\beta = 1/4$, the roots are equal, and

$$y = c_1\, |x|^{1/2} + c_2\, |x|^{1/2}\ \ln|x| .$$

Since $\lim_{x \to 0} \sqrt{|x|}\, \ln|x| = 0$, $y(x) \to 0$ as $x \to 0$. If $\beta < 1/4$, the roots are real, with $r_{1,2} = (1 \pm \sqrt{1 - 4\beta}\,)/2$. Hence the general solution, for $x \neq 0$, is

$$y = c_1\, |x|^{1/2 + \sqrt{1-4\beta}/2} + c_2\, |x|^{1/2 - \sqrt{1-4\beta}/2} .$$

Evidently, solutions approach zero as long as $1/2 - \sqrt{1-4\beta}/2 > 0$. That is,

$$0 < \beta < 1/4 .$$

Hence all solutions approach zero for $\beta > 0$.

37. Substitution of $y = x^r$ results in the quadratic equation $r^2 - r - 2 = 0$. The roots are $r = -1\,,\ 2$. Hence the general solution, for $x \neq 0$, is $y = c_1 x^{-1} + c_2\, x^2$. Invoking the initial conditions, we obtain the system of equations

$$c_1 + c_2 = 1, \qquad -c_1 + 2c_2 = \gamma$$

Hence the solution of the initial value problem is

$$y(x) = \frac{2-\gamma}{3} x^{-1} + \frac{1+\gamma}{3}\, x^2.$$

The solution is bounded, as $x \to 0$, if $\gamma = 2$.

38. Substitution of $y = x^r$ results in the quadratic equation $r^2 + (\alpha - 1)r + 5/2 = 0$. Formally, the roots are given by

$$r = \frac{1-\alpha \,\pm\, \sqrt{\alpha^2 - 2\alpha - 9}}{2} = \frac{1-\alpha \,\pm\, \sqrt{(\alpha - 1 - \sqrt{10})(\alpha - 1 + \sqrt{10})}}{2}.$$

(i) The roots will be complex if $|1-\alpha| < \sqrt{10}$. For solutions to approach zero, as $x \to \infty$, we need $-\sqrt{10} < 1 - \alpha < 0$.

(ii) The roots will be equal if $|1-\alpha| = \sqrt{10}$. In this case, all solutions approach zero as long as $1 - \alpha = -\sqrt{10}$.

(iii) The roots will be real and distinct if $|1-\alpha| > \sqrt{10}$. It follows that

$$r_{max} = \frac{1-\alpha \,+\, \sqrt{\alpha^2 - 2\alpha - 9}}{2} .$$

For solutions to approach zero, we need $1 - \alpha + \sqrt{\alpha^2 - 2\alpha - 9} < 0$. That is, $1 - \alpha < -\sqrt{10}$. Hence all solutions approach zero, as $x \to \infty$, as long as $\alpha > 1$.

42. $x = 0$ is the only singular point. Dividing the ODE by $2x^2$, we have $p(x) = 3/(2x)$ and $q(x) = -x^{-2}(1+x)/2$. It follows that

$$\lim_{x \to 0} x\, p(x) = \lim_{x \to 0} x \frac{3}{2x} = \frac{3}{2} ,$$

$$\lim_{x \to 0} x^2 q(x) = \lim_{x \to 0} x^2 \frac{-(1+x)}{2x^2} = -\frac{1}{2},$$

so $x = 0$ is a regular singular point. Let $y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$. Substitution into the ODE results in

$$2x^2 \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n + 3x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - (1+x) \sum_{n=0}^{\infty} a_n x^n = 0 .$$

That is,

$$2\sum_{n=2}^{\infty} n(n-1)a_n\, x^n + 3\sum_{n=1}^{\infty} n\, a_n x^n - \sum_{n=0}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0\,.$$

It follows that

$$-a_0 + (2a_1 - a_0)x + \sum_{n=2}^{\infty} \left[2n(n-1)a_n + 3n\, a_n - a_n - a_{n-1}\right] x^n = 0\,.$$

Equating the coefficients to zero, we find that $a_0 = 0$, $2a_1 - a_0 = 0$, and

$$(2n-1)(n+1)a_n = a_{n-1}\,, \quad n = 2, 3, \ldots\,.$$

We conclude that all the a_n are equal to zero. Hence $y(x) = 0$ is the only solution that can be obtained.

44. Based on Problem 43, the change of variable, $x = 1/\xi$, transforms the ODE into the form

$$\xi^4 \frac{d^2y}{d\xi^2} + 2\xi^3 \frac{dy}{d\xi} + y = 0\,.$$

Evidently, $\xi = 0$ is a singular point. Now $p(\xi) = 2/\xi$ and $q(\xi) = 1/\xi^4$. Since the value of $\lim_{\xi \to 0} \xi^2 q(\xi)$ does not exist, $\xi = 0$ $(x = \infty)$ is an irregular singular point.

46. Under the transformation $x = 1/\xi$, the ODE becomes

$$\xi^4(1 - \frac{1}{\xi^2})\frac{d^2y}{d\xi^2} + \left[2\xi^3(1 - \frac{1}{\xi^2}) + 2\xi^2 \frac{1}{\xi}\right] \frac{dy}{d\xi} + \alpha(\alpha+1)y = 0\,,$$

that is,

$$(\xi^4 - \xi^2)\frac{d^2y}{d\xi^2} + 2\xi^3 \frac{dy}{d\xi} + \alpha(\alpha+1)y = 0\,.$$

Therefore $\xi = 0$ is a singular point. Note that

$$p(\xi) = \frac{2\xi}{\xi^2 - 1} \text{ and } q(\xi) = \frac{\alpha(\alpha+1)}{\xi^2(\xi^2 - 1)}\,.$$

It follows that

$$\lim_{\xi \to 0} \xi\, p(\xi) = \lim_{\xi \to 0} \xi \frac{2\xi}{\xi^2 - 1} = 0\,,$$

$$\lim_{\xi \to 0} \xi^2 q(\xi) = \lim_{\xi \to 0} \xi^2 \frac{\alpha(\alpha+1)}{\xi^2(\xi^2 - 1)} = -\alpha(\alpha+1)\,.$$

Hence $\xi = 0$ $(x = \infty)$ is a regular singular point.

48. Under the transformation $x = 1/\xi$, the ODE becomes

$$\xi^4 \frac{d^2y}{d\xi^2} + \left[2\xi^3 + 2\xi^2 \frac{1}{\xi}\right] \frac{dy}{d\xi} + \lambda\, y = 0\,,$$

that is,

$$\xi^4 \frac{d^2y}{d\xi^2} + 2(\xi^3 + \xi)\frac{dy}{d\xi} + \lambda\, y = 0\,.$$

Therefore $\xi = 0$ is a singular point. Note that

$$p(\xi) = \frac{2(\xi^2+1)}{\xi^3} \text{ and } q(\xi) = \frac{\lambda}{\xi^4}.$$

It immediately follows that the limit $\lim_{\xi\to 0} \xi\, p(\xi)$ does not exist. Hence $\xi = 0$ $(x = \infty)$ is an irregular singular point.

49. Under the transformation $x = 1/\xi$, the ODE becomes

$$\xi^4 \frac{d^2y}{d\xi^2} + 2\xi^3 \frac{dy}{d\xi} - \frac{1}{\xi} y = 0.$$

Therefore $\xi = 0$ is a singular point. Note that

$$p(\xi) = \frac{2}{\xi} \text{ and } q(\xi) = \frac{-1}{\xi^5}.$$

We find that

$$\lim_{\xi\to 0} \xi\, p(\xi) = \lim_{\xi\to 0} \xi \frac{2}{\xi} = 2,$$

but

$$\lim_{\xi\to 0} \xi^2 q(\xi) = \lim_{\xi\to 0} \xi^2 \frac{(-1)}{\xi^5}.$$

The latter limit does not exist. Hence $\xi = 0$ $(x = \infty)$ is an irregular singular point.

5.5

1.(a) $P(x) = 0$ when $x = 0$. Since the three coefficients have no common factors, $x = 0$ is a singular point. Near $x = 0$,

$$\lim_{x\to 0} x\, p(x) = \lim_{x\to 0} x \frac{1}{2x} = \frac{1}{2}.$$

$$\lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2 \frac{1}{2} = 0.$$

Hence $x = 0$ is a regular singular point.

