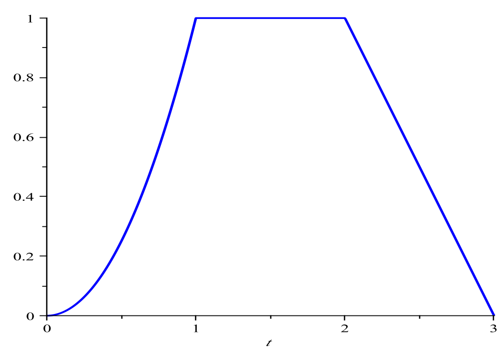
CHAPTER

6

The Laplace Transform

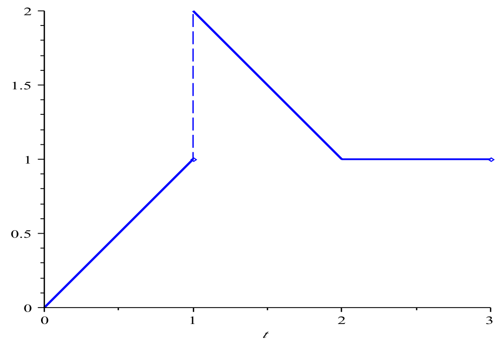
6.1

3.



The function $f(t)$ is continuous.

4.



The function $f(t)$ has a jump discontinuity at $t = 1$, and is thus piecewise continuous.

7. Integration is a linear operation. It follows that

$$\int_0^A \cosh bt \cdot e^{-st}dt = \frac{1}{2}\int_0^A e^{bt} \cdot e^{-st}dt + \frac{1}{2}\int_0^A e^{-bt} \cdot e^{-st}dt =$$

$$= \frac{1}{2}\int_0^A e^{(b-s)t}dt + \frac{1}{2}\int_0^A e^{-(b+s)t}dt\,.$$

Hence

$$\int_0^A \cosh bt \cdot e^{-st}dt = \frac{1}{2}\left[\frac{1-e^{(b-s)A}}{s-b}\right] + \frac{1}{2}\left[\frac{1-e^{-(b+s)A}}{s+b}\right].$$

Taking a limit, as $A \to \infty$,

$$\int_0^\infty \cosh bt \cdot e^{-st}dt = \frac{1}{2}\left[\frac{1}{s-b}\right] + \frac{1}{2}\left[\frac{1}{s+b}\right] = \frac{s}{s^2-b^2}\,.$$

Note that the above is valid for $s > |b|$.

8. Proceeding as in Problem 7,

$$\int_0^A \sinh bt \cdot e^{-st}dt = \frac{1}{2}\left[\frac{1-e^{(b-s)A}}{s-b}\right] - \frac{1}{2}\left[\frac{1-e^{-(b+s)A}}{s+b}\right].$$

Taking a limit, as $A \to \infty$,

$$\int_0^\infty \sinh bt \cdot e^{-st}dt = \frac{1}{2}\left[\frac{1}{s-b}\right] - \frac{1}{2}\left[\frac{1}{s+b}\right] = \frac{b}{s^2-b^2}\,.$$

The limit exists as long as $s > |b|$.

10. Observe that $e^{at}\sinh bt = (e^{(a+b)t} - e^{(a-b)t})/2$. It follows that

$$\int_0^A e^{at}\sinh bt \cdot e^{-st}dt = \frac{1}{2}\left[\frac{1-e^{(a+b-s)A}}{s-a+b}\right] - \frac{1}{2}\left[\frac{1-e^{-(b-a+s)A}}{s+b-a}\right].$$

Taking a limit, as $A \to \infty$,

$$\int_0^\infty e^{at} \sinh bt \cdot e^{-st} dt = \frac{1}{2}\left[\frac{1}{s-a+b}\right] - \frac{1}{2}\left[\frac{1}{s+b-a}\right] = \frac{b}{(s-a)^2 - b^2}\,.$$

The limit exists as long as $s - a > |b|$.

11. Using the linearity of the Laplace transform,

$$\mathcal{L}\left[\sin bt\right] = \frac{1}{2i}\mathcal{L}\left[e^{ibt}\right] - \frac{1}{2i}\mathcal{L}\left[e^{-ibt}\right].$$

Since

$$\int_0^\infty e^{(a+ib)t} e^{-st} dt = \frac{1}{s-a-ib}\ ,$$

we have

$$\int_0^\infty e^{\pm\, ibt}\, e^{-st} dt = \frac{1}{s \mp ib}\,.$$

Therefore

$$\mathcal{L}\left[\sin bt\right] = \frac{1}{2i}\left[\frac{1}{s\, -\, ib} - \frac{1}{s\, +\, ib}\right] = \frac{b}{s^2+b^2}\,.$$

The formula holds for $s > 0$.

12. Using the linearity of the Laplace transform,

$$\mathcal{L}\left[\cos bt\right] = \frac{1}{2}\mathcal{L}\left[e^{ibt}\right] + \frac{1}{2}\mathcal{L}\left[e^{-ibt}\right].$$

From Problem 11, we have

$$\int_0^\infty e^{\pm\, ibt}\, e^{-st} dt = \frac{1}{s \mp ib}\,.$$

Therefore

$$\mathcal{L}\left[\cos bt\right] = \frac{1}{2}\left[\frac{1}{s\, -\, ib} + \frac{1}{s\, +\, ib}\right] = \frac{s}{s^2+b^2}\,.$$

The formula holds for $s > 0$.

14. Using the linearity of the Laplace transform,

$$\mathcal{L}\left[e^{at}\cos bt\right] = \frac{1}{2}\mathcal{L}\left[e^{(a+ib)t}\right] + \frac{1}{2}\mathcal{L}\left[e^{(a-ib)t}\right].$$

Based on the integration in Problem 11,

$$\int_0^\infty e^{(a\,\pm\, ib)t} e^{-st} dt = \frac{1}{s-a \mp ib}\ .$$

Therefore

$$\mathcal{L}\left[e^{at}\cos bt\right] = \frac{1}{2}\left[\frac{1}{s\, -a-\, ib} + \frac{1}{s\, -a+\, ib}\right] = \frac{s-a}{(s-a)^2+b^2}\,.$$

The above is valid for $s > a$.

15. Integrating by parts,

$$\int_0^A te^{at} \cdot e^{-st}dt = -\frac{t\,e^{(a-s)t}}{s-a}\bigg|_0^A + \int_0^A \frac{1}{s-a}e^{(a-s)t}dt =$$

$$= \frac{1 - e^{A(a-s)} + A(a-s)e^{A(a-s)}}{(s-a)^2}\,.$$

Taking a limit, as $A \to \infty$,

$$\int_0^\infty te^{at} \cdot e^{-st}dt = \frac{1}{(s-a)^2}\,.$$

Note that the limit exists as long as $s > a$.

17. Observe that $t \cosh at = (t\,e^{at} + t\,e^{-at})/2$. For any value of c,

$$\int_0^A t\,e^{ct} \cdot e^{-st}dt = -\frac{t\,e^{(c-s)t}}{s-c}\bigg|_0^A + \int_0^A \frac{1}{s-c}e^{(c-s)t}dt =$$

$$= \frac{1 - e^{A(c-s)} + A(c-s)e^{A(c-s)}}{(s-c)^2}\,.$$

Taking a limit, as $A \to \infty$,

$$\int_0^\infty te^{ct} \cdot e^{-st}dt = \frac{1}{(s-c)^2}\,.$$

Note that the limit exists as long as $s > |c|$. Therefore,

$$\int_0^\infty t \cosh at \cdot e^{-st}dt = \frac{1}{2}\left[\frac{1}{(s-a)^2} + \frac{1}{(s+a)^2}\right] = \frac{s^2+a^2}{(s-a)^2(s+a)^2}\,.$$

18. Integrating by parts,

$$\int_0^A t^n e^{at} \cdot e^{-st}dt = -\frac{t^n\,e^{(a-s)t}}{s-a}\bigg|_0^A + \int_0^A \frac{n}{s-a}\,t^{n-1}e^{(a-s)t}dt =$$

$$= -\frac{A^n e^{-(s-a)A}}{s-a} + \int_0^A \frac{n}{s-a}\,t^{n-1}e^{(a-s)t}dt\,.$$

Continuing to integrate by parts, it follows that

$$\int_0^A t^n e^{at} \cdot e^{-st}dt = -\frac{A^n e^{(a-s)A}}{s-a} - \frac{nA^{n-1}e^{(a-s)A}}{(s-a)^2} -$$

$$-\frac{n!Ae^{(a-s)A}}{(n-2)!(s-a)^3} - \cdots - \frac{n!(e^{(a-s)A}-1)}{(s-a)^{n+1}}.$$

That is,

$$\int_0^A t^n e^{at} \cdot e^{-st}dt = p_n(A) \cdot e^{(a-s)A} + \frac{n!}{(s-a)^{n+1}},$$

in which $p_n(\xi)$ is a polynomial of degree n. For any given polynomial,

$$\lim_{A\to\infty} p_n(A) \cdot e^{-(s-a)A} = 0\,,$$

as long as $s > a$. Therefore,

$$\int_0^\infty t^n e^{at} \cdot e^{-st} dt = \frac{n!}{(s-a)^{n+1}}\,.$$

19. First observe that

$$\int \sin at \cdot e^{-st} dt = -\frac{\sin at \cdot e^{-st}}{s} + \frac{1}{s}\int a\cos at \cdot e^{-st} dt =$$

$$= -\frac{\sin at \cdot e^{-st}}{s} + \frac{a}{s}\left[-\frac{\cos at \cdot e^{-st}}{s} - \frac{a}{s}\int \sin at \cdot e^{-st} dt\right].$$

This implies that

$$\int \sin at \cdot e^{-st} dt = -\frac{(s\sin at + a\cos at)e^{-st}}{s^2+a^2}.$$

Integrating by parts we obtain that

$$\int_0^A t^2 \sin at \cdot e^{-st} dt = -t^2 \frac{(s\sin at + a\cos at)e^{-st}}{s^2+a^2}\bigg|_0^A +$$

$$+\int_0^A 2t\frac{(s\sin at + a\cos at)e^{-st}}{s^2+a^2} dt.$$

Taking the limit $A \to \infty$ and using the results of Problem 16 (from the Student Solutions Manual), we obtain that

$$\int_0^\infty t^2 \sin at \cdot e^{-st} dt = \frac{2s}{s^2+a^2}\frac{2as}{(s^2+a^2)^2} + \frac{2a}{s^2+a^2}\frac{s^2-a^2}{(s^2+a^2)^2} = \frac{2a(3s^2-a^2)}{(s^2+a^2)^3}.$$

This is valid for $s > 0$.

20. Observe that $t^2 \sinh at = (t^2e^{at} - t^2e^{-at})/2$. Using the result in Problem 18,

$$\int_0^\infty t^2 \sinh\, at \cdot e^{-st} dt = \frac{1}{2}\left[\frac{2!}{(s-a)^3} - \frac{2!}{(s+a)^3}\right] = \frac{2a(3s^2+a^2)}{(s^2-a^2)^3}\,.$$

The above is valid for $s > |a|$.

22. Using the fact that $f(t) = 0$ when $t \geq 1$ and integration by parts, we obtain that

$$\mathcal{L}\left[f(t)\right] = \int_0^\infty e^{-st} f(t)\, dt = \int_0^1 e^{-st} t\, dt = \left[-\frac{e^{-st}}{s}t\right]_0^1 + \int_0^1 \frac{e^{-st}}{s}\, dt$$

$$= -\frac{e^{-s}}{s} + \left[-\frac{e^{-st}}{s^2}\right]_0^1 = -\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}.$$

23. Using the definition of the Laplace transform and Problem 22, we get that

$$\mathcal{L}\left[f(t)\right] = \int_0^\infty e^{-st} f(t)\, dt = \int_0^1 e^{-st} t\, dt + \int_1^\infty e^{-st}\, dt =$$

$$= -\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} + \frac{e^{-s}}{s} = -\frac{e^{-s}}{s^2} + \frac{1}{s^2}.$$

26. Integrating by parts,

$$\int_0^A t\,e^{-t}dt = -t\,e^{-t}\Big|_0^A + \int_0^A e^{-t}dt = 1 - e^{-A} - Ae^{-A}.$$

Taking a limit, as $A \to \infty$,

$$\int_0^\infty t\,e^{-t}dt = 1.$$

Hence the integral converges.

27. Based on a series expansion, note that for $t > 0$, $e^t > 1 + t + t^2/2 > t^2/2$. It follows that for $t > 0$, $t^{-2}e^t > 1/2$. Hence for any finite $A > 1$,

$$\int_1^A t^{-2}e^t dt > \frac{A-1}{2}.$$

It is evident that the limit as $A \to \infty$ does not exist.

28. Using the fact that $|\cos t| \leq 1$, and the fact that

$$\int_0^\infty e^{-t}dt = 1,$$

it follows that the given integral converges.

30.(a) Let $p > 0$. Integrating by parts,

$$\int_0^A e^{-x}x^p dx = -e^{-x}x^p\Big|_0^A + p\int_0^A e^{-x}x^{p-1}dx = -A^p e^{-A} + p\int_0^A e^{-x}x^{p-1}dx.$$

Taking a limit, as $A \to \infty$,

$$\int_0^\infty e^{-x}x^p dx = p\int_0^\infty e^{-x}x^{p-1}dx.$$

That is, $\Gamma(p+1) = p\,\Gamma(p)$.

(b) Setting $p = 0$,

$$\Gamma(1) = \int_0^\infty e^{-x}dx = 1.$$

(c) Let $p = n$. Using the result in part (a),

$$\begin{aligned}\Gamma(n+1) &= n\,\Gamma(n)\\ &= n(n-1)\Gamma(n-1)\\ &\;\vdots\\ &= n(n-1)(n-2)\cdots 2\cdot 1\cdot\Gamma(1).\end{aligned}$$

Since $\Gamma(1) = 1$, $\Gamma(n+1) = n!$.