(b) Let

$$y = x^r(a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots) = \sum_{n=0}^{\infty} a_n\, x^{r+n}.$$

Then

$$y' = \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1}$$

and

$$y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n-2}.$$

Substitution into the ODE results in

$$2\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\,x^{r+n-1}+\sum_{n=0}^{\infty}(r+n)a_n x^{r+n-1}+\sum_{n=0}^{\infty}a_n x^{r+n+1}=0\,.$$

That is,

$$2\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\,x^{r+n}+\sum_{n=0}^{\infty}(r+n)a_n x^{r+n}+\sum_{n=2}^{\infty}a_{n-2}\,x^{r+n}=0\,.$$

It follows that

$$a_0\,[2r(r-1)+r]\,x^r+a_1\,[2(r+1)r+r+1]\,x^{r+1}$$
$$+\sum_{n=2}^{\infty}\left[2(r+n)(r+n-1)a_n+(r+n)a_n+a_{n-2}\right]x^{r+n}=0\,.$$

Assuming that $a_0\neq 0$, we obtain the indicial equation $2r^2-r=0$, with roots $r_1=1/2$ and $r_2=0$. It immediately follows that $a_1=0$. Setting the remaining coefficients equal to zero, we have

$$a_n=\frac{-a_{n-2}}{(r+n)\,[2(r+n)-1]}\,,\quad n=2,3,\ldots\,.$$

(c) For $r=1/2$, the recurrence relation becomes

$$a_n=\frac{-a_{n-2}}{n(1+2n)}\,,\quad n=2,3,\ldots\,.$$

Since $a_1=0$, the odd coefficients are zero. Furthermore, for $k=1,2,\ldots$,

$$a_{2k}=\frac{-a_{2k-2}}{2k(1+4k)}=\frac{a_{2k-4}}{(2k-2)(2k)(4k-3)(4k+1)}=\frac{(-1)^k a_0}{2^k\,k!\,5\cdot 9\cdot 13\ldots(4k+1)}\,.$$

(d) For $r=0$, the recurrence relation becomes

$$a_n=\frac{-a_{n-2}}{n(2n-1)}\,,\quad n=2,3,\ldots\,.$$

Since $a_1=0$, the odd coefficients are zero, and for $k=1,2,\ldots$,

$$a_{2k}=\frac{-a_{2k-2}}{2k(4k-1)}=\frac{a_{2k-4}}{(2k-2)(2k)(4k-5)(4k-1)}=\frac{(-1)^k a_0}{2^k\,k!\,3\cdot 7\cdot 11\ldots(4k-1)}\,.$$

The two linearly independent solutions are

$$y_1(x)=\sqrt{x}\,\left[1+\sum_{k=1}^{\infty}\frac{(-1)^k\,x^{2k}}{2^k\,k!\,5\cdot 9\cdot 13\ldots(4k+1)}\right]$$

$$y_2(x)=1+\sum_{k=1}^{\infty}\frac{(-1)^k\,x^{2k}}{2^k\,k!\,3\cdot 7\cdot 11\ldots(4k-1)}\,.$$

3.(a) Note that $x\,p(x)=0$ and $x^2q(x)=x$, which are both analytic at $x=0$.

(b) Set $y = x^r(a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\, x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n} = 0\,,$$

and after multiplying both sides of the equation by x,

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\, x^{r+n} + \sum_{n=1}^{\infty} a_{n-1} x^{r+n} = 0\,.$$

It follows that

$$a_0\,[r(r-1)]\,x^r + \sum_{n=1}^{\infty}\left[(r+n)(r+n-1)a_n + a_{n-1}\right] x^{r+n} = 0\,.$$

Setting the coefficients equal to zero, the indicial equation is $r(r-1) = 0$. The roots are $r_1 = 1$ and $r_2 = 0$. Here $r_1 - r_2 = 1$. The recurrence relation is

$$a_n = \frac{-a_{n-1}}{(r+n)(r+n-1)}\,, \quad n = 1, 2, \ldots\,.$$

(c) For $r = 1$,

$$a_n = \frac{-a_{n-1}}{n(n+1)}\,, \quad n = 1, 2, \ldots\,.$$

Hence for $n \geq 1$,

$$a_n = \frac{-a_{n-1}}{n(n+1)} = \frac{a_{n-2}}{(n-1)n^2(n+1)} = \ldots = \frac{(-1)^n a_0}{n!(n+1)!}\,.$$

Therefore one solution is

$$y_1(x) = x \sum_{n=0}^{\infty} \frac{(-1)^n\, x^n}{n!(n+1)!}\,.$$

5.(a) Here $x\,p(x) = 2/3$ and $x^2 q(x) = x^2/3$, which are both analytic at $x = 0$.

(b) Set $y = x^r(a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots)$. Substitution into the ODE results in

$$3\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\, x^{r+n} + 2\sum_{n=0}^{\infty}(r+n)a_n\, x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} = 0\,.$$

It follows that

$$a_0\,[3r(r-1)+2r]\,x^r + a_1\,[3(r+1)r + 2(r+1)]\,x^{r+1}$$
$$+ \sum_{n=2}^{\infty}\left[3(r+n)(r+n-1)a_n + 2(r+n)a_n + a_{n-2}\right] x^{r+n} = 0\,.$$

Assuming $a_0 \neq 0$, the indicial equation is $3r^2 - r = 0$, with roots $r_1 = 1/3$, $r_2 = 0$. Setting the remaining coefficients equal to zero, we have $a_1 = 0$, and

$$a_n = \frac{-a_{n-2}}{(r+n)\,[3(r+n)-1]}\,, \quad n = 2, 3, \ldots\,.$$

It immediately follows that the odd coefficients are equal to zero.

(c) For $r = 1/3$,

$$a_n = \frac{-a_{n-2}}{n(1+3n)}, \quad n = 2, 3, \ldots.$$

So for $k = 1, 2, \ldots$,

$$a_{2k} = \frac{-a_{2k-2}}{2k(6k+1)} = \frac{a_{2k-4}}{(2k-2)(2k)(6k-5)(6k+1)} = \frac{(-1)^k a_0}{2^k\, k!\, 7 \cdot 13 \ldots (6k+1)}.$$

(d) For $r = 0$,

$$a_n = \frac{-a_{n-2}}{n(3n-1)}, \quad n = 2, 3, \ldots.$$

So for $k = 1, 2, \ldots$,

$$a_{2k} = \frac{-a_{2k-2}}{2k(6k-1)} = \frac{a_{2k-4}}{(2k-2)(2k)(6k-7)(6k-1)} = \frac{(-1)^k a_0}{2^k\, k!\, 5 \cdot 11 \ldots (6k-1)}.$$

The two linearly independent solutions are

$$y_1(x) = x^{1/3}\left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!\, 7 \cdot 13 \ldots (6k+1)} (\frac{x^2}{2})^k\right]$$

$$y_2(x) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!\, 5 \cdot 11 \ldots (6k-1)} (\frac{x^2}{2})^k.$$

6.(a) Note that $x\,p(x) = 1$ and $x^2 q(x) = x - 2$, which are both analytic at $x = 0$.

(b) Set $y = x^r(a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (r+n)(r+n-1)a_n\, x^{r+n} + \sum_{n=0}^{\infty} (r+n)a_n\, x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+1} - 2\sum_{n=0}^{\infty} a_n x^{r+n} = 0.$$

After adjusting the indices in the second-to-last series, we obtain

$$a_0\,[r(r-1) + r - 2]\, x^r + \sum_{n=1}^{\infty} [(r+n)(r+n-1)a_n + (r+n)a_n - 2\,a_n + a_{n-1}]\, x^{r+n} = 0.$$

Assuming $a_0 \neq 0$, the indicial equation is $r^2 - 2 = 0$, with roots $r = \pm\sqrt{2}$. Setting the remaining coefficients equal to zero, the recurrence relation is

$$a_n = \frac{-a_{n-1}}{(r+n)^2 - 2}, \quad n = 1, 2, \ldots.$$

Note that $(r+n)^2 - 2 = (r + n + \sqrt{2})(r + n - \sqrt{2})$.

(c) For $r = \sqrt{2}$,

$$a_n = \frac{-a_{n-1}}{n(n+2\sqrt{2})}, \quad n = 1, 2, \ldots .$$

It follows that

$$a_n = \frac{(-1)^n a_0}{n!(1+2\sqrt{2})(2+2\sqrt{2})\ldots(n+2\sqrt{2})}, \quad n = 1, 2, \ldots .$$

(d) For $r = -\sqrt{2}$,

$$a_n = \frac{-a_{n-1}}{n(n-2\sqrt{2})}, \quad n = 1, 2, \ldots ,$$

and therefore

$$a_n = \frac{(-1)^n a_0}{n!(1-2\sqrt{2})(2-2\sqrt{2})\ldots(n-2\sqrt{2})}, \quad n = 1, 2, \ldots .$$

The two linearly independent solutions are

$$y_1(x) = x^{\sqrt{2}}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n!(1+2\sqrt{2})(2+2\sqrt{2})\ldots(n+2\sqrt{2})}\right]$$

$$y_2(x) = x^{-\sqrt{2}}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n!(1-2\sqrt{2})(2-2\sqrt{2})\ldots(n-2\sqrt{2})}\right].$$

7.(a) Here $x\,p(x) = 1 - x$ and $x^2 q(x) = -x$, which are both analytic at $x = 0$.

(b) Set $y = x^r(a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\, x^{r+n-1} + \sum_{n=0}^{\infty}(r+n)a_n\, x^{r+n-1} - \sum_{n=0}^{\infty}(r+n)a_n\, x^{r+n} - \sum_{n=0}^{\infty} a_n x^{r+n} = 0 .$$

After multiplying both sides by x,

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\, x^{r+n} + \sum_{n=0}^{\infty}(r+n)a_n\, x^{r+n} - \sum_{n=0}^{\infty}(r+n)a_n\, x^{r+n+1} - \sum_{n=0}^{\infty} a_n x^{r+n+1} = 0.$$

After adjusting the indices in the last two series, we obtain

$$a_0\,[r(r-1) + r]\,x^r + \sum_{n=1}^{\infty}\left[(r+n)(r+n-1)a_n + (r+n)a_n - (r+n)a_{n-1}\right]x^{r+n} = 0 .$$

Assuming $a_0 \neq 0$, the indicial equation is $r^2 = 0$, with roots $r_1 = r_2 = 0$. Setting the remaining coefficients equal to zero, the recurrence relation is

$$a_n = \frac{a_{n-1}}{r+n}, \quad n = 1, 2, \ldots .$$

(c) With $r = 0$,

$$a_n = \frac{a_{n-1}}{n}, \quad n = 1, 2, \ldots .$$

Hence one solution is

$$y_1(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots = e^x.$$

8.(a) Note that $xp(x) = 3/2$ and $x^2 q(x) = x^2 - 1/2$, which are both analytic at $x = 0$.

(b) Set $y = x^r(a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots)$. Substitution into the ODE results in

$$2\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\, x^{r+n} + 3\sum_{n=0}^{\infty}(r+n)a_n\, x^{r+n} + 2\sum_{n=0}^{\infty} a_n x^{r+n+2} - \sum_{n=0}^{\infty} a_n x^{r+n} = 0\,.$$

After adjusting the indices in the second-to-last series, we obtain

$$a_0\left[2r(r-1) + 3r - 1\right]x^r + a_1\left[2(r+1)r + 3(r+1) - 1\right] + \sum_{n=2}^{\infty}\left[2(r+n)(r+n-1)a_n + 3(r+n)a_n - a_n + 2\,a_{n-2}\right]x^{r+n} = 0.$$

Assuming $a_0 \neq 0$, the indicial equation is $2r^2 + r - 1 = 0$, with roots $r_1 = 1/2$ and $r_2 = -1$. Setting the remaining coefficients equal to zero, the recurrence relation is

$$a_n = \frac{-2a_{n-2}}{(r+n+1)\left[2(r+n) - 1\right]}, \quad n = 2, 3, \ldots .$$

Setting the remaining coefficients equal to zero, we have $a_1 = 0$, which implies that all of the odd coefficients are zero.

(c) With $r = 1/2$,

$$a_n = \frac{-2a_{n-2}}{n(2n+3)}, \quad n = 2, 3, \ldots .$$

So for $k = 1, 2, \ldots$,

$$a_{2k} = \frac{-a_{2k-2}}{k(4k+3)} = \frac{a_{2k-4}}{(k-1)k(4k-5)(4k+3)} = \frac{(-1)^k a_0}{k!\, 7 \cdot 11 \ldots (4k+3)}\,.$$

(d) With $r = -1$,

$$a_n = \frac{-2a_{n-2}}{n(2n-3)}, \quad n = 2, 3, \ldots .$$

So for $k = 1, 2, \ldots,$

$$a_{2k} = \frac{-a_{2k-2}}{k(4k-3)} = \frac{a_{2k-4}}{(k-1)k(4k-11)(4k-3)} = \frac{(-1)^k a_0}{k!\,5\cdot 9\,\ldots\,(4k-3)}\,.$$

The two linearly independent solutions are

$$y_1(x) = x^{1/2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n\, x^{2n}}{n!\,7\cdot 11\,\ldots\,(4n+3)}\right]$$

$$y_2(x) = x^{-1}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n\, x^{2n}}{n!\,5\cdot 9\,\ldots\,(4n-3)}\right].$$

9.(a) Note that $x\,p(x) = -x-3$ and $x^2 q(x) = x+3$, which are both analytic at $x = 0$.

(b) Set $y = x^r(a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\, x^{r+n} - \sum_{n=0}^{\infty}(r+n)a_n\, x^{r+n+1} - 3\sum_{n=0}^{\infty}(r+n)a_n\, x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+1} + 3\sum_{n=0}^{\infty} a_n x^{r+n} = 0\,.$$

After adjusting the indices in the second-to-last series, we obtain

$$a_0\,[r(r-1) - 3r + 3]\,x^r + \sum_{n=1}^{\infty}\left[(r+n)(r+n-1)a_n - (r+n-2)a_{n-1} - 3(r+n-1)a_n\right] x^{r+n} = 0.$$

Assuming $a_0 \neq 0$, the indicial equation is $r^2 - 4r + 3 = 0$, with roots $r_1 = 3$ and $r_2 = 1$. Setting the remaining coefficients equal to zero, the recurrence relation is

$$a_n = \frac{(r+n-2)a_{n-1}}{(r+n-1)(r+n-3)}\,, \quad n = 1, 2, \ldots\,.$$

(c) With $r = 3$,

$$a_n = \frac{(n+1)a_{n-1}}{n(n+2)}\,, \quad n = 1, 2, \ldots\,.$$

It follows that for $n \geq 1$,

$$a_n = \frac{(n+1)a_{n-1}}{n(n+2)} = \frac{a_{n-2}}{(n-1)(n+2)} = \ldots = \frac{2\,a_0}{n!\,(n+2)}\,.$$

Therefore one solution is

$$y_1(x) = x^3\left[1 + \sum_{n=1}^{\infty}\frac{2\,x^n}{n!\,(n+2)}\right].$$

10.(a) Here $x\,p(x) = 0$ and $x^2 q(x) = x^2 + 1/4$, which are both analytic at $x = 0$.

(b) Set $y = x^r(a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\, x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} + \frac{1}{4}\sum_{n=0}^{\infty} a_n x^{r+n} = 0\,.$$

After adjusting the indices in the second series, we obtain

$$a_0\left[r(r-1)+\frac{1}{4}\right]x^r + a_1\left[(r+1)r+\frac{1}{4}\right]x^{r+1} + \sum_{n=2}^{\infty}\left[(r+n)(r+n-1)a_n + \frac{1}{4}a_n + a_{n-2}\right]x^{r+n} = 0\,.$$

Assuming $a_0 \neq 0$, the indicial equation is $r^2 - r + \frac{1}{4} = 0$, with roots $r_1 = r_2 = 1/2$. Setting the remaining coefficients equal to zero, we find that $a_1 = 0$. The recurrence relation is

$$a_n = \frac{-4a_{n-2}}{(2r+2n-1)^2}\,, \quad n = 2, 3, \ldots\,.$$

(c) With $r = 1/2$,

$$a_n = \frac{-a_{n-2}}{n^2}\,, \quad n = 2, 3, \ldots\,.$$

Since $a_1 = 0$, the odd coefficients are zero. So for $k \geq 1$,

$$a_{2k} = \frac{-a_{2k-2}}{4k^2} = \frac{a_{2k-4}}{4^2(k-1)^2k^2} = \ldots = \frac{(-1)^k a_0}{4^k(k!)^2}\,.$$

Therefore one solution is

$$y_1(x) = \sqrt{x}\left[1 + \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}\right].$$

12.(a) Dividing through by the leading coefficient, the ODE can be written as

$$y'' - \frac{x}{1-x^2}\,y' + \frac{\alpha^2}{1-x^2}\,y = 0\,.$$

For $x = 1$,

$$p_0 = \lim_{x\to 1}(x-1)p(x) = \lim_{x\to 1}\frac{x}{x+1} = \frac{1}{2}\,.$$

$$q_0 = \lim_{x\to 1}(x-1)^2 q(x) = \lim_{x\to 1}\frac{\alpha^2(1-x)}{x+1} = 0\,.$$

For $x = -1$,

$$p_0 = \lim_{x\to -1}(x+1)p(x) = \lim_{x\to -1}\frac{x}{x-1} = \frac{1}{2}\,.$$

$$q_0 = \lim_{x\to -1}(x+1)^2 q(x) = \lim_{x\to -1}\frac{\alpha^2(x+1)}{(1-x)} = 0\,.$$

Hence $x = -1$ and $x = 1$ are regular singular points. As shown in Example 1, the indicial equation is given by

$$r(r-1) + p_0 r + q_0 = 0\,.$$

In this case, both sets of roots are $r_1 = 1/2$ and $r_2 = 0$.

(b) Let $t = x - 1$, and $u(t) = y(t+1)$. Under this change of variable, the differential equation becomes

$$(t^2 + 2t)u'' + (t+1)u' - \alpha^2 u = 0 .$$

Based on part (a), $t = 0$ is a regular singular point. Set $u = \sum_{n=0}^{\infty} a_n\, t^{r+n}$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\, t^{r+n} + 2\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\, t^{r+n-1}$$
$$+ \sum_{n=0}^{\infty}(r+n)a_n\, t^{r+n} + \sum_{n=0}^{\infty}(r+n)a_n\, t^{r+n-1} - \alpha^2\sum_{n=0}^{\infty} a_n t^{r+n} = 0 .$$

Upon inspection, we can also write

$$\sum_{n=0}^{\infty}(r+n)^2 a_n\, t^{r+n} + 2\sum_{n=0}^{\infty}(r+n)(r+n-\frac{1}{2})a_n\, t^{r+n-1} - \alpha^2\sum_{n=0}^{\infty} a_n t^{r+n} = 0.$$

After adjusting the indices in the second series, it follows that

$$a_0\left[2r(r-\frac{1}{2})\right] t^{r-1}$$
$$+\sum_{n=0}^{\infty}\left[(r+n)^2 a_n + 2(r+n+1)(r+n+\frac{1}{2})a_{n+1} - \alpha^2 a_n\right] t^{r+n} = 0.$$

Assuming that $a_0 \neq 0$, the indicial equation is $2r^2 - r = 0$, with roots $r = 0\,,\, 1/2$. The recurrence relation is

$$(r+n)^2 a_n + 2(r+n+1)(r+n+\frac{1}{2})a_{n+1} - \alpha^2 a_n = 0\,, \quad n = 0, 1, 2, \ldots .$$

With $r_1 = 1/2$, we find that for $n \geq 1$,

$$a_n = \frac{4\alpha^2 - (2n-1)^2}{4n(2n+1)}\, a_{n-1} = (-1)^n \frac{\left[1-4\alpha^2\right]\left[9-4\alpha^2\right]\ldots\left[(2n-1)^2-4\alpha^2\right]}{2^n(2n+1)!}\, a_0 .$$

With $r_2 = 0$, we find that for $n \geq 1$,

$$a_n = \frac{\alpha^2 - (n-1)^2}{n(2n-1)}\, a_{n-1} = (-1)^n \frac{\alpha(-\alpha)\left[1-\alpha^2\right]\left[4-\alpha^2\right]\ldots\left[(n-1)^2-\alpha^2\right]}{n!\cdot 3\cdot 5\,\ldots(2n-1)}\, a_0 .$$

The two linearly independent solutions of the Chebyshev equation are

$$y_1(x) = |x-1|^{1/2}\left(1 + \sum_{n=1}^{\infty}(-1)^n \frac{(1-4\alpha^2)(9-4\alpha^2)\ldots((2n-1)^2-4\alpha^2)}{2^n(2n+1)!}(x-1)^n\right)$$

$$y_2(x) = 1 + \sum_{n=1}^{\infty}(-1)^n \frac{\alpha(-\alpha)(1-\alpha^2)(4-\alpha^2)\ldots((n-1)^2-\alpha^2)}{n!\cdot 3\cdot 5\,\ldots(2n-1)}(x-1)^n .$$

13.(a) Here $xp(x) = 1 - x$ and $x^2 q(x) = \lambda x$, which are both analytic at $x = 0$. In fact,

$$p_0 = \lim_{x \to 0} x\, p(x) = 1 \text{ and } \; q_0 = \lim_{x \to 0} x^2 q(x) = 0\,.$$

(b) The indicial equation is $r(r-1) + r = 0$, with roots $r_{1,2} = 0$.