(d) Using the result in part (a),

$$\begin{aligned}\Gamma(p+n) =& (p+n-1)\,\Gamma(p+n-1)\\ =& (p+n-1)(p+n-2)\Gamma(p+n-2)\\ &\vdots\\ =& (p+n-1)(p+n-2)\cdots(p+1)p\,\Gamma(p)\,.\end{aligned}$$

Hence

$$\frac{\Gamma(p+n)}{\Gamma(p)} = p(p+1)(p+2)\cdots(p+n-1)\,.$$

Given that $\Gamma(1/2) = \sqrt{\pi}$, it follows that

$$\Gamma(\frac{3}{2}) = \frac{1}{2}\,\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}$$

and

$$\Gamma(\frac{11}{2}) = \frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\,\Gamma(\frac{3}{2}) = \frac{945\sqrt{\pi}}{32}\,.$$

6.2

1. Write the function as

$$\frac{3}{s^2+4} = \frac{3}{2}\,\frac{2}{s^2+4}\,.$$

Hence $\mathcal{L}^{-1}\,[Y(s)] = 3\sin 2t\,/2.$

3. Using partial fractions,

$$\frac{2}{s^2+3s-4} = \frac{2}{5}\left[\frac{1}{s-1} - \frac{1}{s+4}\right].$$

Hence $\mathcal{L}^{-1}\,[Y(s)] = 2(e^t - e^{-4t})/5.$

5. Note that the denominator s^2+2s+5 is irreducible over the reals. Completing the square, $s^2+2s+5 = (s+1)^2+4$. Now convert the function to a rational function of the variable $\xi = s+1$. That is,

$$\frac{2s+2}{s^2+2s+5} = \frac{2(s+1)}{(s+1)^2+4}\,.$$

We know that

$$\mathcal{L}^{-1}\left[\frac{2\,\xi}{\xi^2+4}\right] = 2\,\cos\,2t\,.$$

Using the fact that $\mathcal{L}\,[e^{at}f(t)] = \mathcal{L}\,[f(t)]_{s\to s-a}$,

$$\mathcal{L}^{-1}\left[\frac{2s+2}{s^2+2s+5}\right] = 2e^{-t}\cos\,2t\,.$$

6. Using partial fractions,

$$\frac{2s-3}{s^2-4} = \frac{1}{4}\left[\frac{1}{s-2} + \frac{7}{s+2}\right].$$

Hence $\mathcal{L}^{-1}[Y(s)] = (e^{2t} + 7e^{-2t})/4$. Note that we can also write

$$\frac{2s-3}{s^2-4} = 2\,\frac{s}{s^2-4} - \frac{3}{2}\,\frac{2}{s^2-4}.$$

8. Using partial fractions,

$$\frac{8s^2-4s+12}{s(s^2+4)} = 3\,\frac{1}{s} + 5\,\frac{s}{s^2+4} - 2\,\frac{2}{s^2+4}.$$

Hence $\mathcal{L}^{-1}[Y(s)] = 3 + 5\cos 2t - 2\sin 2t$.

9. The denominator s^2+4s+5 is irreducible over the reals. Completing the square, $s^2+4s+5 = (s+2)^2+1$. Now convert the function to a rational function of the variable $\xi = s+2$. That is,

$$\frac{1-2s}{s^2+4s+5} = \frac{5-2(s+2)}{(s+2)^2+1}.$$

We find that

$$\mathcal{L}^{-1}\left[\frac{5}{\xi^2+1} - \frac{2\,\xi}{\xi^2+1}\right] = 5\sin t - 2\cos t\,.$$

Using the fact that $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]_{s\to s-a}$,

$$\mathcal{L}^{-1}\left[\frac{1-2s}{s^2+4s+5}\right] = e^{-2t}(5\sin t - 2\cos t)\,.$$

10. Note that the denominator $s^2+2s+10$ is irreducible over the reals. Completing the square, $s^2+2s+10 = (s+1)^2+9$. Now convert the function to a rational function of the variable $\xi = s+1$. That is,

$$\frac{2s-3}{s^2+2s+10} = \frac{2(s+1)-5}{(s+1)^2+9}.$$

We find that

$$\mathcal{L}^{-1}\left[\frac{2\,\xi}{\xi^2+9} - \frac{5}{\xi^2+9}\right] = 2\cos 3t - \frac{5}{3}\sin 3t\,.$$

Using the fact that $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]_{s\to s-a}$,

$$\mathcal{L}^{-1}\left[\frac{2s-3}{s^2+2s+10}\right] = e^{-t}(2\cos 3t - \frac{5}{3}\sin 3t)\,.$$

12. Taking the Laplace transform of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + 3\,[s\,Y(s) - y(0)] + 2\,Y(s) = 0\,.$$

Applying the initial conditions,

$$s^2\,Y(s) + 3s\,Y(s) + 2\,Y(s) - s - 3 = 0\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{s+3}{s^2+3s+2}\,.$$

Using partial fractions,

$$\frac{s+3}{s^2+3s+2} = \frac{2}{s+1} - \frac{1}{s+2}.$$

Hence $y(t) = \mathcal{L}^{-1}\left[Y(s)\right] = 2\,e^{-t} - e^{-2t}$.

13. Taking the Laplace transform of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) - 2\left[s\,Y(s) - y(0)\right] + 2\,Y(s) = 0\,.$$

Applying the initial conditions,

$$s^2\,Y(s) - 2s\,Y(s) + 2\,Y(s) - 1 = 0\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{1}{s^2-2s+2}\,.$$

Since the denominator is irreducible, write the transform as a function of $\xi = s-1$. That is,

$$\frac{1}{s^2-2s+2} = \frac{1}{(s-1)^2+1}.$$

First note that

$$\mathcal{L}^{-1}\left[\frac{1}{\xi^2+1}\right] = \sin\,t\,.$$

Using the fact that $\mathcal{L}\left[e^{at}f(t)\right] = \mathcal{L}\left[f(t)\right]_{s\to s-a}$,

$$\mathcal{L}^{-1}\left[\frac{1}{s^2-2s+2}\right] = e^{t}\sin\,t\,.$$

Hence $y(t) = e^{t}\sin\,t\,.$

16. Taking the Laplace transform of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + 2\left[s\,Y(s) - y(0)\right] + 5\,Y(s) = 0\,.$$

Applying the initial conditions,

$$s^2\,Y(s) + 2s\,Y(s) + 5\,Y(s) - 2s - 3 = 0\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{2s+3}{s^2+2s+5}\,.$$

Since the denominator is irreducible, write the transform as a function of $\xi = s + 1$. That is,

$$\frac{2s+3}{s^2+2s+5} = \frac{2(s+1)+1}{(s+1)^2+4}.$$

We know that

$$\mathcal{L}^{-1}\left[\frac{2\,\xi}{\xi^2+4} + \frac{1}{\xi^2+4}\right] = 2\cos 2t + \frac{1}{2}\sin 2t\,.$$

Using the fact that $\mathcal{L}\left[e^{at}f(t)\right] = \mathcal{L}\left[f(t)\right]_{s\to s-a}$, the solution of the IVP is

$$y(t) = \mathcal{L}^{-1}\left[\frac{2s+3}{s^2+2s+5}\right] = e^{-t}(2\cos 2t + \frac{1}{2}\sin 2t\,)\,.$$

18. Taking the Laplace transform of the ODE, we obtain

$$s^4\,Y(s) - s^3y(0) - s^2y'(0) - s\,y''(0) - y'''(0) - Y(s) = 0\;.$$

Applying the initial conditions,

$$s^4Y(s) - Y(s) - s^3 - s = 0\,.$$

Solving for the transform of the solution,

$$Y(s) = \frac{s}{s^2-1}\,.$$

By inspection, it follows that $y(t) = \mathcal{L}^{-1}\left[Y(s)\right] = \cosh\,t\,$.

19. Taking the Laplace transform of the ODE, we obtain

$$s^4\,Y(s) - s^3y(0) - s^2y'(0) - s\,y''(0) - y'''(0) - 4\,Y(s) = 0\;.$$

Applying the initial conditions,

$$s^4Y(s) - 4Y(s) - s^3 + 2s = 0\,.$$

Solving for the transform of the solution,

$$Y(s) = \frac{s}{s^2+2}\,.$$

It follows that $y(t) = \mathcal{L}^{-1}\left[Y(s)\right] = \cos\sqrt{2}\,t\,$.

21. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) - 2\left[s\,Y(s) - y(0)\right] + 2\,Y(s) = \frac{s}{s^2+1}\,.$$

Applying the initial conditions,

$$s^2\,Y(s) - 2s\,Y(s) + 2\,Y(s) - s + 2 = \frac{s}{s^2+1}\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{s}{(s^2-2s+2)(s^2+1)} + \frac{s-2}{s^2-2s+2}\,.$$

Using partial fractions on the first term,

$$\frac{s}{(s^2-2s+2)(s^2+1)} = \frac{1}{5}\left[\frac{s-2}{s^2+1} - \frac{s-4}{s^2-2s+2}\right].$$

Thus we can write

$$Y(s) = \frac{1}{5}\,\frac{s}{s^2+1} - \frac{2}{5}\,\frac{1}{s^2+1} + \frac{2}{5}\,\frac{2s-3}{s^2-2s+2}.$$

For the last term, we note that $s^2-2s+2 = (s-1)^2+1$. So that

$$\frac{2s-3}{s^2-2s+2} = \frac{2(s-1)-1}{(s-1)^2+1}.$$

We know that

$$\mathcal{L}^{-1}\left[\frac{2\,\xi}{\xi^2+1} - \frac{1}{\xi^2+1}\right] = 2\,\cos\,t - \sin\,t\,.$$

Based on the translation property of the Laplace transform,

$$\mathcal{L}^{-1}\left[\frac{2s-3}{s^2-2s+2}\right] = e^t(2\,\cos\,t - \sin\,t).$$

Combining the above, the solution of the IVP is

$$y(t) = \frac{1}{5}\,\cos\,t - \frac{2}{5}\,\sin\,t + \frac{2}{5}\,e^t(2\,\cos\,t - \sin\,t)\,.$$

23. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + 2\,[s\,Y(s) - y(0)] + Y(s) = \frac{4}{s+1}.$$

Applying the initial conditions,

$$s^2\,Y(s) + 2s\,Y(s) + Y(s) - 2s - 3 = \frac{4}{s+1}.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{4}{(s+1)^3} + \frac{2s+3}{(s+1)^2}.$$

First write

$$\frac{2s+3}{(s+1)^2} = \frac{2(s+1)+1}{(s+1)^2} = \frac{2}{s+1} + \frac{1}{(s+1)^2}.$$

We note that

$$\mathcal{L}^{-1}\left[\frac{4}{\xi^3} + \frac{2}{\xi} + \frac{1}{\xi^2}\right] = 2\,t^2 + 2 + t\,.$$

So based on the translation property of the Laplace transform, the solution of the IVP is

$$y(t) = 2\,t^2e^{-t} + t\,e^{-t} + 2\,e^{-t}.$$

25. Let $f(t)$ be the forcing function on the right-hand-side. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + \,Y(s) = \mathcal{L}\left[f(t)\right].$$

Applying the initial conditions,

$$s^2\,Y(s) + \,Y(s) = \mathcal{L}\left[f(t)\right].$$

Based on the definition of the Laplace transform,

$$\mathcal{L}\left[f(t)\right] = \int_0^\infty f(t)\,e^{-st}dt = \int_0^1 t\,e^{-st}dt = \frac{1}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-s}}{s^2}\,.$$

Solving for the transform,

$$Y(s) = \frac{1}{s^2(s^2+1)} - e^{-s}\frac{s+1}{s^2(s^2+1)}\,.$$

26. Let $f(t)$ be the forcing function on the right-hand-side. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + 4\,Y(s) = \mathcal{L}\left[f(t)\right].$$

Applying the initial conditions,

$$s^2\,Y(s) + \,4\,Y(s) = \mathcal{L}\left[f(t)\right].$$

Based on the definition of the Laplace transform,

$$\mathcal{L}\left[f(t)\right] = \int_0^\infty f(t)\,e^{-st}dt = \int_0^1 t\,e^{-st}dt + \int_1^\infty e^{-st}dt = \frac{1}{s^2} - \frac{e^{-s}}{s^2}\,.$$

Solving for the transform,

$$Y(s) = \frac{1}{s^2(s^2+4)} - e^{-s}\frac{1}{s^2(s^2+4)}\,.$$

29.(a) Assuming that the conditions of Theorem 6.2.1 are satisfied,

$$F'(s) = \frac{d}{ds}\int_0^\infty e^{-st}f(t)dt = \int_0^\infty \frac{\partial}{\partial s}\left[e^{-st}f(t)\right]dt =$$
$$= \int_0^\infty \left[-t\,e^{-st}f(t)\right]dt = \int_0^\infty e^{-st}\left[-tf(t)\right]dt\,.$$

(b) Using mathematical induction, suppose that for some $k \geq 1$,

$$F^{(k)}(s) = \int_0^\infty e^{-st}\left[(-t)^k f(t)\right]dt\,.$$

Differentiating both sides,

$$F^{(k+1)}(s) = \frac{d}{ds}\int_0^\infty e^{-st}\left[(-t)^k f(t)\right]dt = \int_0^\infty \frac{\partial}{\partial s}\left[e^{-st}(-t)^k f(t)\right]dt =$$
$$= \int_0^\infty \left[-t\,e^{-st}(-t)^k f(t)\right]dt = \int_0^\infty e^{-st}\left[(-t)^{k+1}f(t)\right]dt\,.$$

30. We know that

$$\mathcal{L}\left[e^{at}\right] = \frac{1}{s-a}.$$

Based on Problem 29,

$$\mathcal{L}\left[-t\,e^{at}\right] = \frac{d}{ds}\left[\frac{1}{s-a}\right].$$

Therefore,

$$\mathcal{L}\left[t\,e^{at}\right] = \frac{1}{(s-a)^2}.$$

32. Based on Problem 29,

$$\mathcal{L}\left[(-t)^n\right] = \frac{d^n}{ds^n}\mathcal{L}\left[\,1\,\right] = \frac{d^n}{ds^n}\left[\frac{1}{s}\right].$$

Therefore,

$$\mathcal{L}\left[\,t^n\,\right] = (-1)^n\frac{(-1)^n n!}{s^{n+1}} = \frac{n!}{s^{n+1}}.$$

34. Using the translation property of the Laplace transform,

$$\mathcal{L}\left[e^{at}\,\sin\,bt\right] = \frac{b}{(s-a)^2+b^2}.$$

Therefore,

$$\mathcal{L}\left[t\,e^{at}\,\sin\,bt\right] = -\frac{d}{ds}\left[\frac{b}{(s-a)^2+b^2}\right] = \frac{2b(s-a)}{((s-a)^2+b^2)^2}.$$

35. Using the translation property of the Laplace transform,

$$\mathcal{L}\left[e^{at}\,\cos\,bt\right] = \frac{s-a}{(s-a)^2+b^2}.$$

Therefore,

$$\mathcal{L}\left[t\,e^{at}\,\cos\,bt\right] = -\frac{d}{ds}\left[\frac{s-a}{(s-a)^2+b^2}\right] = \frac{(s-a)^2-b^2}{((s-a)^2+b^2)^2}.$$

36.(a) Taking the Laplace transform of the given Bessel equation,

$$\mathcal{L}\left[\,t\,y''\right] + \mathcal{L}\left[\,y'\right] + \mathcal{L}\left[\,t\,y\,\right] = 0\,.$$

Using the differentiation property of the transform,

$$-\frac{d}{ds}\mathcal{L}\left[\,y''\right] + \mathcal{L}\left[\,y'\right] - \frac{d}{ds}\mathcal{L}\left[\,y\,\right] = 0\,.$$

That is,

$$-\frac{d}{ds}\left[s^2Y(s) - s\,y(0) - y'(0)\right] + s\,Y(s) - \,y(0) - \frac{d}{ds}Y(s) = 0\,.$$

It follows that

$$(1+s^2)Y'(s) + s\,Y(s) = 0\,.$$

(b) We obtain a first-order linear ODE in $Y(s)$:

$$Y'(s) + \frac{s}{s^2+1} Y(s) = 0\,,$$

with integrating factor

$$\mu(s) = e^{\int \frac{s}{s^2+1} ds} = \sqrt{s^2+1}\,.$$

The first-order ODE can be written as

$$\frac{d}{ds}\left[\sqrt{s^2+1} \cdot Y(s)\right] = 0\,,$$

with solution

$$Y(s) = \frac{c}{\sqrt{s^2+1}}\,.$$

(c) In order to obtain negative powers of s, first write

$$\frac{1}{\sqrt{s^2+1}} = \frac{1}{s}\left[1 + \frac{1}{s^2}\right]^{-1/2}.$$

Expanding $(1+1/s^2)^{-1/2}$ in a binomial series,

$$\frac{1}{\sqrt{1+(1/s^2)}} = 1 - \frac{1}{2}\,s^{-2} + \frac{1\cdot 3}{2\cdot 4}\,s^{-4} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\,s^{-6} + \cdots\,,$$

valid for $s^{-2} < 1$. Hence, we can formally express $Y(s)$ as

$$Y(s) = c\left[\frac{1}{s} - \frac{1}{2}\,\frac{1}{s^3} + \frac{1\cdot 3}{2\cdot 4}\,\frac{1}{s^5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\,\frac{1}{s^7} + \cdots\right].$$

Assuming that term-by-term inversion is valid,

$$\begin{aligned} y(t) &= c\left[1 - \frac{1}{2}\,\frac{t^2}{2!} + \frac{1\cdot 3}{2\cdot 4}\,\frac{t^4}{4!} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\,\frac{t^6}{6!} + \cdots\right] \\ &= c\left[1 - \frac{2!}{2^2}\,\frac{t^2}{2!} + \frac{4!}{2^2\cdot 4^2}\,\frac{t^4}{4!} - \frac{6!}{2^2\cdot 4^2\cdot 6^2}\,\frac{t^6}{6!} + \cdots\right]. \end{aligned}$$

It follows that

$$y(t) = c\left[1 - \frac{1}{2^2}\,t^2 + \frac{1}{2^2\cdot 4^2}\,t^4 - \frac{1}{2^2\cdot 4^2\cdot 6^2}\,t^6 + \cdots\right] = c\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2}\,t^{2n}.$$

The series is evidently the expansion, about $x = 0$, of $J_0(t)$.