(c) Set

$$y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots.$$

Substitution into the ODE results in

$$\sum_{n=2}^{\infty} n(n-1)a_n\, x^{n-1} + \sum_{n=1}^{\infty} n a_n\, x^{n-1} - \sum_{n=0}^{\infty} n a_n\, x^n + \lambda \sum_{n=0}^{\infty} a_n x^n = 0\,.$$

That is,

$$\sum_{n=1}^{\infty} n(n+1)a_{n+1}\, x^n + \sum_{n=0}^{\infty} (n+1)a_{n+1}\, x^n - \sum_{n=1}^{\infty} n a_n\, x^n + \lambda \sum_{n=0}^{\infty} a_n x^n = 0\,.$$

It follows that

$$a_1 + \lambda\, a_0 + \sum_{n=1}^{\infty} \left[(n+1)^2 a_{n+1} - (n-\lambda)a_n\right] x^n = 0\,.$$

Setting the coefficients equal to zero, we find that $a_1 = -\lambda a_0$, and

$$a_n = \frac{(n-1-\lambda)}{n^2}\, a_{n-1}\,, \quad n = 2, 3, \ldots\,.$$

That is, for $n \geq 2$,

$$a_n = \frac{(n-1-\lambda)}{n^2}\, a_{n-1} = \ldots = \frac{(-\lambda)(1-\lambda)\ldots(n-1-\lambda)}{(n!)^2}\, a_0\,.$$

Therefore one solution of the Laguerre equation is

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{(-\lambda)(1-\lambda)\ldots(n-1-\lambda)}{(n!)^2}\, x^n.$$

Note that if $\lambda = m$, a positive integer, then $a_n = 0$ for $n \geq m+1$. In that case, the solution is a polynomial

$$y_1(x) = 1 + \sum_{n=1}^{m} \frac{(-\lambda)(1-\lambda)\ldots(n-1-\lambda)}{(n!)^2}\, x^n\,.$$

5.6

2.(a) $P(x) = 0$ only for $x = 0$. Furthermore, $xp(x) = -2 - x$ and $x^2q(x) = 2 + x^2$. It follows that

$$p_0 = \lim_{x \to 0} (-2 - x) = -2$$
$$q_0 = \lim_{x \to 0} (2 + x^2) = 2$$

and therefore $x = 0$ is a regular singular point.

(b) The indicial equation is given by $r(r-1) - 2r + 2 = 0$, that is, $r^2 - 3r + 2 = 0$, with roots $r_1 = 2$ and $r_2 = 1$.

4. The coefficients $P(x)$, $Q(x)$, and $R(x)$ are analytic for all $x \in \mathbb{R}$. Hence there are no singular points.

5.(a) $P(x) = 0$ only for $x = 0$. Furthermore, $xp(x) = 3\sin x / x$ and $x^2q(x) = -2$. It follows that

$$p_0 = \lim_{x \to 0} 3\,\frac{\sin x}{x} = 3$$
$$q_0 = \lim_{x \to 0} -2 = -2$$

and therefore $x = 0$ is a regular singular point.

(b) The indicial equation is given by $r(r-1) + 3r - 2 = 0$, that is, $r^2 + 2r - 2 = 0$, with roots $r_1 = -1 + \sqrt{3}$ and $r_2 = -1 - \sqrt{3}$.

6.(a) $P(x) = 0$ for $x = 0$ and $x = -2$. We note that $p(x) = x^{-1}(x+2)^{-1}/2$, and $q(x) = -(x+2)^{-1}/2$. For the singularity at $x = 0$,

$$p_0 = \lim_{x \to 0} \frac{1}{2(x+2)} = \frac{1}{4}$$
$$q_0 = \lim_{x \to 0} \frac{-x^2}{2(x+2)} = 0$$

and therefore $x = 0$ is a regular singular point.

For the singularity at $x = -2$,

$$p_0 = \lim_{x \to -2} (x+2)p(x) = \lim_{x \to -2} \frac{1}{2x} = -\frac{1}{4}$$
$$q_0 = \lim_{x \to -2} (x+2)^2 q(x) = \lim_{x \to -2} \frac{-(x+2)}{2} = 0$$

and therefore $x = -2$ is a regular singular point.

(b) For $x = 0$: the indicial equation is given by $r(r-1) + r/4 = 0$, that is, $r^2 - 3r/4 = 0$, with roots $r_1 = 3/4$ and $r_2 = 0$.

For $x = -2$: the indicial equation is given by $r(r-1) - r/4 = 0$, that is, $r^2 - 5r/4 = 0$, with roots $r_1 = 5/4$ and $r_2 = 0$.

7.(a) $P(x) = 0$ only for $x = 0$. Furthermore, $xp(x) = 1/2 + \sin x/\,2x$ and $x^2q(x) = 1$. It follows that

$$p_0 = \lim_{x\to 0} xp(x) = 1$$
$$q_0 = \lim_{x\to 0} x^2q(x) = 1$$

and therefore $x = 0$ is a regular singular point.

(b) The indicial equation is given by

$$r(r-1) + r + 1 = 0\,,$$

that is, $r^2 + 1 = 0$, with complex conjugate roots $r = \pm\, i$.

8.(a) Note that $P(x) = 0$ only for $x = -1$. We find that $p(x) = 3(x-1)/(x+1)$, and $q(x) = 3/(x+1)^2$. It follows that

$$p_0 = \lim_{x\to -1}(x+1)p(x) = \lim_{x\to -1} 3(x-1) = -6$$
$$q_0 = \lim_{x\to -1}(x+1)^2q(x) = \lim_{x\to -1} 3 = 3$$

and therefore $x = -1$ is a regular singular point.

(b) The indicial equation is given by

$$r(r-1) - 6r + 3 = 0,$$

that is, $r^2 - 7r + 3 = 0$, with roots $r_1 = (7+\sqrt{37}\,)/2$ and $r_2 = (7-\sqrt{37}\,)/2$.

10.(a) $P(x) = 0$ for $x = 2$ and $x = -2$. We note that $p(x) = 2x(x-2)^{-2}(x+2)^{-1}$, and $q(x) = 3(x-2)^{-1}(x+2)^{-1}$. For the singularity at $x = 2$,

$$\lim_{x\to 2}(x-2)p(x) = \lim_{x\to 2}\frac{2x}{x^2-4}\,,$$

which is undefined. Therefore $x = 2$ is an irregular singular point. For the singularity at $x = -2$,

$$p_0 = \lim_{x\to -2}(x+2)p(x) = \lim_{x\to -2}\frac{2x}{(x-2)^2} = -\frac{1}{4}$$
$$q_0 = \lim_{x\to -2}(x+2)^2q(x) = \lim_{x\to -2}\frac{3(x+2)}{x-2} = 0$$

and therefore $x = -2$ is a regular singular point.

(b) The indicial equation is given by $r(r-1) - r/4 = 0$, that is, $r^2 - 5r/4 = 0$, with roots $r_1 = 5/4$ and $r_2 = 0$.

11.(a) $P(x)=0$ for $x=2$ and $x=-2$. We note that $p(x)=2x/(4-x^2)$, and $q(x)=3/(4-x^2)$. For the singularity at $x=2$,

$$p_0=\lim_{x\to 2}(x-2)p(x)=\lim_{x\to 2}\frac{-2x}{x+2}=-1$$
$$q_0=\lim_{x\to 2}(x-2)^2q(x)=\lim_{x\to 2}\frac{3(2-x)}{x+2}=0$$

and therefore $x=2$ is a regular singular point.
For the singularity at $x=-2$,

$$p_0=\lim_{x\to -2}(x+2)p(x)=\lim_{x\to -2}\frac{2x}{2-x}=-1$$
$$q_0=\lim_{x\to -2}(x+2)^2q(x)=\lim_{x\to -2}\frac{3(x+2)}{2-x}=0$$

and therefore $x=-2$ is a regular singular point.

(b) For $x=2$: the indicial equation is given by $r(r-1)-r=0$, that is, $r^2-2r=0$, with roots $r_1=2$ and $r_2=0$.
For $x=-2$: the indicial equation is given by $r(r-1)-r=0$, that is, $r^2-2r=0$, with roots $r_1=2$ and $r_2=0$.