38. By definition of the Laplace transform, given the appropriate conditions,

$$\mathcal{L}\,[g(t)] = \int_0^\infty e^{-st}\left[\int_0^t f(\tau)d\tau\right]dt = \int_0^\infty \int_0^t e^{-st} f(\tau)d\tau dt\,.$$

Assuming that the order of integration can be exchanged,

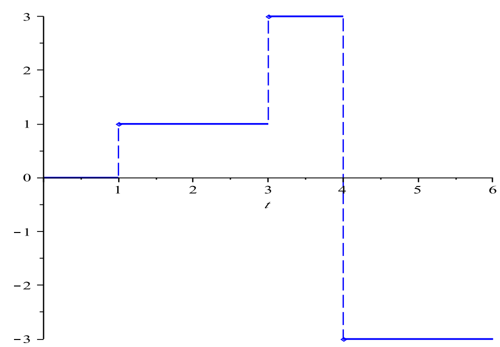
$$\mathcal{L}\,[g(t)] = \int_0^\infty f(\tau)\left[\int_\tau^\infty e^{-st}dt\right]d\tau = \int_0^\infty f(\tau)\left[\frac{e^{-s\tau}}{s}\right]d\tau.$$

(Note the region of integration is the area between the lines $\tau(t) = t$ and $\tau(t) = 0$.) Hence

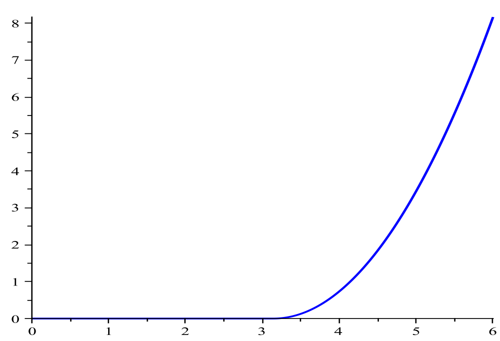
$$\mathcal{L}\left[\, g(t)\right] = \frac{1}{s}\int_0^\infty f(\tau)\, e^{-s\tau} d\tau = \frac{1}{s}\,\mathcal{L}\left[\, f(t)\right].$$

6.3

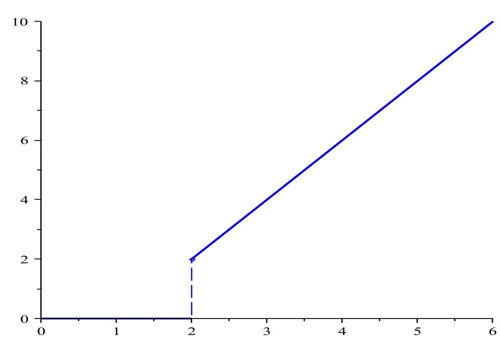
1.



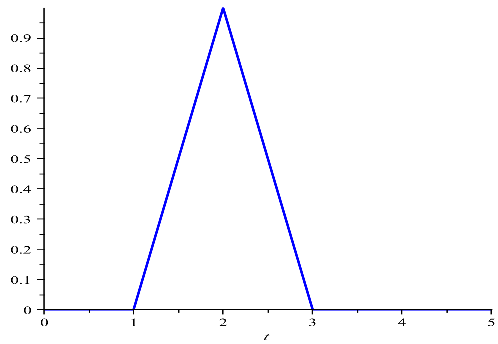
3.



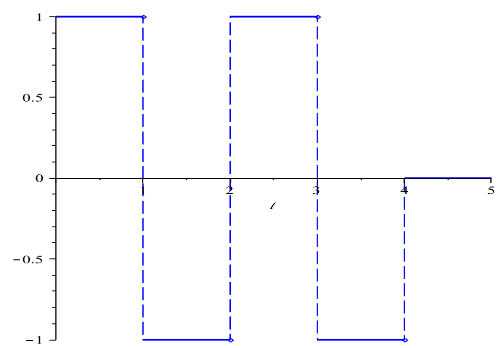
5.



6.

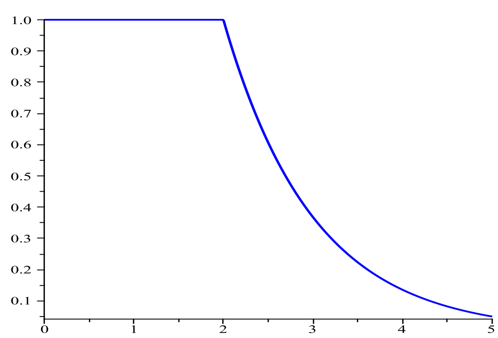


8.(a)



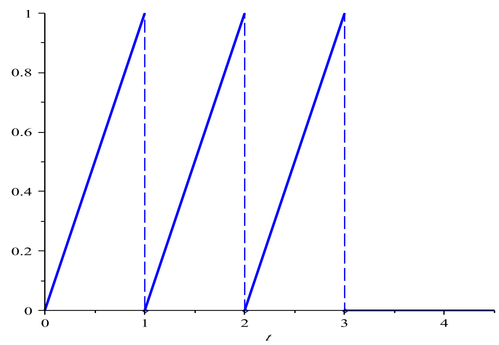
(b) $f(t) = 1 - 2u_1(t) + 2u_2(t) - 2u_3(t) + u_4(t)$.

9.(a)



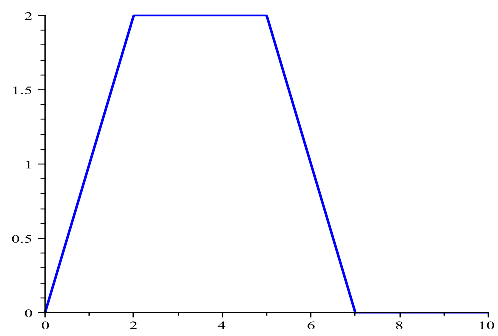
(b) $f(t) = 1 + (e^{-(t-2)} - 1)u_2(t)$.

11.(a)



(b) $f(t) = t - u_1(t) - u_2(t) + (2 - t)u_3(t)$.

12.(a)



(b) $f(t) = t + (2 - t)u_2(t) + (5 - t)u_5(t) + (t - 7)u_7(t)$.

13. Using the Heaviside function, we can write $f(t) = (t - 2)^2\, u_2(t)$. The Laplace transform has the property that $\mathcal{L}\,[\,u_c(t)f(t - c)] = e^{-cs}\mathcal{L}\,[f(t)]$. Hence

$$\mathcal{L}\left[\,(t-2)^2\,u_2(t)\right] = \frac{2\,e^{-2s}}{s^3}.$$

15. The function can be expressed as $f(t) = (t - \pi)\,[u_\pi(t) - u_{2\pi}(t)]$. Before invoking the translation property of the transform, write the function as

$$f(t) = (t-\pi)\,u_\pi(t) - (t-2\pi)\,u_{2\pi}(t) - \pi\,u_{2\pi}(t)\,.$$

It follows that

$$\mathcal{L}\,[\,f(t)] = \frac{e^{-\pi s}}{s^2} - \frac{e^{-2\pi s}}{s^2} - \frac{\pi e^{-2\pi s}}{s}.$$

16. It follows directly from the translation property of the transform that

$$\mathcal{L}\,[\,f(t)] = \frac{e^{-s}}{s} + 2\frac{e^{-3s}}{s} - 6\frac{e^{-4s}}{s}.$$

17. Before invoking the translation property of the transform, write the function as

$$f(t) = (t-2)\,u_2(t) - \,u_2(t) - (t-3)\,u_3(t) - u_3(t)\,.$$

It follows that

$$\mathcal{L}\left[\,f(t)\right] = \frac{e^{-2s}}{s^2} - \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s^2} - \frac{e^{-3s}}{s}.$$

18. It follows directly from the translation property of the transform that

$$\mathcal{L}\left[\,f(t)\right] = \frac{1}{s^2} - \frac{e^{-s}}{s^2}.$$

19. Using the fact that $\mathcal{L}\left[e^{at}f(t)\right] = \mathcal{L}\left[f(t)\right]_{s\to s-a}$,

$$\mathcal{L}^{-1}\left[\frac{3!}{(s-2)^4}\right] = t^3e^{2t}\,.$$

22. The inverse transform of the function $2/(s^2-4)$ is $f(t) = \sinh 2t$. Using the translation property of the transform,

$$\mathcal{L}^{-1}\left[\frac{2\,e^{-2s}}{s^2-4}\right] = \sinh\left(2(t-2)\right)\cdot\,u_2(t)\,.$$

23. First consider the function

$$G(s) = \frac{(s-2)}{s^2-4s+3}\,.$$

Completing the square in the denominator,

$$G(s) = \frac{(s-2)}{(s-2)^2-1}\,.$$

It follows that $\mathcal{L}^{-1}\left[G(s)\right] = e^{2t}\cosh t$. Hence

$$\mathcal{L}^{-1}\left[\frac{(s-2)e^{-s}}{s^2-4s+3}\right] = e^{2(t-1)}\cosh\left(t-1\right)u_1(t)\,.$$

24. Write the function as

$$F(s) = \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} - \frac{e^{-4s}}{s}\,.$$

It follows from the translation property of the transform, that

$$\mathcal{L}^{-1}\left[\frac{e^{-s}+e^{-2s}-e^{-3s}-e^{-4s}}{s}\right] = u_1(t)+u_2(t)-u_3(t)-u_4(t)\,.$$

25.(a) By definition of the Laplace transform,

$$\mathcal{L}\left[\,f(ct)\right] = \int_0^\infty e^{-st}f(ct)dt\,.$$

Making a change of variable, $\tau = ct$, we have

$$\mathcal{L}[f(ct)] = \frac{1}{c}\int_0^\infty e^{-s(\tau/c)} f(\tau)d\tau = \frac{1}{c}\int_0^\infty e^{-(s/c)\tau} f(\tau)d\tau\,.$$

Hence $\mathcal{L}[f(ct)] = (1/c)\,F(s/c)$, where $s/c > a$.

(b) Using the result in part (a),

$$\mathcal{L}\left[f\left(\frac{t}{k}\right)\right] = k\,F(ks).$$

Hence

$$\mathcal{L}^{-1}[F(ks)] = \frac{1}{k}\,f\left(\frac{t}{k}\right).$$

(c) From part (b), $\mathcal{L}^{-1}[F(as)] = (1/a)f(t/a)$ Note that $as + b = a(s + b/a)$. Using the fact that $\mathcal{L}[e^{ct}f(t)] = \mathcal{L}[f(t)]_{s\to s-c}$,

$$\mathcal{L}^{-1}[F(as+b)] = e^{-bt/a}\,\frac{1}{a}f\left(\frac{t}{a}\right).$$

26. First write

$$F(s) = \frac{n!}{(\frac{s}{2})^{n+1}}\,.$$

Let $G(s) = n!/s^{n+1}$. Based on the results in Problem 25,

$$\frac{1}{2}\,\mathcal{L}^{-1}\left[G\left(\frac{s}{2}\right)\right] = g(2t),$$

in which $g(t) = t^n$. Hence $\mathcal{L}^{-1}[F(s)] = 2\,(2t)^n = 2^{n+1}t^n$.

29. First write

$$F(s) = \frac{e^{-4(s-1/2)}}{2(s-1/2)}\,.$$

Now consider

$$G(s) = \frac{e^{-2s}}{s}\,.$$

Using the result in Problem 25(b),

$$\mathcal{L}^{-1}[G(2s)] = \frac{1}{2}g\left(\frac{t}{2}\right),$$

in which $g(t) = u_2(t)$. Hence $\mathcal{L}^{-1}[G(2s)] = u_2(t/2)/2 = u_4(t)/2$. It follows that

$$\mathcal{L}^{-1}[F(s)] = \frac{1}{2}e^{t/2}\,u_4(t).$$

30. By definition of the Laplace transform,

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st}u_1(t)dt\,.$$

That is,

$$\mathcal{L}\,[\,f(t)] = \int_0^1 e^{-st}\,dt = \frac{1-e^{-s}}{s}\,.$$

31. First write the function as $f(t) = u_0(t) - u_1(t)\, +\, u_2(t) - u_3(t)$. It follows that

$$\mathcal{L}\,[\,f(t)] = \int_0^1 e^{-st}dt\, + \int_2^3 e^{-st}dt\,.$$

That is,

$$\mathcal{L}\,[\,f(t)] = \frac{1-e^{-s}}{s} + \frac{e^{-2s}-e^{-3s}}{s} = \frac{1-e^{-s}+e^{-2s}-e^{-3s}}{s}\,.$$

32. The transform may be computed directly. On the other hand, using the translation property of the transform,

$$\mathcal{L}\,[f(t)] = \frac{1}{s} + \sum_{k=1}^{2n+1}(-1)^k\,\frac{e^{-ks}}{s} = \frac{1}{s}\left[\sum_{k=0}^{2n+1}(-e^{-s})^k\right] = \frac{1}{s}\,\frac{1-(-e^{-s})^{2n+2}}{1+e^{-s}}\,.$$

That is,

$$\mathcal{L}\,[f(t)] = \frac{1-(e^{-2s})^{n+1}}{s(1+e^{-s})}\,.$$

35. The given function is periodic, with $T = 2$. Using the result of Problem 34,

$$\mathcal{L}\,[f(t)] = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st}f(t)dt = \frac{1}{1-e^{-2s}}\int_0^1 e^{-st}dt\,.$$

That is,

$$\mathcal{L}\,[f(t)] = \frac{1-e^{-s}}{s(1-e^{-2s})} = \frac{1}{s(1+e^{-s})}\,.$$

37. The function is periodic, with $T = 1$. Using the result of Problem 34,

$$\mathcal{L}\,[f(t)] = \frac{1}{1-e^{-s}}\int_0^1 t\,e^{-st}dt\,.$$

It follows that

$$\mathcal{L}\,[f(t)] = \frac{1-e^{-s}(1+s)}{s^2(1-e^{-s})}\,.$$

38. The function is periodic, with $T = \pi$. Using the result of Problem 34,

$$\mathcal{L}\,[f(t)] = \frac{1}{1-e^{-\pi s}}\int_0^\pi \sin\,t\cdot e^{-st}dt\,.$$

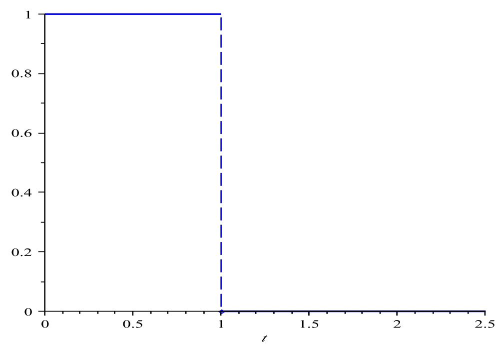
We first calculate

$$\int_0^\pi \sin\,t\cdot e^{-st}dt = \frac{1+e^{-\pi s}}{1+s^2}\,.$$

Hence

$$\mathcal{L}\,[f(t)] = \frac{1+e^{-\pi s}}{(1-e^{-\pi s})(1+s^2)}\,.$$

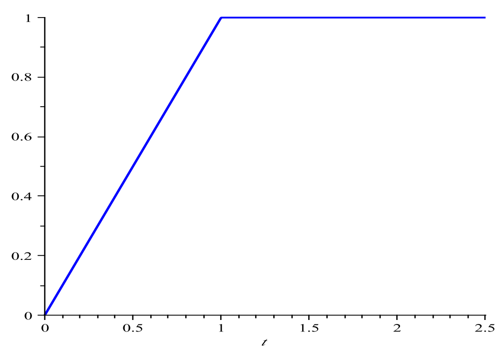
39.(a)



We get that

$$\mathcal{L}\left[f(t)\right] = \mathcal{L}\left[1\right] - \mathcal{L}\left[u_1(t)\right] = \frac{1}{s} - \frac{e^{-s}}{s}\,.$$

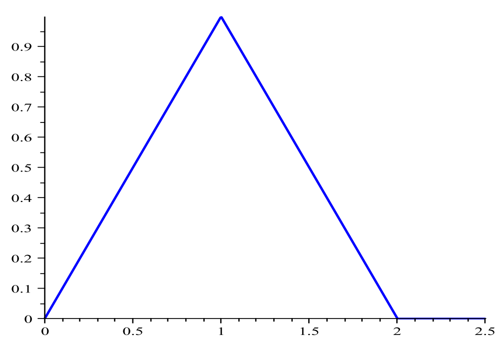
(b)



Let $F(s) = \mathcal{L}\left[1 - u_1(t)\right]$. Then

$$\mathcal{L}\left[\int_0^t \left[1 - u_1(\tau)\right] d\tau\right] = \frac{1}{s}\,F(s) = \frac{1 - e^{-s}}{s^2}\,.$$

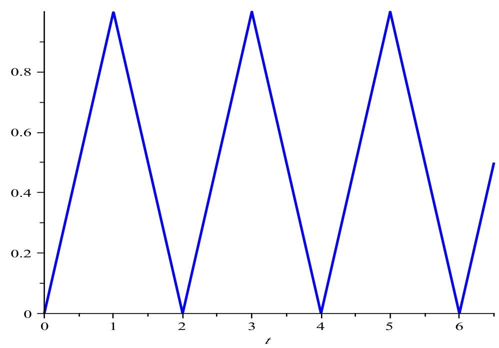
(c)



Let $G(s) = \mathcal{L}[g(t)]$. Then

$$\mathcal{L}[h(t)] = G(s) - e^{-s}\, G(s) = \frac{1-e^{-s}}{s^2} - e^{-s}\,\frac{1-e^{-s}}{s^2} = \frac{(1-e^{-s})^2}{s^2}\,.$$

40.(a)



(b) The given function is periodic, with $T = 2$. Using the result of Problem 34,

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st}p(t)dt\,.$$

Based on the piecewise definition of $p(t)$,

$$\int_0^2 e^{-st}p(t)dt = \int_0^1 t\,e^{-st}dt + \int_1^2 (2-t)e^{-st}dt = \frac{1}{s^2}(1-e^{-s})^2.$$

Hence

$$\mathcal{L}[p(t)] = \frac{(1-e^{-s})}{s^2(1+e^{-s})}\,.$$

(c) Since $p(t)$ satisfies the hypotheses of Theorem 6.2.1, $\mathcal{L}[p'(t)] = s\,\mathcal{L}[p(t)] - p(0)$. Using the result of Problem 36 from the Student Solutions Manual,

$$\mathcal{L}[p'(t)] = \frac{(1-e^{-s})}{s(1+e^{-s})}\,.$$

We note the $p(0) = 0$, hence

$$\mathcal{L}[p(t)] = \frac{1}{s}\left[\frac{(1-e^{-s})}{s(1+e^{-s})}\right].$$

6.4

2.(a) Let $h(t)$ be the forcing function on the right-hand-side. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + 2\,[s\,Y(s) -\, y(0)] + 2\,Y(s) = \mathcal{L}[h(t)]\,.$$

Applying the initial conditions,

$$s^2\,Y(s)+2s\,Y(s)+2\,Y(s)-1=\mathcal{L}\,[h(t)]\,.$$

The forcing function can be written as $h(t)=u_\pi(t)-u_{2\pi}(t)$. Its transform is

$$\mathcal{L}\,[h(t)]=\frac{e^{-\pi s}-e^{-2\pi s}}{s}\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s)=\frac{1}{s^2+2s+2}+\frac{e^{-\pi s}-e^{-2\pi s}}{s(s^2+2s+2)}\,.$$

First note that

$$\frac{1}{s^2+2s+2}=\frac{1}{(s+1)^2+1}.$$

Using partial fractions,

$$\frac{1}{s(s^2+2s+2)}=\frac{1}{2}\frac{1}{s}-\frac{1}{2}\frac{(s+1)+1}{(s+1)^2+1}\,.$$

Taking the inverse transform, term-by-term,

$$\mathcal{L}\left[\frac{1}{s^2+2s+2}\right]=\mathcal{L}\left[\frac{1}{(s+1)^2+1}\right]=e^{-t}\sin\,t\,.$$

Now let

$$G(s)=\frac{1}{s(s^2+2s+2)}\,.$$

Then

$$\mathcal{L}^{-1}\,[G(s)]=\frac{1}{2}-\frac{1}{2}e^{-t}\cos\,t-\frac{1}{2}e^{-t}\sin\,t\,.$$

Using Theorem 6.3.1,

$$\mathcal{L}^{-1}\left[e^{-cs}G(s)\right]=\frac{1}{2}u_c(t)-\frac{1}{2}e^{-(t-c)}\left[\cos(t-c)+\sin(t-c)\right]u_c(t)\,.$$

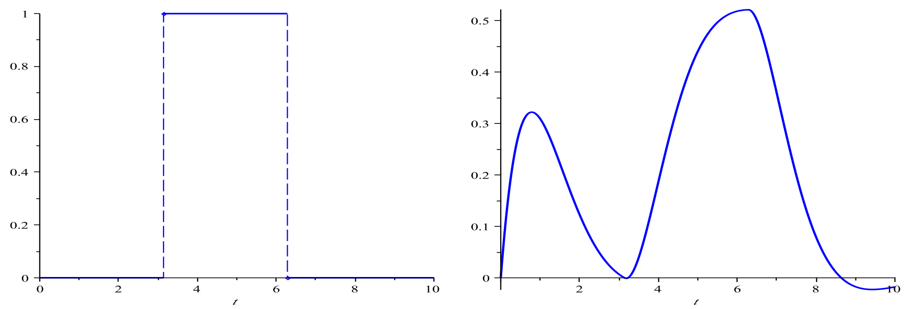
Hence the solution of the IVP is

$$\begin{aligned}y(t)\ &=e^{-t}\sin\,t+\frac{1}{2}u_\pi(t)-\frac{1}{2}e^{-(t-\pi)}\left[\cos(t-\pi)+\sin(t-\pi)\right]u_\pi(t)-\\&-\frac{1}{2}u_{2\pi}(t)+\frac{1}{2}e^{-(t-2\pi)}\left[\cos(t-2\pi)+\sin(t-2\pi)\right]u_{2\pi}(t)\,.\end{aligned}$$

That is,

$$\begin{aligned}y(t)\ &=e^{-t}\sin\,t+\frac{1}{2}\left[u_\pi(t)-u_{2\pi}(t)\right]+\frac{1}{2}e^{-(t-\pi)}\left[\cos\,t+\sin\,t\right]\,u_\pi(t)+\\&+\frac{1}{2}e^{-(t-2\pi)}\left[\cos\,t+\sin\,t\right]u_{2\pi}(t)\,.\end{aligned}$$

(b)



The solution starts out as free oscillation, due to the initial conditions. The amplitude increases, as long as the forcing is present. Thereafter, the solution rapidly decays.

4.(a) Let $h(t)$ be the forcing function on the right-hand-side. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + 4\,Y(s) = \mathcal{L}\left[h(t)\right].$$

Applying the initial conditions,

$$s^2\,Y(s) + 4\,Y(s) = \mathcal{L}\left[h(t)\right].$$

The transform of the forcing function is

$$\mathcal{L}\left[h(t)\right] = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1}\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{1}{(s^2+4)(s^2+1)} + \frac{e^{-\pi s}}{(s^2+4)(s^2+1)}\,.$$

Using partial fractions,

$$\frac{1}{(s^2+4)(s^2+1)} = \frac{1}{3}\left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right].$$

It follows that

$$\mathcal{L}^{-1}\left[\frac{1}{(s^2+4)(s^2+1)}\right] = \frac{1}{3}\left[\sin\,t - \frac{1}{2}\,\sin\,2t\right].$$

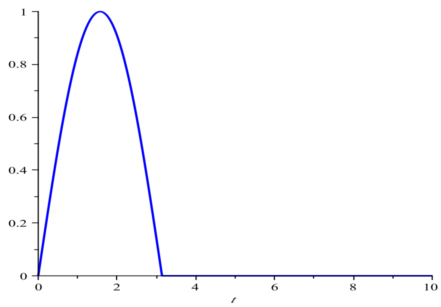
Based on Theorem 6.3.1,

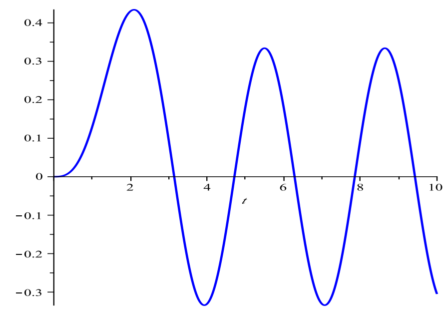
$$\mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{(s^2+4)(s^2+1)}\right] = \frac{1}{3}\left[\sin(t-\pi) - \frac{1}{2}\,\sin(2t-2\pi)\right]u_\pi(t)\,.$$

Hence the solution of the IVP is

$$y(t) = \frac{1}{3}\left[\sin\,t - \frac{1}{2}\,\sin\,2t\right] - \frac{1}{3}\left[\sin\,t + \frac{1}{2}\,\sin\,2t\right]u_\pi(t)\,.$$

(b)





Since there is no damping term, the solution follows the forcing function, after which the response is a steady oscillation about $y = 0$.

5.(a) Let $f(t)$ be the forcing function on the right-hand-side. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\, Y(s) - s\, y(0) - y'(0) + 3\,[s\, Y(s) - \; y(0)] + 2\, Y(s) = \mathcal{L}\,[f(t)]\,.$$

Applying the initial conditions,

$$s^2\, Y(s) + 3s\, Y(s) + 2\, Y(s) = \mathcal{L}\,[f(t)]\,.$$

The transform of the forcing function is

$$\mathcal{L}\,[f(t)] = \frac{1}{s} - \frac{e^{-10s}}{s}\,.$$

Solving for the transform,

$$Y(s) = \frac{1}{s(s^2+3s+2)} - \frac{e^{-10s}}{s(s^2+3s+2)}\,.$$

Using partial fractions,

$$\frac{1}{s(s^2+3s+2)} = \frac{1}{2}\left[\frac{1}{s} + \frac{1}{s+2} - \frac{2}{s+1}\right].$$

Hence

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2+3s+2)}\right] = \frac{1}{2} + \frac{e^{-2t}}{2} - e^{-t}\,.$$

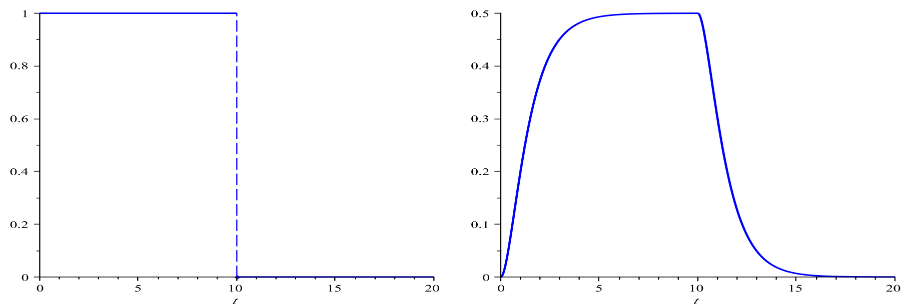
Based on Theorem 6.3.1,

$$\mathcal{L}^{-1}\left[\frac{e^{-10s}}{s(s^2+3s+2)}\right] = \frac{1}{2}\left[1 + e^{-2(t-10)} - 2e^{-(t-10)}\right] u_{10}(t)\,.$$

Hence the solution of the IVP is

$$y(t) = \frac{1}{2}\,[1 - u_{10}(t)] + \frac{e^{-2t}}{2} - e^{-t} - \frac{1}{2}\left[e^{-(2t-20)} - 2e^{-(t-10)}\right] u_{10}(t)\,.$$

(b)



The solution increases to a temporary steady value of $y = 1/2$. After the forcing ceases, the response decays exponentially to $y = 0$.