12.(a) $P(x)=0$ for $x=0$ and $x=-3$. We note that $p(x)=-2x^{-1}(x+3)^{-1}$, and $q(x)=-1/(x+3)^2$. For the singularity at $x=0$,

$$p_0=\lim_{x\to 0}x\,p(x)=\lim_{x\to 0}\frac{-2}{x+3}=-\frac{2}{3}$$
$$q_0=\lim_{x\to 0}x^2q(x)=\lim_{x\to 0}\frac{-x^2}{(x+3)^2}=0$$

and therefore $x=0$ is a regular singular point.
For the singularity at $x=-3$,

$$p_0=\lim_{x\to -3}(x+3)p(x)=\lim_{x\to -3}\frac{-2}{x}=\frac{2}{3}$$
$$q_0=\lim_{x\to -3}(x+3)^2q(x)=\lim_{x\to -3}(-1)=-1$$

and therefore $x=-3$ is a regular singular point.

(b) For $x=0$: the indicial equation is given by $r(r-1)-2r/3=0$, that is, $r^2-5r/3=0$, with roots $r_1=5/3$ and $r_2=0$.
For $x=-3$: the indicial equation is given by $r(r-1)+2r/3-1=0$, that is, $r^2-r/3-1=0$, with roots $r_1=(1+\sqrt{37})/6$ and $r_2=(1-\sqrt{37})/6$.

14.(a) Here $x\,p(x)=2x$ and $x^2q(x)=6\,xe^x$. Both of these functions are analytic at $x=0$, therefore $x=0$ is a regular singular point. Note that $p_0=q_0=0$.

(b) The indicial equation is given by $r(r-1)=0$, that is, $r^2-r=0$, with roots $r_1=1$ and $r_2=0$.

(c) In order to find the solution corresponding to $r_1 = 1$, set $y = x\sum_{n=0}^{\infty} a_n x^n$. Upon substitution into the ODE, we have

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+1}\,x^{n+1} + 2\sum_{n=0}^{\infty}(n+1)a_n x^{n+1} + 6\,e^x \sum_{n=0}^{\infty} a_n x^{n+1} = 0\,.$$

After adjusting the indices in the first two series, and expanding the exponential function,

$$\sum_{n=1}^{\infty} n(n+1)a_n\,x^n + 2\sum_{n=1}^{\infty} n\,a_{n-1}x^n + 6\,a_0 x + (6a_0 + 6a_1)x^2$$

$$+(6a_2 + 6a_1 + 3a_0)x^3 + (6a_3 + 6a_2 + 3a_1 + a_0)x^4 + \ldots = 0\,.$$

Equating the coefficients, we obtain the system of equations

$$\begin{aligned} 2a_1 + 2a_0 + 6a_0 &= 0 \\ 6a_2 + 4a_1 + 6a_0 + 6a_1 &= 0 \\ 12a_3 + 6a_2 + 6a_2 + 6a_1 + 3a_0 &= 0 \\ 20a_4 + 8a_3 + 6a_3 + 6a_2 + 3a_1 + a_0 &= 0 \\ &\vdots \end{aligned}$$

Setting $a_0 = 1$, solution of the system results in $a_1 = -4$, $a_2 = 17/3$, $a_3 = -47/12$, $a_4 = 191/120$, ... Therefore one solution is

$$y_1(x) = x - 4x^2 + \frac{17}{3}x^3 - \frac{47}{12}x^4 + \ldots\,.$$

The exponents differ by an integer. So for a second solution, set

$$y_2(x) = a\,y_1(x)\,\ln\,x\, + 1 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots\,.$$

Substituting into the ODE, we obtain

$$a\,L\,[y_1(x)]\cdot\ln\,x + 2a\,y_1'(x) + 2a\,y_1(x) - a\frac{y_1(x)}{x} + L\left[1 + \sum_{n=1}^{\infty} c_n\,x^n\right] = 0\,.$$

Since $L\,[y_1(x)] = 0$, it follows that

$$L\left[1 + \sum_{n=1}^{\infty} c_n\,x^n\right] = -2a\,y_1'(x) - 2a\,y_1(x) + a\frac{y_1(x)}{x}\,.$$

More specifically,

$$\sum_{n=1}^{\infty} n(n+1)c_{n+1}x^n + 2\sum_{n=1}^{\infty} n\,c_n x^n + 6\, + (6 + 6c_1)x$$

$$+(6c_2 + 6c_1 + 3)x^2 + \ldots = -a + 10ax - \frac{61}{3}ax^2 + \frac{193}{12}ax^3 + \ldots\,.$$

Equating the coefficients, we obtain the system of equations

$$\begin{aligned}
6 &= -a \\
2c_2 + 8c_1 + 6 &= 10a \\
6c_3 + 10c_2 + 6c_1 + 3 &= -\frac{61}{3}a \\
12c_4 + 12c_3 + 6c_2 + 3c_1 + 1 &= \frac{193}{12}a \\
&\vdots
\end{aligned}$$

Solving these equations for the coefficients, $a = -6$. In order to solve the remaining equations, set $c_1 = 0$. Then $c_2 = -33$, $c_3 = 449/6$, $c_4 = -1595/24, \ldots$. Therefore a second solution is

$$y_2(x) = -6\, y_1(x) \ln x + \left[1 - 33x^2 + \frac{449}{6}x^3 - \frac{1595}{24}x^4 + \ldots\right].$$

15.(a) Note the $p(x) = 6x/(x-1)$ and $q(x) = 3x^{-1}(x-1)^{-1}$. Furthermore, $x\,p(x) = 6x^2/(x-1)$ and $x^2 q(x) = 3x/(x-1)$. It follows that

$$\begin{aligned}
p_0 &= \lim_{x \to 0} \frac{6x^2}{x-1} = 0 \\
q_0 &= \lim_{x \to 0} \frac{3x}{x-1} = 0
\end{aligned}$$

and therefore $x = 0$ is a regular singular point.

(b) The indicial equation is given by $r(r-1) = 0$, that is, $r^2 - r = 0$, with roots $r_1 = 1$ and $r_2 = 0$.

(c) In order to find the solution corresponding to $r_1 = 1$, set $y = x \sum_{n=0}^{\infty} a_n x^n$. Upon substitution into the ODE, we have

$$\sum_{n=1}^{\infty} n(n+1)a_n\, x^{n+1} - \sum_{n=1}^{\infty} n(n+1)a_n\, x^n + 6\sum_{n=0}^{\infty}(n+1)a_n x^{n+2} + 3\sum_{n=0}^{\infty} a_n x^{n+1} = 0\,.$$

After adjusting the indices, it follows that

$$\sum_{n=2}^{\infty} n(n-1)a_{n-1}\, x^n - \sum_{n=1}^{\infty} n(n+1)a_n\, x^n + 6\sum_{n=2}^{\infty}(n-1)a_{n-2} x^n + 3\sum_{n=1}^{\infty} a_{n-1} x^n = 0\,.$$

That is,

$$-2a_1 + 3a_0 + \sum_{n=2}^{\infty}\left[-n(n+1)a_n + (n^2 - n + 3)a_{n-1} + 6(n-1)a_{n-2}\right] x^n = 0.$$

Setting the coefficients equal to zero, we have $a_1 = 3a_0/2$, and for $n \geq 2$,

$$n(n+1)a_n = (n^2 - n + 3)a_{n-1} + 6(n-1)a_{n-2}\,.$$

If we assign $a_0 = 1$, then we obtain $a_1 = 3/2$, $a_2 = 9/4$, $a_3 = 51/16$, ... Hence one solution is

$$y_1(x) = x + \frac{3}{2}x^2 + \frac{9}{4}x^3 + \frac{51}{16}x^4 + \frac{111}{40}x^5 + \ldots .$$

The exponents differ by an integer. So for a second solution, set

$$y_2(x) = a\, y_1(x)\, \ln x + 1 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots .$$

Substituting into the ODE, we obtain

$$2ax\, y_1'(x) - 2a\, y_1'(x) + 6ax\, y_1(x) - a\, y_1(x) + a\frac{y_1(x)}{x} + L\left[1 + \sum_{n=1}^{\infty} c_n\, x^n\right] = 0,$$

since $L\left[y_1(x)\right] = 0$. It follows that

$$L\left[1 + \sum_{n=1}^{\infty} c_n\, x^n\right] = 2a\, y_1'(x) - 2ax\, y_1'(x) + a\, y_1(x) - 6ax\, y_1(x) - a\frac{y_1(x)}{x}.$$

Now

$$\begin{aligned} L\left[1 + \sum_{n=1}^{\infty} c_n\, x^n\right] &= 3 + (-2c_2 + 3c_1)x + (-6c_3 + 5c_2 + 6c_1)x^2 + \\ &+ (-12c_4 + 9c_3 + 12c_2)x^3 + (-20c_5 + 15c_4 + 18c_3)x^4 + \ldots . \end{aligned}$$

Substituting for $y_1(x)$, the right hand side of the ODE is

$$a + \frac{7}{2}ax + \frac{3}{4}ax^2 + \frac{33}{16}ax^3 - \frac{867}{80}ax^4 - \frac{441}{10}ax^5 + \ldots .$$

Equating the coefficients, we obtain the system of equations

$$\begin{aligned} 3 &= a \\ -2c_2 + 3c_1 &= \frac{7}{2}a \\ -6c_3 + 5c_2 + 6c_1 &= \frac{3}{4}a \\ -12c_4 + 9c_3 + 12c_2 &= \frac{33}{16}a \\ &\vdots \end{aligned}$$

We find that $a = 3$. In order to solve the second equation, set $c_1 = 0$. Solution of the remaining equations results in $c_2 = -21/4$, $c_3 = -19/4$, $c_4 = -597/64$, ... Hence a second solution is

$$y_2(x) = 3\, y_1(x)\, \ln x + \left[1 - \frac{21}{4}x^2 - \frac{19}{4}x^3 - \frac{597}{64}x^4 + \ldots\right].$$

16.(a) After multiplying both sides of the ODE by x, we find that $x\, p(x) = 0$ and $x^2 q(x) = x$. Both of these functions are analytic at $x = 0$, hence $x = 0$ is a regular singular point.