6.(a) Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + 3\,[s\,Y(s) - \,y(0)] + 2\,Y(s) = \frac{e^{-2s}}{s}.$$

Applying the initial conditions,

$$s^2\,Y(s) + 3s\,Y(s) + 2\,Y(s) - 1 = \frac{e^{-2s}}{s}.$$

Solving for the transform,

$$Y(s) = \frac{1}{s^2 + 3s + 2} + \frac{e^{-2s}}{s(s^2 + 3s + 2)}\,.$$

Using partial fractions,

$$\frac{1}{s^2 + 3s + 2} = \frac{1}{s+1} - \frac{1}{s+2}$$

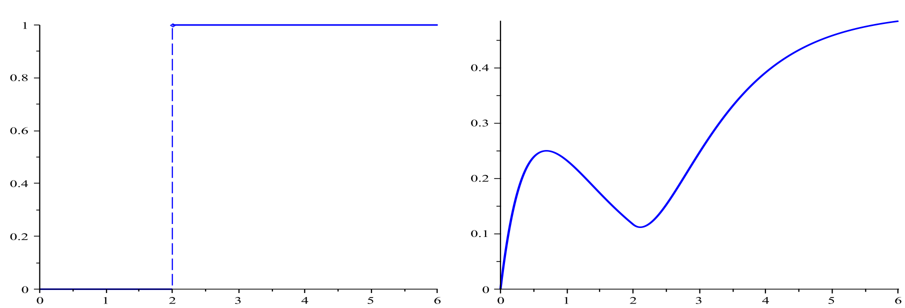
and

$$\frac{1}{s(s^2 + 3s + 2)} = \frac{1}{2}\left[\frac{1}{s} + \frac{1}{s+2} - \frac{2}{s+1}\right].$$

Taking the inverse transform. term-by-term, the solution of the IVP is

$$y(t) = e^{-t} - e^{-2t} + \left[\frac{1}{2} - e^{-(t-2)} + \frac{1}{2}e^{-2(t-2)}\right]u_2(t)\,.$$

(b)



Due to the initial conditions, the response has a transient overshoot, followed by an exponential convergence to a steady value of $y_s = 1/2$.

7.(a) Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y\,'(0) + Y(s) = \frac{e^{-3\pi s}}{s}.$$

Applying the initial conditions,

$$s^2\,Y(s) + \,Y(s) - s = \frac{e^{-3\pi s}}{s}.$$

Solving for the transform,

$$Y(s) = \frac{s}{s^2+1} + \frac{e^{-3\pi s}}{s(s^2+1)}\,.$$

Using partial fractions,

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}\,.$$

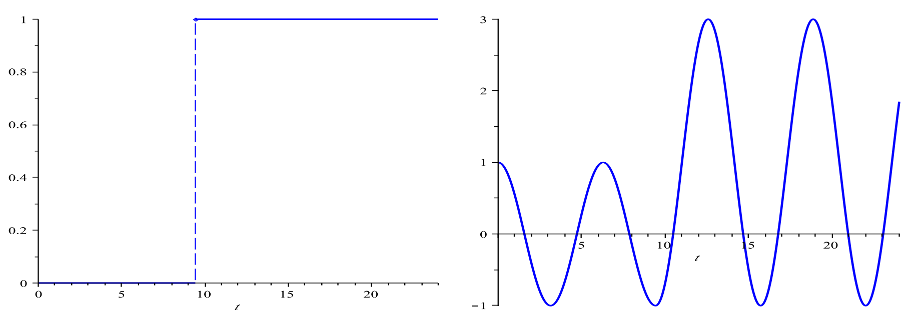
Hence

$$Y(s) = \frac{s}{s^2+1} + e^{-3\pi s}\left[\frac{1}{s} - \frac{s}{s^2+1}\right]\,.$$

Taking the inverse transform, the solution of the IVP is

$$y(t) = \cos\,t + [1 - \cos(t - 3\pi)]\,u_{3\pi}(t) = \cos\,t + [1 + \cos\,t\,]\,u_{3\pi}(t)\,.$$

(b)



Due to initial conditions, the solution temporarily oscillates about $y = 0$. After the forcing is applied, the response is a steady oscillation about $y_m = 1$.

9.(a) Let $g(t)$ be the forcing function on the right-hand-side. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y\,'(0) + Y(s) = \mathcal{L}\,[g(t)]\,.$$

Applying the initial conditions,

$$s^2\,Y(s) + Y(s) - 1 = \mathcal{L}\,[g(t)]\,.$$

The forcing function can be written as

$$g(t) = \frac{t}{2}\,[1 - u_6(t)] + 3\,u_6(t) = \frac{t}{2} - \frac{1}{2}(t-6)u_6(t)$$

with Laplace transform

$$\mathcal{L}\,[g(t)] = \frac{1}{2s^2} - \frac{e^{-6s}}{2s^2}\,.$$

Solving for the transform,

$$Y(s) = \frac{1}{s^2+1} + \frac{1}{2s^2(s^2+1)} - \frac{e^{-6s}}{2s^2(s^2+1)}\,.$$

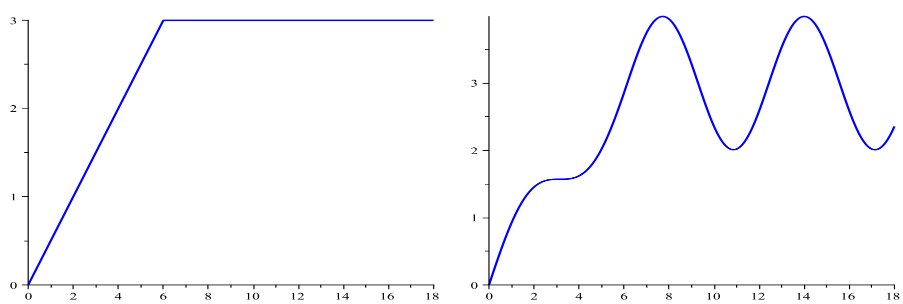
Using partial fractions,

$$\frac{1}{2s^2(s^2+1)} = \frac{1}{2}\left[\frac{1}{s^2} - \frac{1}{s^2+1}\right].$$

Taking the inverse transform, and using Theorem 6.3.1, the solution of the IVP is

$$\begin{aligned} y(t) &= \sin\, t + \frac{1}{2}\,[t - \sin\, t] - \frac{1}{2}\,[(t-6) - \sin(t-6)]\,u_6(t) \\ &= \frac{1}{2}\,[t + \sin\, t] - \frac{1}{2}\,[(t-6) - \sin(t-6)]\,u_6(t). \end{aligned}$$

(b)



The solution increases, in response to the ramp input, and thereafter oscillates about a mean value of $y_m = 3$.

11.(a) Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + 4\,Y(s) = \frac{e^{-\pi s}}{s} - \frac{e^{-3\pi s}}{s}.$$

Applying the initial conditions,

$$s^2\,Y(s) + 4\,Y(s) = \frac{e^{-\pi s}}{s} - \frac{e^{-3\pi s}}{s}.$$

Solving for the transform,

$$Y(s) = \frac{e^{-\pi s}}{s(s^2+4)} - \frac{e^{-3\pi s}}{s(s^2+4)}\,.$$

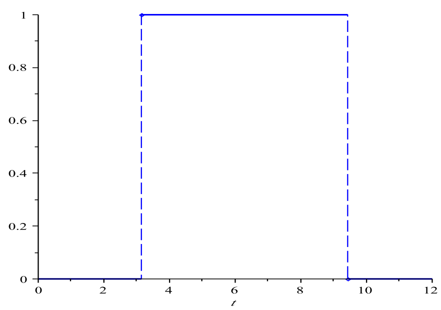
Using partial fractions,

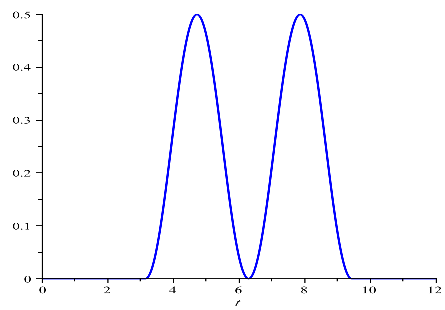
$$\frac{1}{s(s^2+4)} = \frac{1}{4}\left[\frac{1}{s} - \frac{s}{s^2+4}\right].$$

Taking the inverse transform, and applying Theorem 6.3.1,

$$\begin{aligned} y(t) &= \frac{1}{4}\left[1-\cos(2t-2\pi)\right]u_\pi(t) - \frac{1}{4}\left[1-\cos(2t-6\pi)\right]u_{3\pi}(t) \\ &= \frac{1}{4}\left[u_\pi(t)-u_{3\pi}(t)\right] - \frac{1}{4}\cos\,2t\cdot\left[u_\pi(t)-u_{3\pi}(t)\right]. \end{aligned}$$

(b)





Since there is no damping term, the solution responds immediately to the forcing input. There is a temporary oscillation about $y = 1/4$.

12.(a) Taking the Laplace transform of the ODE, we obtain

$$s^4\,Y(s) - s^3y(0) - s^2y'(0) - s\,y''(0) - y'''(0) - Y(s) = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}.$$

Applying the initial conditions,

$$s^4Y(s) - Y(s) = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}.$$

Solving for the transform of the solution,

$$Y(s) = \frac{e^{-s}}{s(s^4-1)} - \frac{e^{-2s}}{s(s^4-1)}.$$

Using partial fractions,

$$\frac{1}{s(s^4-1)} = \frac{1}{4}\left[-\frac{4}{s} + \frac{1}{s+1} + \frac{1}{s-1} + \frac{2s}{s^2+1}\right].$$

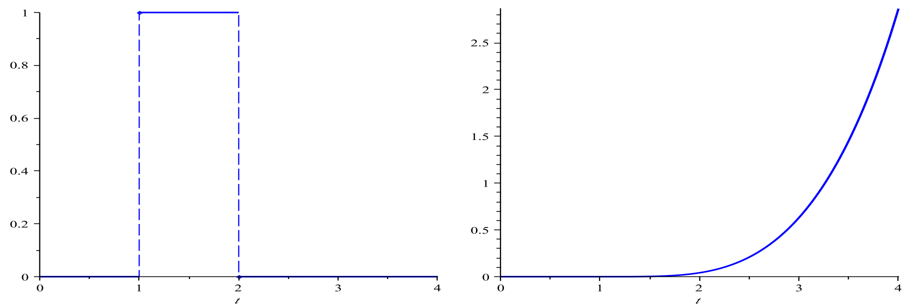
It follows that

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^4-1)}\right] = \frac{1}{4}\left[-4 + e^{-t} + e^t + 2\,\cos\,t\right].$$

Based on Theorem 6.3.1, the solution of the IVP is

$$\begin{aligned} y(t) &= -\left[u_1(t)-u_2(t)\right] + \frac{1}{4}\left[e^{-(t-1)} + e^{(t-1)} + 2\,\cos(t-1)\right]u_1(t) - \\ &- \frac{1}{4}\left[e^{-(t-2)} + e^{(t-2)} + 2\,\cos(t-2)\right]u_2(t). \end{aligned}$$

(b)



The solution increases without bound, exponentially.

13.(a) Taking the Laplace transform of the ODE, we obtain

$$s^4\,Y(s) - s^3y(0) - s^2y'(0) - s\,y''(0) - y'''(0) +$$
$$+\,5\left[s^2Y(s) - s\,y(0) - y'(0)\right] + 4\,Y(s) = \frac{1}{s} - \frac{e^{-\pi s}}{s}.$$

Applying the initial conditions,

$$s^4Y(s) + 5s^2Y(s) + \,4Y(s) = \frac{1}{s} - \frac{e^{-\pi s}}{s}\,.$$

Solving for the transform of the solution,

$$Y(s) = \frac{1}{s(s^4+5s^2+4)} - \frac{e^{-\pi s}}{s(s^4+5s^2+4)}\,.$$

Using partial fractions,

$$\frac{1}{s(s^4+5s^2+4)} = \frac{1}{12}\left[\frac{3}{s} + \frac{s}{s^2+4} - \frac{4s}{s^2+1}\right].$$

It follows that

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^4+5s^2+4)}\right] = \frac{1}{12}\left[3 + \cos\,2t - 4\,\cos\,t\right].$$

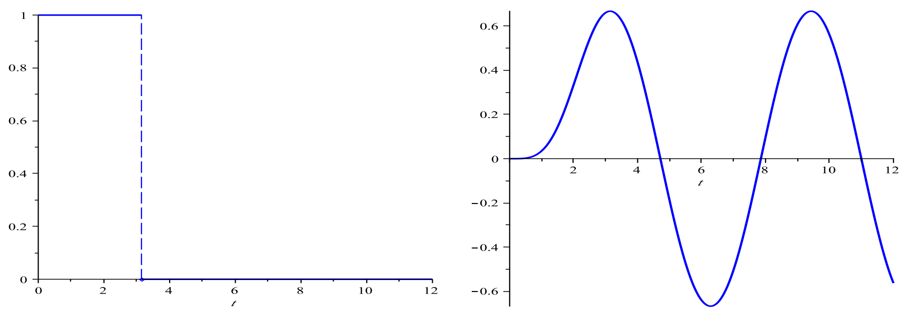
Based on Theorem 6.3.1, the solution of the IVP is

$$y(t) = \frac{1}{4}\left[1 - u_\pi(t)\right] + \frac{1}{12}\left[\cos\,2t - 4\,\cos\,t\right] - \frac{1}{12}\left[\cos\,2(t-\pi) - 4\,\cos(t-\pi)\right]u_\pi(t)\,.$$

That is,

$$y(t) = \frac{1}{4}\left[1 - u_\pi(t)\right] + \frac{1}{12}\left[\cos\,2t - 4\,\cos\,t\right] - \frac{1}{12}\left[\cos\,2t + 4\,\cos\,t\right]u_\pi(t)\,.$$

(b)



After an initial transient, the solution oscillates about $y_m = 0$.

14. The specified function is defined by

$$f(t) = \begin{cases} 0, & 0 \le t < t_0 \\ \frac{h}{k}(t - t_0), & t_0 \le t < t_0 + k \\ h, & t \ge t_0 + k \end{cases}$$

which can conveniently be expressed as

$$f(t) = \frac{h}{k}(t - t_0)\, u_{t_0}(t) - \frac{h}{k}(t - t_0 - k)\, u_{t_0+k}(t)\,.$$

15. The function is defined by

$$g(t) = \begin{cases} 0, & 0 \le t < t_0 \\ \frac{h}{k}(t - t_0), & t_0 \le t < t_0 + k \\ -\frac{h}{k}(t - t_0 - 2k), & t_0 + k \le t < t_0 + 2k \\ 0, & t \ge t_0 + 2k \end{cases}$$

which can also be written as

$$g(t) = \frac{h}{k}(t - t_0)\, u_{t_0}(t) - \frac{2h}{k}(t - t_0 - k)\, u_{t_0+k}(t) + \frac{h}{k}(t - t_0 - 2k)\, u_{t_0+2k}(t).$$

17. We consider the initial value problem

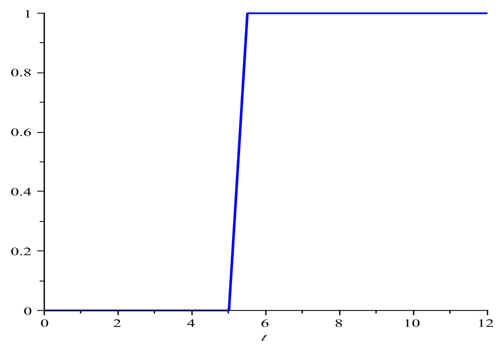
$$y'' + 4y = \frac{1}{k}\left[(t-5)\, u_5(t) - (t - 5 - k)\, u_{5+k}(t)\right],$$

with $y(0) = y'(0) = 0$.