(b) Furthermore, $p_0 = q_0 = 0$. So the indicial equation is $r(r-1) = 0$, with roots $r_1 = 1$ and $r_2 = 0$.

(c) In order to find the solution corresponding to $r_1 = 1$, set $y = x \sum_{n=0}^{\infty} a_n x^n$. Upon substitution into the ODE, we have

$$\sum_{n=1}^{\infty} n(n+1)a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+1} = 0 .$$

That is,

$$\sum_{n=1}^{\infty} \left[n(n+1)a_n + a_{n-1} \right] x^n = 0 .$$

Setting the coefficients equal to zero, we find that for $n \geq 1$,

$$a_n = \frac{-a_{n-1}}{n(n+1)} .$$

It follows that

$$a_n = \frac{-a_{n-1}}{n(n+1)} = \frac{a_{n-2}}{(n-1)n^2(n+1)} = \ldots = \frac{(-1)^n a_0}{(n!)^2(n+1)} .$$

Hence one solution is

$$y_1(x) = x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \frac{1}{2880}x^5 + \ldots .$$

The exponents differ by an integer. So for a second solution, set

$$y_2(x) = a\, y_1(x) \ln x + 1 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots .$$

Substituting into the ODE, we obtain

$$a\, L\left[y_1(x) \right] \cdot \ln x + 2a\, y_1'(x) - a\frac{y_1(x)}{x} + L\left[1 + \sum_{n=1}^{\infty} c_n x^n \right] = 0 .$$

Since $L\left[y_1(x) \right] = 0$, it follows that

$$L\left[1 + \sum_{n=1}^{\infty} c_n x^n \right] = -2a\, y_1'(x) + a\frac{y_1(x)}{x} .$$

Now

$$L\left[1 + \sum_{n=1}^{\infty} c_n x^n \right] = 1 + (2c_2 + c_1)x + (6c_3 + c_2)x^2 + (12c_4 + c_3)x^3$$

$$+ (20c_5 + c_4)x^4 + (30c_6 + c_5)x^5 + \ldots .$$

Substituting for $y_1(x)$, the right hand side of the ODE is

$$-a + \frac{3}{2}ax - \frac{5}{12}ax^2 + \frac{7}{144}ax^3 - \frac{1}{320}ax^4 + \ldots .$$

Equating the coefficients, we obtain the system of equations

$$\begin{aligned} 1 &= -a \\ 2c_2 + c_1 &= \frac{3}{2}a \\ 6c_3 + c_2 &= -\frac{5}{12}a \\ 12c_4 + c_3 &= \frac{7}{144}a \\ &\vdots \end{aligned}$$

Evidently, $a = -1$. In order to solve the second equation, set $c_1 = 0$. We then find that $c_2 = -3/4$, $c_3 = 7/36$, $c_4 = -35/1728$, ... Therefore a second solution is

$$y_2(x) = -y_1(x)\ \ln x + \left[1 - \frac{3}{4}x^2 + \frac{7}{36}x^3 - \frac{35}{1728}x^4 + \ldots\right].$$

19.(a) After dividing by the leading coefficient, we find that

$$p_0 = \lim_{x\to 0} x\,p(x) = \lim_{x\to 0} \frac{\gamma - (1+\alpha+\beta)x}{1-x} = \gamma\,.$$

$$q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} \frac{-\alpha\beta\, x}{1-x} = 0\,.$$

Hence $x = 0$ is a regular singular point. The indicial equation is $r(r-1) + \gamma\, r = 0$, with roots $r_1 = 1-\gamma$ and $r_2 = 0$.

(b) For $x = 1$,

$$p_0 = \lim_{x\to 1} (x-1)p(x) = \lim_{x\to 1} \frac{-\gamma + (1+\alpha+\beta)x}{x} = 1 - \gamma + \alpha + \beta\,.$$

$$q_0 = \lim_{x\to 1} (x-1)^2 q(x) = \lim_{x\to 1} \frac{\alpha\beta(x-1)}{x} = 0\,.$$

Hence $x = 1$ is a regular singular point. The indicial equation is

$$r^2 - (\gamma - \alpha - \beta)\, r = 0\,,$$

with roots $r_1 = \gamma - \alpha - \beta$ and $r_2 = 0$.

(c) Given that $r_1 - r_2$ is not a positive integer, we can set $y = \sum_{n=0}^{\infty} a_n x^n$. Substitution into the ODE results in

$$x(1-x)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + [\gamma - (1+\alpha+\beta)x]\sum_{n=1}^{\infty} n\, a_n x^{n-1} - \alpha\beta \sum_{n=0}^{\infty} a_n x^n = 0.$$

That is,

$$\sum_{n=1}^{\infty} n(n+1)a_{n+1}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n + \gamma \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$-(1+\alpha+\beta)\sum_{n=1}^{\infty} n\, a_n x^n - \alpha\beta \sum_{n=0}^{\infty} a_n x^n = 0.$$

Combining the series, we obtain

$$\gamma\, a_1 - \alpha\beta\, a_0 + [(2+2\gamma)a_2 - (1+\alpha+\beta+\alpha\beta)a_1]\, x + \sum_{n=2}^{\infty} A_n\, x^n = 0\,,$$

in which

$$A_n = (n+1)(n+\gamma)a_{n+1} - [n(n-1) + (1+\alpha+\beta)n + \alpha\beta]\, a_n\,.$$

Note that $n(n-1)+(1+\alpha+\beta)n+\alpha\beta = (n+\alpha)(n+\beta)$. Setting the coefficients equal to zero, we have $\gamma\, a_1 - \alpha\beta\, a_0 = 0$, and

$$a_{n+1} = \frac{(n+\alpha)(n+\beta)}{(n+1)(n+\gamma)}\; a_n$$

for $n \geq 1$. Hence one solution is

$$y_1(x) = 1 + \frac{\alpha\beta}{\gamma \cdot 1!}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)\cdot 2!}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)\cdot 3!}x^3 + \ldots\,.$$

Since the nearest other singularity is at $x = 1$, the radius of convergence of $y_1(x)$ will be at least $\rho = 1$.

(d) Given that $r_1 - r_2$ is not a positive integer, we can set $y = x^{1-\gamma}\sum_{n=0}^{\infty} b_n x^n$. Then substitution into the ODE results in

$$x(1-x)\sum_{n=0}^{\infty}(n+1-\gamma)(n-\gamma)a_n x^{n-\gamma-1}$$

$$+\,[\gamma-(1+\alpha+\beta)x]\sum_{n=0}^{\infty}(n+1-\gamma)a_n x^{n-\gamma} - \alpha\beta\sum_{n=0}^{\infty}a_n x^{n+1-\gamma} = 0.$$

That is,

$$\sum_{n=0}^{\infty}(n+1-\gamma)(n-\gamma)a_n x^{n-\gamma} - \sum_{n=0}^{\infty}(n+1-\gamma)(n-\gamma)a_n x^{n+1-\gamma}$$

$$+\gamma\sum_{n=0}^{\infty}(n+1-\gamma)a_n x^{n-\gamma} - (1+\alpha+\beta)\sum_{n=0}^{\infty}(n+1-\gamma)a_n x^{n+1-\gamma}$$

$$-\,\alpha\beta\sum_{n=0}^{\infty}a_n x^{n+1-\gamma} = 0.$$

After adjusting the indices,

$$\sum_{n=0}^{\infty}(n+1-\gamma)(n-\gamma)a_n x^{n-\gamma} - \sum_{n=1}^{\infty}(n-\gamma)(n-1-\gamma)a_{n-1}x^{n-\gamma}$$

$$+\gamma\sum_{n=0}^{\infty}(n+1-\gamma)a_n x^{n-\gamma} - (1+\alpha+\beta)\sum_{n=1}^{\infty}(n-\gamma)a_{n-1}x^{n-\gamma} - \alpha\beta\sum_{n=1}^{\infty}a_{n-1}x^{n-\gamma} = 0.$$

Combining the series, we obtain

$$\sum_{n=1}^{\infty} B_n\, x^{n-\gamma} = 0\,,$$

in which

$$B_n = n(n+1-\gamma)b_n - [(n-\gamma)(n-\gamma+\alpha+\beta)+\alpha\beta]\, b_{n-1}.$$

Note that $(n-\gamma)(n-\gamma+\alpha+\beta)+\alpha\beta = (n+\alpha-\gamma)(n+\beta-\gamma)$. Setting $B_n = 0$, it follows that for $n \geq 1$,

$$b_n = \frac{(n+\alpha-\gamma)(n+\beta-\gamma)}{n(n+1-\gamma)}\, b_{n-1}.$$

Therefore a second solution is

$$y_2(x) = x^{1-\gamma}\left[1 + \frac{(1+\alpha-\gamma)(1+\beta-\gamma)}{(2-\gamma)1!}x \right.$$
$$\left. + \frac{(1+\alpha-\gamma)(2+\alpha-\gamma)(1+\beta-\gamma)(2+\beta-\gamma)}{(2-\gamma)(3-\gamma)2\,!}x^2 + \ldots\right].$$