(a) The specified function is defined by

$$f(t) = \begin{cases} 0, & 0 \le t < 5 \\ \frac{1}{k}(t - 5), & 5 \le t < 5 + k \\ 1, & t \ge 5 + k \end{cases}$$

so k controls the point at which $f(t)$ reaches 1. When $k = 5$, $f(t) = g(t)$ in Ex.2.



(b) Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + 4\,Y(s) = \frac{e^{-5s}}{ks^2} - \frac{e^{-(5+k)s}}{ks^2}.$$

Applying the initial conditions,

$$s^2\,Y(s) + 4\,Y(s) = \frac{e^{-5s}}{ks^2} - \frac{e^{-(5+k)s}}{ks^2}.$$

Solving for the transform,

$$Y(s) = \frac{e^{-5s}}{ks^2(s^2+4)} - \frac{e^{-(5+k)s}}{ks^2(s^2+4)}\,.$$

Using partial fractions,

$$\frac{1}{s^2(s^2+4)} = \frac{1}{4}\left[\frac{1}{s^2} - \frac{1}{s^2+4}\right].$$

It follows that

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s^2+4)}\right] = \frac{1}{4}t - \frac{1}{8}\sin\,2t\,.$$

Using Theorem 6.3.1, the solution of the IVP is

$$y(t) = \frac{1}{k}\left[h(t-5)\,u_5(t) - h(t-5-k)\,u_{5+k}(t)\right]\,,$$

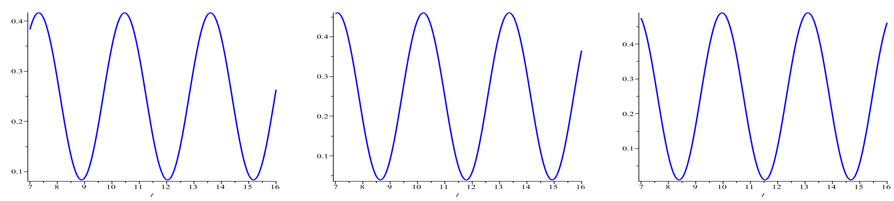
in which $h(t) = t/4 - \sin\,2t\,/8$.

(c) Note that for $t > 5+k$, the solution is given by

$$y(t) = \frac{1}{4} - \frac{1}{8k}\sin(2t-10) + \frac{1}{8k}\,\sin(2t-10-2k) = \frac{1}{4} - \frac{\sin\,k}{4k}\,\cos(2t-10-k)\,.$$

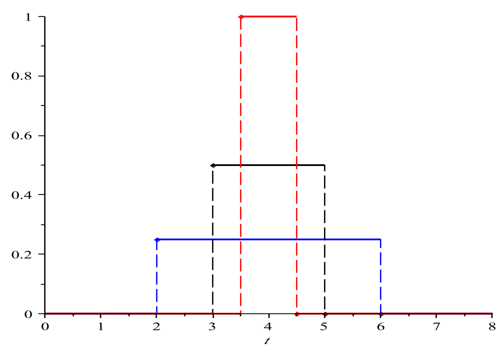
So for $t > 5+k$, the solution oscillates about $y_m = 1/4$, with an amplitude of

$$A = \frac{|\sin(k)|}{4k}\,.$$



(a) $k = 3/2$ (b) $k = 1$ (c) $k = 1/2$

18.(a) The graph shows f_k for $k = 2$, $k = 1$ and $k = 1/2$.



(b) The forcing function can be expressed as

$$f_k(t) = \frac{1}{2k}\left[u_{4-k}(t) - u_{4+k}(t)\right].$$

Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + \frac{1}{3}\left[s\,Y(s) - \,y(0)\right] + 4\,Y(s) = \frac{e^{-(4-k)s}}{2ks} - \frac{e^{-(4+k)s}}{2ks}.$$

Applying the initial conditions,

$$s^2\,Y(s) + \frac{1}{3}s\,Y(s) + 4\,Y(s) = \frac{e^{-(4-k)s}}{2ks} - \frac{e^{-(4+k)s}}{2ks}.$$

Solving for the transform,

$$Y(s) = \frac{3\,e^{-(4-k)s}}{2ks(3s^2+s+12)} - \frac{3\,e^{-(4+k)s}}{2ks(3s^2+s+12)}.$$

Using partial fractions,

$$\frac{1}{s(3s^2+s+12)} = \frac{1}{12}\left[\frac{1}{s} - \frac{1+3s}{3s^2+s+12}\right] = \frac{1}{12}\left[\frac{1}{s} - \frac{1}{6}\,\frac{1+6(s+\frac{1}{6})}{(s+\frac{1}{6})^2+\frac{143}{36}}\right].$$

Let

$$H(s) = \frac{1}{8k}\left[\frac{1}{s} - \frac{\frac{1}{6}}{(s+\frac{1}{6})^2+\frac{143}{36}} - \frac{s+\frac{1}{6}}{(s+\frac{1}{6})^2+\frac{143}{36}}\right].$$

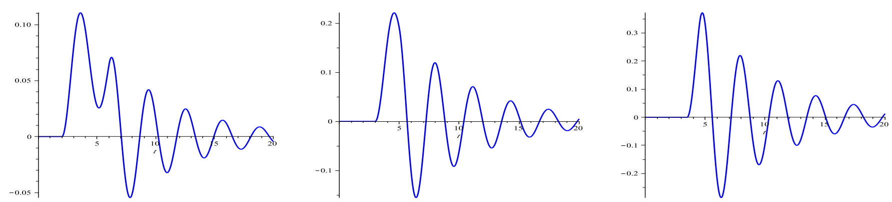
It follows that

$$h(t) = \mathcal{L}^{-1}\left[H(s)\right] = \frac{1}{8k} - \frac{e^{-t/6}}{8k}\left[\frac{1}{\sqrt{143}}\sin\left(\frac{\sqrt{143}\,t}{6}\right) + \cos\left(\frac{\sqrt{143}\,t}{6}\right)\right].$$

Based on Theorem 6.3.1, the solution of the IVP is

$$y(t) = h(t-4+k)\,u_{4-k}(t) - h(t-4-k)\,u_{4+k}(t)\,.$$

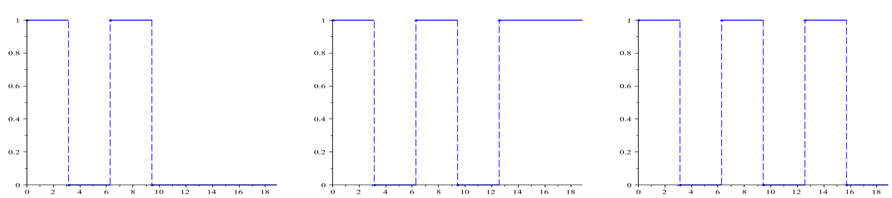
(c)



(a) $k = 2$ (b) $k = 1$ (c) $k = 1/2$

As the parameter k decreases, the solution remains null for a longer period of time. Since the magnitude of the impulsive force increases, the initial overshoot of the response also increases. The duration of the impulse decreases. All solutions eventually decay to $y = 0$.

21.(a)



(a) $n = 3$ (b) $n = 4$ (c) $n = 5$

(b) Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + Y(s) = \frac{1}{s} + \sum_{k=1}^{n}\frac{(-1)^k e^{-k\pi s}}{s}.$$

Applying the initial conditions,

$$s^2\,Y(s) + Y(s) = \frac{1}{s} + \sum_{k=1}^{n}\frac{(-1)^k e^{-k\pi s}}{s}.$$

Solving for the transform,

$$Y(s) = \frac{1}{s(s^2+1)} + \sum_{k=1}^{n} \frac{(-1)^k e^{-k\pi s}}{s(s^2+1)}\,.$$

Using partial fractions,

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}\,.$$

Let

$$h(t) = \mathcal{L}^{-1}\left[\frac{1}{s(s^2+1)}\right] = 1 - \cos\, t\,.$$

Applying Theorem 6.3.1, term-by-term, the solution of the IVP is

$$y(t) = h(t) + \sum_{k=1}^{n} (-1)^k h(t-k\pi)\, u_{k\pi}(t)\,.$$

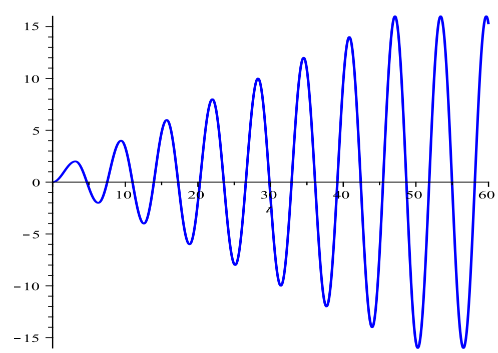
Note that

$$h(t-k\pi) = u_0(t-k\pi) - \cos(t-k\pi) = u_{k\pi}(t) - (-1)^k \cos\, t\,.$$

Hence

$$y(t) = 1 - \cos\, t + \sum_{k=1}^{n} (-1)^k\, u_{k\pi}(t) - (\cos\, t) \sum_{k=1}^{n} u_{k\pi}(t)\,.$$

(c)



The ODE has no damping term. Each interval of forcing adds to the energy of the system, so the amplitude will increase. For $n = 15$, $g(t) = 0$ when $t > 15\pi$. Therefore the oscillation will eventually become steady, with an amplitude depending on the values of $y(15\pi)$ and $y'(15\pi)$.

(d) As n increases, the interval of forcing also increases. Hence the amplitude of the transient will increase with n. Eventually, the forcing function will be constant. In fact, for large values of t,

$$g(t) = \begin{cases} 1\,, & n \text{ even} \\ 0\,, & n \text{ odd} \end{cases}$$

Further, for $t > n\pi$,

$$y(t) = 1 - \cos t - n\cos t - \frac{1-(-1)^n}{2}.$$

Hence the steady state solution will oscillate about 0 or 1, depending on n, with an amplitude of $A = n + 1$. In the limit, as $n \to \infty$, the forcing function will be a periodic function, with period 2π. From Problem 33, in Section 6.3,

$$\mathcal{L}[g(t)] = \frac{1}{s(1+e^{-s})}.$$

As n increases, the duration and magnitude of the transient will increase without bound.

22.(a) Taking the initial conditions into consideration, the transform of the ODE is

$$s^2 Y(s) + 0.1\, s\, Y(s) + Y(s) = \frac{1}{s} + \sum_{k=1}^{n} \frac{(-1)^k e^{-k\pi s}}{s}.$$

Solving for the transform,

$$Y(s) = \frac{1}{s(s^2+0.1s+1)} + \sum_{k=1}^{n} \frac{(-1)^k e^{-k\pi s}}{s(s^2+0.1s+1)}.$$

Using partial fractions,

$$\frac{1}{s(s^2+0.1s+1)} = \frac{1}{s} - \frac{s+0.1}{s^2+0.1s+1}.$$

Since the denominator in the second term is irreducible, write

$$\frac{s+0.1}{s^2+0.1s+1} = \frac{(s+0.05)+0.05}{(s+0.05)^2+(399/400)}.$$

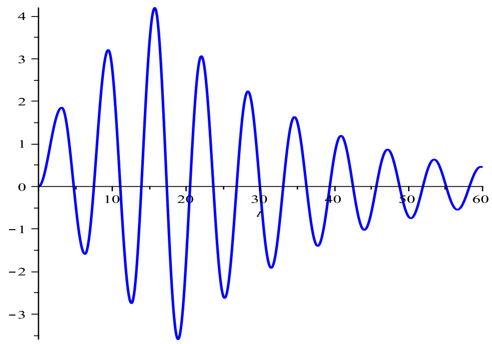
Let

$$\begin{aligned} h(t) &= \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{(s+0.05)}{(s+0.05)^2+(399/400)} - \frac{0.05}{(s+0.05)^2+(399/400)}\right] \\ &= 1 - e^{-t/20}\left[\cos\left(\frac{\sqrt{399}}{20}\, t\right) + \frac{1}{\sqrt{399}} \sin\left(\frac{\sqrt{399}}{20}\, t\right)\right]. \end{aligned}$$

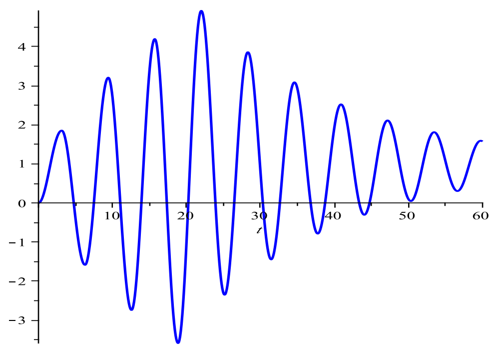
Applying Theorem 6.3.1, term-by-term, the solution of the IVP is

$$y(t) = h(t) + \sum_{k=1}^{n} (-1)^k h(t-k\pi)\, u_{k\pi}(t).$$

For odd values of n, the solution approaches $y = 0$. (On the next figure, $n = 5$.)



For even values of n, the solution approaches $y = 1$. (On the next figure, $n = 6$.)



(b) The solution is a sum of damped sinusoids, each of frequency $\omega = \sqrt{399}/20 \approx 1$. Each term has an initial amplitude of approximately 1. For any given n, the solution contains $n+1$ such terms. Although the amplitude will increase with n, the amplitude will also be bounded by $n+1$.

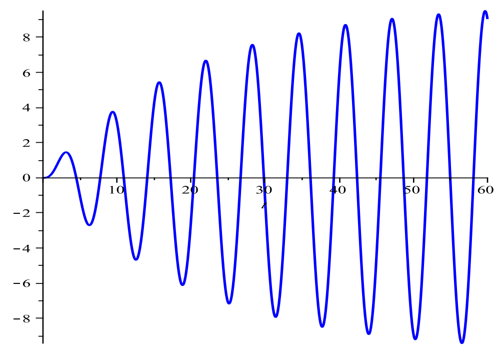
(c) Suppose that the forcing function is replaced by $g(t) = \sin t$. Based on the methods in Chapter 3, the general solution of the differential equation is

$$y(t) = e^{-t/20}\left[c_1 \cos(\frac{\sqrt{399}}{20}\, t) + c_2 \sin(\frac{\sqrt{399}}{20}\, t)\right] + y_p(t)\,.$$

Note that $y_p(t) = A\cos t + B\sin t$. Using the method of undetermined coefficients, $A = -10$ and $B = 0$. Based on the initial conditions, the solution of the IVP is

$$y(t) = 10\,e^{-t/20}\left[\cos(\frac{\sqrt{399}}{20}\, t) + \frac{1}{\sqrt{399}} \sin(\frac{\sqrt{399}}{20}\, t)\right] - 10\cos t\,.$$

Observe that both solutions have the same frequency, $\omega = \sqrt{399}/20 \approx 1$.



23.(a) Taking the initial conditions into consideration, the transform of the ODE is

$$s^2 Y(s) + Y(s) = \frac{1}{s} + 2\sum_{k=1}^{n} \frac{(-1)^k e^{-(11k/4)s}}{s}.$$

Solving for the transform,

$$Y(s) = \frac{1}{s(s^2+1)} + 2\sum_{k=1}^{n} \frac{(-1)^k e^{-(11k/4)s}}{s(s^2+1)}.$$

Using partial fractions,

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}.$$

Let

$$h(t) = \mathcal{L}^{-1}\left[\frac{1}{s(s^2+1)}\right] = 1 - \cos t.$$

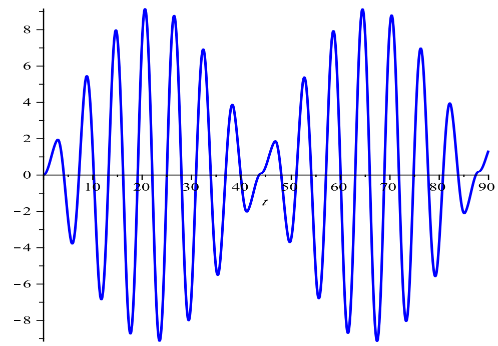
Applying Theorem 6.3.1, term-by-term, the solution of the IVP is

$$y(t) = h(t) + 2\sum_{k=1}^{n} (-1)^k h(t - \frac{11k}{4})\, u_{11k/4}(t).$$

That is,

$$y(t) = 1 - \cos t + 2\sum_{k=1}^{n} (-1)^k \left[1 - \cos(t - \frac{11k}{4})\right] u_{11k/4}(t).$$

(b) On the figure we see the solution for $n = 35$.



(c) Based on the plot, the slow period appears to be 88. The fast period appears to be about 6. These values correspond to a slow frequency of $\omega_s = 0.0714$ and a fast frequency $\omega_f = 1.0472$.

(d) The natural frequency of the system is $\omega_0 = 1$. The forcing function is initially periodic, with period $T = 11/2 = 5.5$. Hence the corresponding forcing frequency is $w = 1.1424$. Using the results in Section 3.8, the slow frequency is given by

$$\omega_s = \frac{|\omega - \omega_0|}{2} = 0.0712$$

and the fast frequency is given by

$$\omega_f = \frac{|\omega + \omega_0|}{2} = 1.0712\,.$$

Based on theses values, the slow period is predicted as 88.247 and the fast period is given as 5.8656.

6.5

2.(a) Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + 4Y(s) = e^{-\pi s} - e^{-2\pi s}.$$

Applying the initial conditions,

$$s^2\,Y(s) + 4Y(s) = e^{-\pi s} - e^{-2\pi s}.$$

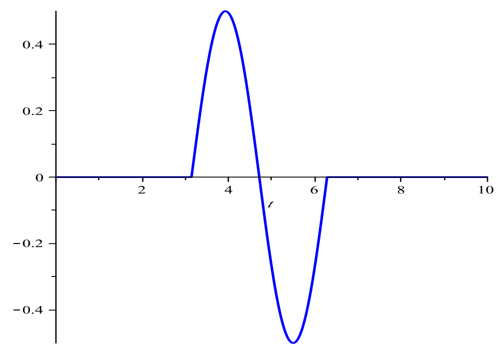
Solving for the transform,

$$Y(s) = \frac{e^{-\pi s} - e^{-2\pi s}}{s^2+4} = \frac{e^{-\pi s}}{s^2+4} - \frac{e^{-2\pi s}}{s^2+4}\,.$$

Applying Theorem 6.3.1, the solution of the IVP is

$$y(t) = \frac{1}{2}\sin(2t-2\pi)u_\pi(t) - \frac{1}{2}\sin(2t-4\pi)u_{2\pi}(t) = \frac{1}{2}\sin(2t)\left[\,u_\pi(t) - u_{2\pi}(t)\,\right]\,.$$

(b)



4.(a) Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) - Y(s) = -20\,e^{-3s}.$$

Applying the initial conditions,

$$s^2\,Y(s) - Y(s) - s = -20\,e^{-3s}.$$

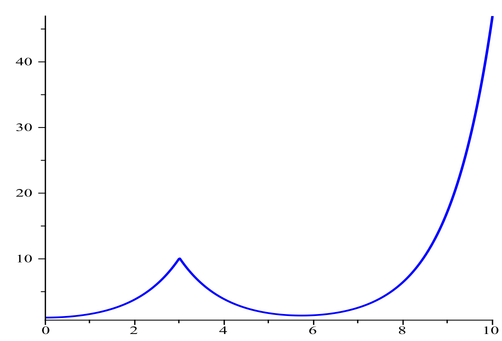
Solving for the transform,

$$Y(s) = \frac{s}{s^2-1} - \frac{20\,e^{-3s}}{s^2-1}\,.$$

Using a table of transforms, and Theorem 6.3.1, the solution of the IVP is

$$y(t) = \cosh t - 20\,\sinh(t-3)u_3(t)\,.$$

(b)



6.(a) Taking the initial conditions into consideration, the transform of the ODE is

$$s^2\,Y(s) + 4Y(s) - s/2 = \,e^{-4\pi s}.$$

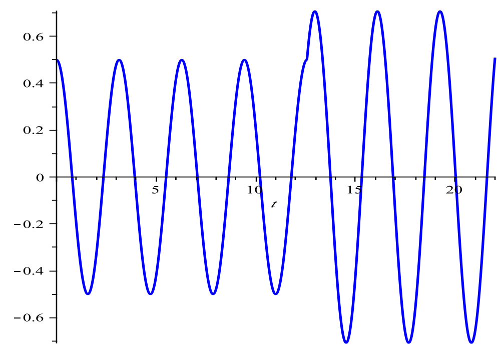
Solving for the transform,

$$Y(s) = \frac{s/2}{s^2+4} + \frac{e^{-4\pi s}}{s^2+4}\,.$$

Using a table of transforms, and Theorem 6.3.1, the solution of the IVP is

$$y(t) = \frac{1}{2}\cos 2t + \frac{1}{2}\sin(2t - 8\pi)u_{4\pi}(t) = \frac{1}{2}\cos 2t + \frac{1}{2}\sin(2t)\, u_{4\pi}(t)\,.$$

(b)



8.(a) Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\, Y(s) - s\, y(0) - y'(0) + 4Y(s) = 2\, e^{-(\pi/4)s}.$$

Applying the initial conditions,

$$s^2\, Y(s) + 4Y(s) = 2\, e^{-(\pi/4)s}.$$

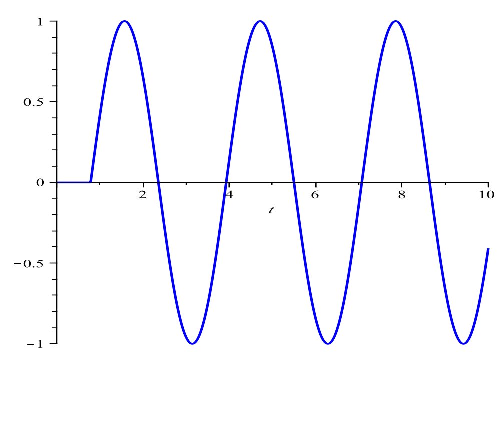
Solving for the transform,

$$Y(s) = \frac{2\, e^{-(\pi/4)s}}{s^2 + 4}\,.$$

Applying Theorem 6.3.1, the solution of the IVP is

$$y(t) = \sin(2t - \frac{\pi}{2})u_{\pi/4}(t) = -\cos(2t)\, u_{\pi/4}(t)\,.$$

(b)



9.(a) Taking the initial conditions into consideration, the transform of the ODE is

$$s^2\,Y(s) + Y(s) = \frac{e^{-(\pi/2)s}}{s} + 3\,e^{-(3\pi/2)s} - \frac{e^{-2\pi s}}{s}.$$

Solving for the transform,

$$Y(s) = \frac{e^{-(\pi/2)s}}{s(s^2+1)} + \frac{3\,e^{-(3\pi/2)s}}{s^2+1} - \frac{e^{-2\pi s}}{s(s^2+1)}\,.$$

Using partial fractions,

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}\,.$$

Hence

$$Y(s) = \frac{e^{-(\pi/2)s}}{s} - \frac{s\,e^{-(\pi/2)s}}{s^2+1} + \frac{3\,e^{-(3\pi/2)s}}{s^2+1} - \frac{e^{-2\pi s}}{s} + \frac{s\,e^{-2\pi s}}{s^2+1}\,.$$

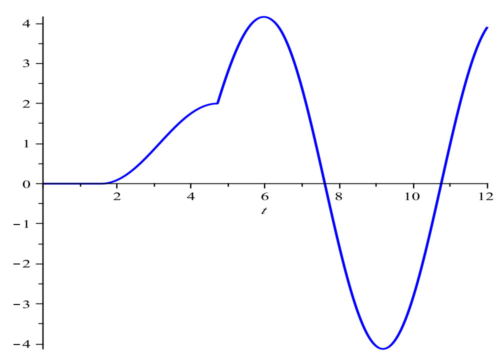
Based on Theorem 6.3.1, the solution of the IVP is

$$y(t) = u_{\pi/2}(t) - \cos(t-\frac{\pi}{2})u_{\pi/2}(t) + 3\,\sin(t-\frac{3\pi}{2})u_{3\pi/2}(t) - u_{2\pi}(t) + \cos(t-2\pi)u_{2\pi}(t).$$

That is,

$$y(t) = [1-\sin(t)]\;u_{\pi/2}(t) + 3\,\cos(t)\,u_{3\pi/2}(t) - [1-\cos(t)]\;u_{2\pi}(t)\,.$$

(b)



10.(a) Taking the transform of both sides of the ODE,

$$2s^2Y(s) + sY(s) + 4Y(s) = \int_0^\infty e^{-st}\,\delta(t-\frac{\pi}{6})\sin t\,dt = \frac{1}{2}\,e^{-(\pi/6)s}.$$

Solving for the transform,

$$Y(s) = \frac{e^{-(\pi/6)s}}{2(2s^2+s+4)}\,.$$

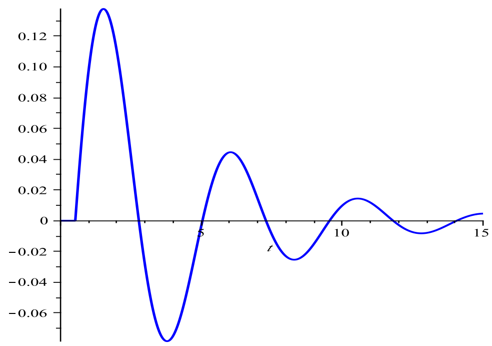
First write

$$\frac{1}{2(2s^2+s+4)} = \frac{\frac{1}{4}}{(s+\frac{1}{4})^2 + \frac{31}{16}}\,.$$

It follows that

$$y(t) = \mathcal{L}^{-1}\left[Y(s)\right] = \frac{1}{\sqrt{31}} e^{-(t-\pi/6)/4} \cdot \sin\frac{\sqrt{31}}{4}(t - \frac{\pi}{6}) u_{\pi/6}(t)\,.$$

(b)



11.(a) Taking the initial conditions into consideration, the transform of the ODE is

$$s^2\,Y(s) + 2s\,Y(s) + 2\,Y(s) = \frac{s}{s^2+1} + e^{-(\pi/2)s}.$$

Solving for the transform,

$$Y(s) = \frac{s}{(s^2+1)(s^2+2s+2)} + \frac{e^{-(\pi/2)s}}{s^2+2s+2}\,.$$

Using partial fractions,

$$\frac{s}{(s^2+1)(s^2+2s+2)} = \frac{1}{5}\left[\frac{s}{s^2+1} + \frac{2}{s^2+1} - \frac{s+4}{s^2+2s+2}\right].$$

We can also write

$$\frac{s+4}{s^2+2s+2} = \frac{(s+1)+3}{(s+1)^2+1}\,.$$

Let

$$Y_1(s) = \frac{s}{(s^2+1)(s^2+2s+2)}\,.$$

Then

$$\mathcal{L}^{-1}\left[Y_1(s)\right] = \frac{1}{5}\,\cos\,t + \frac{2}{5}\,\sin\,t - \frac{1}{5}\,e^{-t}\left[\cos\,t + 3\,\sin\,t\right].$$

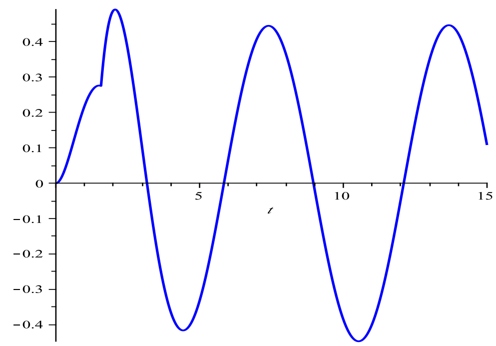
Applying Theorem 6.3.1,

$$\mathcal{L}^{-1}\left[\frac{e^{-(\pi/2)s}}{s^2+2s+2}\right] = e^{-(t-\frac{\pi}{2})}\,\sin\left(t - \frac{\pi}{2}\right) u_{\pi/2}(t)\,.$$

Hence the solution of the IVP is

$$y(t) = \frac{1}{5}\,\cos\,t + \frac{2}{5}\,\sin\,t - \frac{1}{5}\,e^{-t}\left[\cos\,t + 3\,\sin\,t\right] - e^{-(t-\frac{\pi}{2})}\,\cos(t)\,u_{\pi/2}(t)\,.$$

(b)



12.(a) Taking the initial conditions into consideration, the transform of the ODE is

$$s^4\,Y(s) - Y(s) = e^{-s}.$$

Solving for the transform,

$$Y(s) = \frac{e^{-s}}{s^4 - 1}\,.$$

Using partial fractions,

$$\frac{1}{s^4 - 1} = \frac{1}{2}\left[\frac{1}{s^2 - 1} - \frac{1}{s^2 + 1}\right].$$

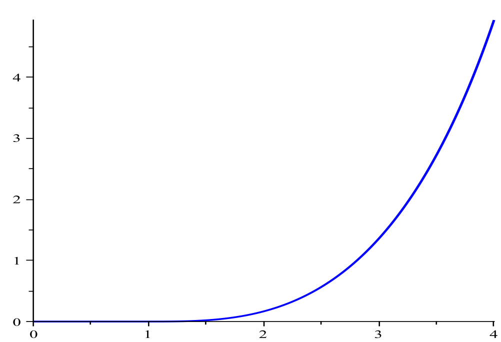
It follows that

$$\mathcal{L}^{-1}\left[\frac{1}{s^4 - 1}\right] = \frac{1}{2}\sinh t - \frac{1}{2}\sin t\,.$$