(e) Under the transformation $x = 1/\xi$, the ODE becomes

$$\xi^4 \frac{1}{\xi}(1-\frac{1}{\xi})\frac{d^2y}{d\xi^2} + \left\{2\xi^3\frac{1}{\xi}(1-\frac{1}{\xi}) - \xi^2\left[\gamma - (1+\alpha+\beta)\frac{1}{\xi}\right]\right\}\frac{dy}{d\xi} - \alpha\beta\, y = 0\,.$$

That is,

$$(\xi^3-\xi^2)\frac{d^2y}{d\xi^2} + \left[2\xi^2 - \gamma\,\xi^2 + (-1+\alpha+\beta)\xi\right]\frac{dy}{d\xi} - \alpha\beta\; y = 0\,.$$

Therefore $\xi = 0$ is a singular point. Note that

$$p(\xi) = \frac{(2-\gamma)\,\xi + (-1+\alpha+\beta)}{\xi^2-\xi} \text{ and } q(\xi) = \frac{-\alpha\beta}{\xi^3-\xi^2}\,.$$

It follows that

$$p_0 = \lim_{\xi\to 0}\xi\, p(\xi) = \lim_{\xi\to 0}\frac{(2-\gamma)\,\xi + (-1+\alpha+\beta)}{\xi-1} = 1-\alpha-\beta\,,$$

$$q_0 = \lim_{\xi\to 0}\xi^2 q(\xi) = \lim_{\xi\to 0}\frac{-\alpha\beta}{\xi-1} = \alpha\beta\,.$$

Hence $\xi = 0$ $(x = \infty)$ is a regular singular point. The indicial equation is

$$r(r-1) + (1-\alpha-\beta)r + \alpha\beta = 0\,,$$

or $r^2 - (\alpha+\beta)r + \alpha\beta = 0$. Evidently, the roots are $r = \alpha$ and $r = \beta$.

5.7

3. Here $x\,p(x) = 1$ and $x^2 q(x) = 2x$, which are both analytic everywhere. We set $y = x^r(a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\,x^{r+n} + \sum_{n=0}^{\infty}(r+n)a_n\,x^{r+n} + 2\sum_{n=0}^{\infty}a_n x^{r+n+1} = 0\,.$$

After adjusting the indices in the last series, we obtain

$$a_0\,[r(r-1)+r]\,x^r + \sum_{n=1}^{\infty}\left[(r+n)(r+n-1)a_n + (r+n)a_n + 2\,a_{n-1}\right]x^{r+n} = 0.$$

Assuming $a_0 \neq 0$, the indicial equation is $r^2 = 0$, with double root $r = 0$. Setting the remaining coefficients equal to zero, we have for $n \geq 1$,

$$a_n(r) = -\frac{2}{(n+r)^2}\,a_{n-1}(r)\,.$$

It follows that

$$a_n(r) = \frac{(-1)^n\,2^n}{\left[(n+r)(n+r-1)\ldots(1+r)\right]^2}\,a_0\,, \quad n \geq 1\,.$$

Since $r = 0$, one solution is given by

$$y_1(x) = \sum_{n=0}^{\infty}\frac{(-1)^n\,2^n}{(n!)^2}\,x^n\,.$$

For a second linearly independent solution, we follow the discussion in Section 5.6. First note that

$$\frac{a_n'(r)}{a_n(r)} = -2\left[\frac{1}{n+r} + \frac{1}{n+r-1} + \ldots + \frac{1}{1+r}\right].$$

Setting $r = 0$,

$$a_n'(0) = -2\,H_n\,a_n(0) = -2\,H_n\frac{(-1)^n\,2^n}{(n!)^2}\,.$$

Therefore,

$$y_2(x) = y_1(x)\,\ln x - 2\sum_{n=0}^{\infty}\frac{(-1)^n\,2^n H_n}{(n!)^2}\,x^n\,.$$

4. Here $x\,p(x) = 4$ and $x^2 q(x) = 2 + x$, which are both analytic everywhere. We set $y = x^r(a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\,x^{r+n} + 4\sum_{n=0}^{\infty}(r+n)a_n\,x^{r+n}$$

$$+\sum_{n=0}^{\infty}a_n x^{r+n+1} + 2\sum_{n=0}^{\infty}a_n x^{r+n} = 0\,.$$

After adjusting the indices in the second-to-last series, we obtain

$$a_0\,[r(r-1)+4r+2]\,x^r$$
$$+\sum_{n=1}^{\infty}\left[(r+n)(r+n-1)a_n+4(r+n)a_n+2\,a_n+a_{n-1}\right]x^{r+n}=0.$$

Assuming $a_0 \neq 0$, the indicial equation is $r^2+3r+2=0$, with roots $r_1=-1$ and $r_2=-2$. Setting the remaining coefficients equal to zero, we have for $n \geq 1$,

$$a_n(r)=-\frac{1}{(n+r+1)(n+r+2)}\,a_{n-1}(r)\,.$$

It follows that

$$a_n(r)=\frac{(-1)^n}{[(n+r+1)(n+r)\ldots(2+r)]\,[(n+r+2)(n+r)\ldots(3+r)]}\,a_0\,,\ n\geq 1\,.$$

Since $r_1=-1$, one solution is given by

$$y_1(x)=x^{-1}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n)!(n+1)!}\,x^n\,.$$

For a second linearly independent solution, we follow the discussion in Section 5.6. Since $r_1-r_2=N=1$, we find that

$$a_1(r)=-\frac{1}{(r+2)(r+3)}\,,$$

with $a_0=1$. Hence the leading coefficient in the solution is

$$a=\lim_{r\to -2}(r+2)\ a_1(r)=-1\,.$$

Further,

$$(r+2)\,a_n(r)=\frac{(-1)^n}{(n+r+2)\ [(n+r+1)(n+r)\ldots(3+r)]^2}\,.$$

Let $A_n(r)=(r+2)\,a_n(r)$. It follows that

$$\frac{A_n'(r)}{A_n(r)}=-\frac{1}{n+r+2}-2\left[\frac{1}{n+r+1}+\frac{1}{n+r}+\ldots+\frac{1}{3+r}\right].$$

Setting $r=r_2=-2$,

$$\frac{A_n'(-2)}{A_n(-2)}=-\frac{1}{n}-2\left[\frac{1}{n-1}+\frac{1}{n-2}+\ldots+1\right]=-H_n-H_{n-1}\,.$$

Hence

$$c_n(-2)=-(H_n+H_{n-1})\,A_n(-2)=-(H_n+H_{n-1})\frac{(-1)^n}{n!(n-1)!}\,.$$

Therefore,

$$y_2(x)=-y_1(x)\,\ln\,x+x^{-2}\left[1-\sum_{n=1}^{\infty}\frac{(-1)^n(H_n+H_{n-1})}{n!(n-1)!}\,x^n\right].$$

6. Let $y(x) = v(x)/\sqrt{x}$. Then $y' = x^{-1/2}\,v' - x^{-3/2}\,v/2$ and $y'' = x^{-1/2}\,v'' - x^{-3/2}\,v' + 3\,x^{-5/2}\,v/4$. Substitution into the ODE results in

$$\left[x^{3/2}\,v'' - x^{1/2}\,v' + 3\,x^{-1/2}\,v/4\right] + \left[x^{1/2}\,v' - x^{-1/2}\,v/2\right] + (x^2 - \frac{1}{4})x^{-1/2}\,v = 0\,.$$

Simplifying, we find that

$$v'' + v = 0\,,$$

with general solution $v(x) = c_1 \cos\, x + c_2 \sin\, x$. Hence

$$y(x) = c_1 x^{-1/2} \cos\, x + c_2\, x^{-1/2} \sin\, x\,.$$

8. The absolute value of the ratio of consecutive terms is

$$\left|\frac{a_{2m+2}\,x^{2m+2}}{a_{2m}\,x^{2m}}\right| = \frac{|x|^{2m+2}\;2^{2m}(m+1)!\,m!}{|x|^{2m}\;2^{2m+2}(m+2)!(m+1)!} = \frac{|x|^2}{4(m+2)(m+1)}\,.$$

Applying the ratio test,

$$\lim_{m\to\infty}\left|\frac{a_{2m+2}\,x^{2m+2}}{a_{2m}\,x^{2m}}\right| = \lim_{m\to\infty}\frac{|x|^2}{4(m+2)(m+1)} = 0\,.$$

Hence the series for $J_1(x)$ converges absolutely for all values of x . Furthermore, since the series for $J_0(x)$ also converges absolutely for all x, term-by-term differentiation results in

$$J_0'(x) = \sum_{m=1}^{\infty}\frac{(-1)^m x^{2m-1}}{2^{2m-1}\,m!(m-1)!} = \sum_{m=0}^{\infty}\frac{(-1)^{m+1}\,x^{2m+1}}{2^{2m+1}(m+1)!\,m!} =$$

$$= -\frac{x}{2}\sum_{m=0}^{\infty}\frac{(-1)^m\,x^{2m}}{2^{2m}(m+1)!\,m!}\,.$$

Therefore, $J_0'(x) = -J_1(x)$.

9.(a) Note that $x\,p(x) = 1$ and $x^2 q(x) = x^2 - \nu^2$, which are both analytic at $x = 0$. Thus $x = 0$ is a regular singular point. Furthermore, $p_0 = 1$ and $q_0 = -\nu^2$. Hence the indicial equation is $r^2 - \nu^2 = 0$, with roots $r_1 = \nu$ and $r_2 = -\nu$.

(b) Set $y = x^r(a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\,x^{r+n} + \sum_{n=0}^{\infty}(r+n)a_n\,x^{r+n}$$

$$+ \sum_{n=0}^{\infty} a_n x^{r+n+2} - \nu^2\sum_{n=0}^{\infty} a_n x^{r+n} = 0\,.$$

After adjusting the indices in the second-to-last series, we obtain

$$a_0\left[r(r-1) + r - \nu^2\right]x^r + a_1\left[(r+1)r + (r+1) - \nu^2\right]$$

$$+ \sum_{n=2}^{\infty}\left[(r+n)(r+n-1)a_n + (r+n)a_n - \nu^2 a_n + a_{n-2}\right]x^{r+n} = 0.$$

Setting the coefficients equal to zero, we find that $a_1 = 0$, and

$$a_n = \frac{-1}{(r+n)^2 - \nu^2}\, a_{n-2}\,,$$

for $n \geq 2$. It follows that $a_3 = a_5 = \ldots = a_{2m+1} = \ldots = 0$. Furthermore, with $r = \nu$,

$$a_n = \frac{-1}{n(n+2\nu)}\, a_{n-2}\,.$$

So for $m = 1, 2, \ldots$,

$$a_{2m} = \frac{-1}{2m(2m+2\nu)}\, a_{2m-2} = \frac{(-1)^m}{2^{2m}\, m!(1+\nu)(2+\nu)\ldots(m-1+\nu)(m+\nu)}\, a_0\,.$$

Hence one solution is

$$y_1(x) = x^{\nu}\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(1+\nu)(2+\nu)\ldots(m-1+\nu)(m+\nu)} (\frac{x}{2})^{2m}\right].$$

(c) Assuming that $r_1 - r_2 = 2\nu$ is not an integer, simply setting $r = -\nu$ in the above results in a second linearly independent solution

$$y_2(x) = x^{-\nu}\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(1-\nu)(2-\nu)\ldots(m-1-\nu)(m-\nu)} (\frac{x}{2})^{2m}\right].$$

(d) The absolute value of the ratio of consecutive terms in $y_1(x)$ is

$$\begin{aligned}\left|\frac{a_{2m+2}\, x^{2m+2}}{a_{2m}\, x^{2m}}\right| &= \frac{|x|^{2m+2}\; 2^{2m}\, m!(1+\nu)\ldots(m+\nu)}{|x|^{2m}\; 2^{2m+2}(m+1)!(1+\nu)\ldots(m+1+\nu)} \\ &= \frac{|x|^2}{4(m+1)(m+1+\nu)}\,.\end{aligned}$$

Applying the ratio test,

$$\lim_{m\to\infty}\left|\frac{a_{2m+2}\, x^{2m+2}}{a_{2m}\, x^{2m}}\right| = \lim_{m\to\infty} \frac{|x|^2}{4(m+1)(m+1+\nu)} = 0\,.$$

Hence the series for $y_1(x)$ converges absolutely for all values of x. The same can be shown for $y_2(x)$. Note also, that if ν is a positive integer, then the coefficients in the series for $y_2(x)$ are undefined.

10.(a) It suffices to calculate $L\,[J_0(x)\,\ln x]$. Indeed,

$$[J_0(x)\,\ln x]' = J_0'(x)\,\ln x + \frac{J_0(x)}{x}$$

and

$$[J_0(x)\,\ln x]'' = J_0''(x)\,\ln x + 2\frac{J_0'(x)}{x} - \frac{J_0(x)}{x^2}\,.$$

Hence

$$\begin{aligned}L\,[J_0(x)\,\ln x] &= x^2 J_0''(x)\,\ln x + 2x\,J_0'(x) - J_0(x) \\ &\quad + x\,J_0'(x)\,\ln x + J_0(x) + x^2 J_0(x)\,\ln x\,.\end{aligned}$$

Since $x^2 J_0''(x) + x\, J_0'(x) + x^2 J_0(x) = 0\,,$

$$L\,[J_0(x)\,\ln\,x] = 2x\,J_0'(x)\,.$$

(b) Given that $L\,[y_2(x)] = 0\,,$ after adjusting the indices in part (a), we have

$$b_1 x + 2^2 b_2\, x^2 + \sum_{n=3}^{\infty}(n^2 b_n + b_{n-2})x^n = -2x\,J_0'(x)\,.$$

Using the series representation of $J_0'(x)$ in Problem 8 ,

$$b_1 x + 2^2 b_2\, x^2 + \sum_{n=3}^{\infty}(n^2 b_n + b_{n-2})x^n = -2\sum_{n=1}^{\infty}\frac{(-1)^n(2n)x^{2n}}{2^{2n}(n!)^2}\,.$$

(c) Equating the coefficients on both sides of the equation, we find that

$$b_1 = b_3 = \ldots = b_{2m+1} = \ldots = 0\,.$$

Also, with $n = 1$, $2^2 b_2 = 1/(1!)^2$, that is, $b_2 = 1/\left[2^2(1!)^2\right]$. Furthermore, for $m \geq 2$,

$$(2m)^2 b_{2m} + b_{2m-2} = -2\frac{(-1)^m(2m)}{2^{2m}(m!)^2}\,.$$

More explicitly,

$$\begin{aligned} b_4 &= -\frac{1}{2^2\,4^2}(1+\frac{1}{2}) \\ b_6 &= \frac{1}{2^2\,4^2\,6^2}(1+\frac{1}{2}+\frac{1}{3}) \\ &\vdots \end{aligned}$$

It can be shown, in general, that

$$b_{2m} = (-1)^{m+1}\frac{H_m}{2^{2m}(m!)^2}\,.$$

11. Bessel's equation of order one is

$$x^2\,y'' + x\,y' + (x^2 - 1)y = 0\,.$$

Based on Problem 9, the roots of the indicial equation are $r_1 = 1$ and $r_2 = -1$. Set $y = x^r(a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\,x^{r+n} + \sum_{n=0}^{\infty}(r+n)a_n\,x^{r+n}$$
$$+\sum_{n=0}^{\infty}a_n x^{r+n+2} - \sum_{n=0}^{\infty}a_n x^{r+n} = 0\,.$$

After adjusting the indices in the second-to-last series, we obtain

$a_0\,[r(r-1) + r - 1]\,x^r + a_1\,[(r+1)r + (r+1) - 1]$

$$+\sum_{n=2}^{\infty}\left[(r+n)(r+n-1)a_n+(r+n)a_n-a_n+\,a_{n-2}\right]x^{r+n}=0.$$

Setting the coefficients equal to zero, we find that $a_1 = 0$, and

$$a_n(r)=\frac{-1}{(r+n)^2-1}\,a_{n-2}(r)=\frac{-1}{(n+r+1)(n+r-1)}\,a_{n-2}(r),$$

for $n \geq 2$. It follows that $a_3 = a_5 = \ldots = a_{2m+1} = \ldots = 0$. Solving the recurrence relation,

$$a_{2m}(r)=\frac{(-1)^m}{(2m+r+1)(2m+r-1)^2\ldots(r+3)^2(r+1)}\,a_0.$$

With $r = r_1 = 1$,

$$a_{2m}(1)=\frac{(-1)^m}{2^{2m}(m+1)!\,m!}\,a_0.$$

For a second linearly independent solution, we follow the discussion in Section 5.6. Since $r_1 - r_2 = N = 2$, we find that

$$a_2(r)=-\frac{1}{(r+3)(r+1)}\,,$$

with $a_0 = 1$. Hence the leading coefficient in the solution is

$$a=\lim_{r\to -1}(r+1)\,a_2(r)=-\frac{1}{2}\,.$$

Further,

$$(r+1)\,a_{2m}(r)=\frac{(-1)^m}{(2m+r+1)\;\left[(2m+r-1)\ldots(3+r)\right]^2}\,.$$

Let $A_n(r) = (r+1)\,a_n(r)$. It follows that

$$\frac{A'_{2m}(r)}{A_{2m}(r)}=-\frac{1}{2m+r+1}-2\left[\frac{1}{2m+r-1}+\ldots+\frac{1}{3+r}\right].$$

Setting $r = r_2 = -1$, we calculate

$$c_{2m}(-1)=-\frac{1}{2}(H_m+H_{m-1})A_{2m}(-1)$$

$$=-\frac{1}{2}(H_m+H_{m-1})\frac{(-1)^m}{2m\;\left[(2m-2)\ldots 2\right]^2}=-\frac{1}{2}(H_m+H_{m-1})\frac{(-1)^m}{2^{2m-1}\,m!(m-1)!}\,.$$

Note that $a_{2m+1}(r) = 0$ implies that $A_{2m+1}(r) = 0$, so

$$c_{2m+1}(-1)=\left[\frac{d}{dr}A_{2m+1}(r)\right]_{r=r_2}=0\,.$$

Therefore,

$$y_2(x)=-\frac{1}{2}\left[x\sum_{m=0}^{\infty}\frac{(-1)^m}{(m+1)!\,m!}(\frac{x}{2})^{2m}\right]\ln x$$

$$+\frac{1}{x}\left[1-\sum_{m=1}^{\infty}\frac{(-1)^m(H_m+H_{m-1})}{m!(m-1)!}(\frac{x}{2})^{2m}\right].$$

Based on the definition of $J_1(x)$,

$$y_2(x)=-J_1(x)\ \ln x+\frac{1}{x}\left[1-\sum_{m=1}^{\infty}\frac{(-1)^m(H_m+H_{m-1})}{m!(m-1)!}(\frac{x}{2})^{2m}\right].$$