Applying Theorem 6.3.1, the solution of the IVP is

$$y(t) = \frac{1}{2}\left[\sinh(t-1) - \sin(t-1)\right]u_1(t)\,.$$

(b)



14.(a) The Laplace transform of the ODE is

$$s^2\,Y(s) + \frac{1}{2}s\,Y(s) + Y(s) = e^{-s}.$$

Solving for the transform of the solution,

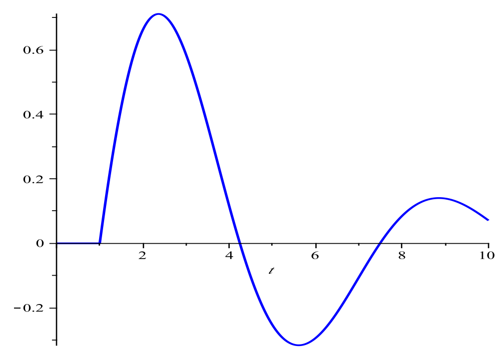
$$Y(s) = \frac{e^{-s}}{s^2 + s/2 + 1}\,.$$

First write

$$\frac{1}{s^2 + s/2 + 1} = \frac{1}{(s + \frac{1}{4})^2 + \frac{15}{16}}\,.$$

Taking the inverse transform and applying both shifting theorems,

$$y(t) = \frac{4}{\sqrt{15}}e^{-(t-1)/4}\,\sin\frac{\sqrt{15}}{4}(t-1)\,u_1(t)\,.$$



(b) As shown on the graph, the maximum is attained at some $t_1 > 2$. Note that for $t > 2$,

$$y(t) = \frac{4}{\sqrt{15}}e^{-(t-1)/4}\,\sin\frac{\sqrt{15}}{4}(t-1)\,.$$

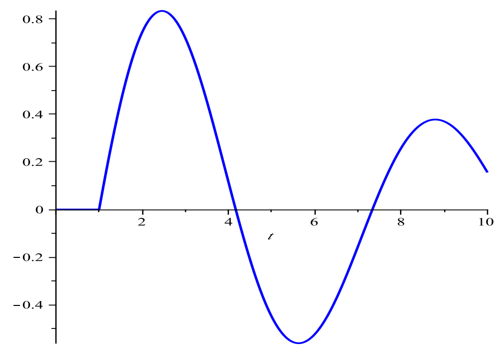
Setting $y'(t) = 0$, we find that $t_1 \approx 2.3613$. The maximum value is calculated as $y(2.3613) \approx 0.71153$.

(c) Setting $\gamma = 1/4$, the transform of the solution is

$$Y(s) = \frac{e^{-s}}{s^2 + s/4 + 1}\,.$$

Following the same steps, it follows that

$$y(t) = \frac{8}{3\sqrt{7}}e^{-(t-1)/8}\,\sin\frac{3\sqrt{7}}{8}(t-1)\,u_1(t)\,.$$



Once again, the maximum is attained at some $t_1 > 2$. Setting $y'(t) = 0$, we find that $t_1 \approx 2.4569$, with $y(t_1) \approx 0.8335$.

(d) Now suppose that $0 < \gamma < 1$. Then the transform of the solution is

$$Y(s) = \frac{e^{-s}}{s^2 + \gamma s + 1}.$$

First write

$$\frac{1}{s^2 + \gamma s + 1} = \frac{1}{(s + \gamma/2)^2 + (1 - \gamma^2/4)}.$$

It follows that

$$h(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2 + \gamma s + 1}\right] = \frac{2}{\sqrt{4 - \gamma^2}}\, e^{-\gamma t/2} \sin(\sqrt{1 - \gamma^2/4} \cdot t).$$

Hence the solution is

$$y(t) = h(t-1)\, u_1(t).$$

The solution is nonzero only if $t > 1$, in which case $y(t) = h(t-1)$. Setting $y'(t) = 0$, we obtain

$$\tan\left[\sqrt{1 - \gamma^2/4} \cdot (t-1)\right] = \frac{1}{\gamma}\sqrt{4 - \gamma^2},$$

that is,

$$\frac{\tan\left[\sqrt{1 - \gamma^2/4} \cdot (t-1)\right]}{\sqrt{1 - \gamma^2/4}} = \frac{2}{\gamma}.$$

As $\gamma \to 0$, we obtain the formal equation $\tan(t-1) = \infty$. Hence $t_1 \to 1 + \frac{\pi}{2}$. Setting $t = \pi/2$ in $h(t)$, and letting $\gamma \to 0$, we find that $y_1 \to 1$. These conclusions agree with the case $\gamma = 0$, for which it is easy to show that the solution is

$$y(t) = \sin(t-1)\, u_1(t).$$

15.(a) See Problem 14. It follows that the solution of the IVP is

$$y(t) = \frac{4k}{\sqrt{15}} e^{-(t-1)/4} \sin\frac{\sqrt{15}}{4}(t-1)\, u_1(t).$$

This function is a multiple of the answer in Problem 14(a). Hence the peak value occurs at $t_1 \approx 2.3613$. The maximum value is calculated as $y(2.3613) \approx 0.71153\,k$. We find that the appropriate value of k is $k_1 = 2/0.71153 \approx 2.8108$.

(b) Based on Problem 14(c), the solution is

$$y(t) = \frac{8\,k}{3\sqrt{7}} e^{-(t-1)/8} \sin \frac{3\sqrt{7}}{8}(t-1)\, u_1(t)\,.$$

Since this function is a multiple of the solution in Problem 14(c), we have that $t_1 \approx 2.4569$, with $y(t_1) \approx 0.8335\,k$. The solution attains a value of $y = 2$ when $k_1 = 2/0.8335$, that is, $k_1 \approx 2.3995$.

(c) Similar to Problem 14(d), for $0 < \gamma < 1$, the solution is

$$y(t) = h(t-1)\, u_1(t)\,,$$

in which

$$h(t) = \frac{2\,k}{\sqrt{4-\gamma^2}}\, e^{-\gamma t/2} \sin(\sqrt{1-\gamma^2/4} \cdot t).$$

It follows that $t_1 - 1 \to \pi/2$. Setting $t = \pi/2$ in $h(t)$, and letting $\gamma \to 0$, we find that $y_1 \to k$. Requiring that the peak value remains at $y = 2$, the limiting value of k is $k_1 = 2$. These conclusions agree with the case $\gamma = 0$, for which it is easy to show that the solution is

$$y(t) = k \sin(t-1)\, u_1(t)\,.$$

16.(a) Taking the initial conditions into consideration, the transformation of the ODE is

$$s^2\, Y(s) + Y(s) = \frac{1}{2k}\left[\frac{e^{-(4-k)s}}{s} - \frac{e^{-(4+k)s}}{s}\right].$$

Solving for the transform of the solution,

$$Y(s) = \frac{1}{2k}\left[\frac{e^{-(4-k)s}}{s(s^2+1)} - \frac{e^{-(4+k)s}}{s(s^2+1)}\right].$$

Using partial fractions,

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}\,.$$

Now let

$$h(t) = \mathcal{L}^{-1}\left[\frac{1}{s(s^2+1)}\right] = 1 - \cos t\,.$$

Applying Theorem 6.3.1, the solution is

$$\phi(t\,,k) = \frac{1}{2k}\left[h(t-4+k)\, u_{4-k}(t) - h(t-4-k)\, u_{4+k}(t)\right].$$

That is,

$$\begin{aligned}\phi(t\,,k) \;&= \frac{1}{2k}\,[\,u_{4-k}(t) -\; u_{4+k}(t)] - \\ &- \frac{1}{2k}\,[\cos(t-4+k)\,u_{4-k}(t) - \cos(t-4-k)\,u_{4+k}(t)]\,.\end{aligned}$$

(b) Consider various values of t. For any fixed $t<4$, $\phi(t,k)=0$, as long as $4-k>t$. If $t \geq 4$, then for $4+k<t$,

$$\phi(t\,,k) = -\frac{1}{2k}\,[\cos(t-4+k) - \cos(t-4-k)\,]\,.$$

It follows that

$$\lim_{k\to 0}\phi(t\,,k) = \lim_{k\to 0} -\frac{\cos(t-4+k)-\cos(t-4-k)}{2k} = \sin(t-4)\,.$$

Hence

$$\lim_{k\to 0}\phi(t\,,k) = \sin(t-4)\,u_4(t)\,.$$

(c) The Laplace transform of the differential equation

$$y'' + y = \delta(t-4),$$

with $y(0)=y'(0)=0$, is

$$s^2\,Y(s) + Y(s) = e^{-4s}.$$

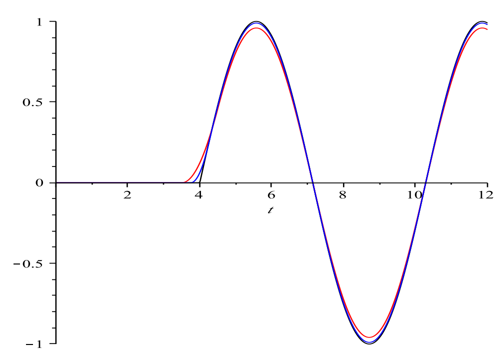
Solving for the transform of the solution,

$$Y(s) = \frac{e^{-4s}}{s^2+1}\,.$$

It follows that the solution is $\phi_0(t) = \sin(t-4)\,u_4(t)$, and this means that

$$\lim_{k\to 0}\phi(t,k) = \phi_0(t).$$

(d) We can see the convergence on the graphs.



18.(b) The transform of the ODE (given the specified initial conditions) is

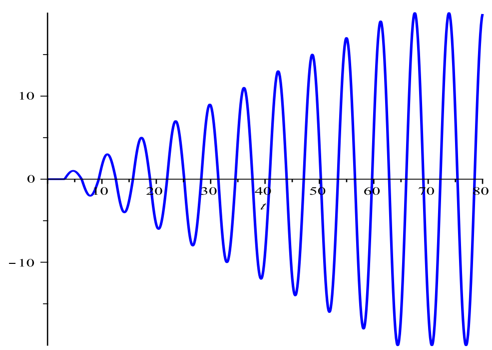
$$s^2\,Y(s) + Y(s) = \sum_{k=1}^{20} (-1)^{k+1} e^{-k\pi s}.$$

Solving for the transform of the solution,

$$Y(s) = \frac{1}{s^2+1}\,\sum_{k=1}^{20} (-1)^{k+1} e^{-k\pi s}.$$

Applying Theorem 6.3.1, term-by-term,

$$y(t) = \sum_{k=1}^{20} (-1)^{k+1} \sin(t - k\pi)\, u_{k\pi}(t) = -\sin(t) \cdot \sum_{k=1}^{20} u_{k\pi}(t)\,.$$



(c) For $t > 20\pi$, $y(t) = -20\sin t$.

19.(b) Taking the initial conditions into consideration, the transform of the ODE is

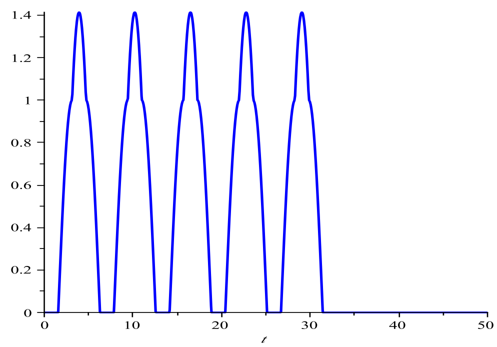
$$s^2\,Y(s) + Y(s) = \sum_{k=1}^{20} e^{-(k\pi/2)s}.$$

Solving for the transform of the solution,

$$Y(s) = \frac{1}{s^2+1}\,\sum_{k=1}^{20} e^{-(k\pi/2)s}.$$

Applying Theorem 6.3.1, term-by-term,

$$y(t) = \sum_{k=1}^{20} \sin(t - \frac{k\pi}{2})\, u_{k\pi/2}(t)\,.$$



(c) For $t > 10\pi$, $y(t) = 0$.

20.(b) The transform of the ODE (given the specified initial conditions) is

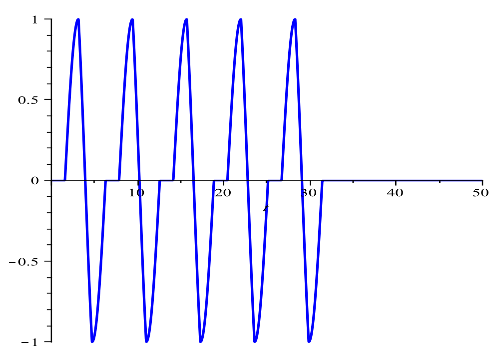
$$s^2\,Y(s) + Y(s) = \sum_{k=1}^{20} (-1)^{k+1} e^{-(k\pi/2)s}.$$

Solving for the transform of the solution,

$$Y(s) = \sum_{k=1}^{20} (-1)^{k+1} \frac{e^{-(k\pi/2)s}}{s^2+1}.$$

Applying Theorem 6.3.1, term-by-term,

$$y(t) = \sum_{k=1}^{20} (-1)^{k+1} \sin(t - \frac{k\pi}{2})\, u_{k\pi/2}(t)\,.$$



(c) For $t > 10\pi$, $y(t) = 0$.

22.(b) Taking the initial conditions into consideration, the transform of the ODE is

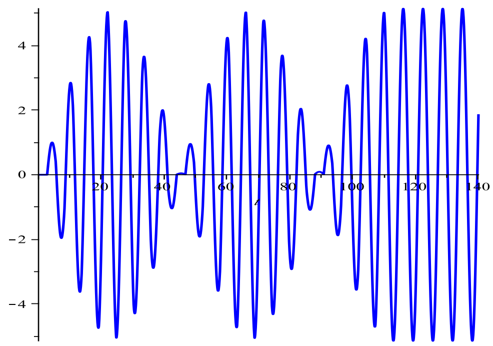
$$s^2\,Y(s) + Y(s) = \sum_{k=1}^{40} (-1)^{k+1} e^{-(11k/4)s}.$$

Solving for the transform of the solution,

$$Y(s) = \sum_{k=1}^{40} (-1)^{k+1} \frac{e^{-(11k/4)s}}{s^2+1}.$$

Applying Theorem 6.3.1, term-by-term,

$$y(t) = \sum_{k=1}^{40} (-1)^{k+1} \sin(t - \frac{11k}{4})\, u_{11k/4}(t)\,.$$



(c) For $t > 110$,

$$y(t) = \sum_{k=1}^{40} (-1)^{k+1} \sin\left(t - \frac{11k}{4}\right) \approx -5.13887 \cos(56.375 - t).$$

23.(b) The transform of the ODE (given the specified initial conditions) is

$$s^2\, Y(s) + 0.1s\, Y(s) + Y(s) = \sum_{k=1}^{20} (-1)^{k+1} e^{-k\pi s}.$$

Solving for the transform of the solution,

$$Y(s) = \sum_{k=1}^{20} \frac{e^{-k\pi s}}{s^2 + 0.1s + 1}.$$

First write

$$\frac{1}{s^2+0.1s+1} = \frac{1}{(s+\frac{1}{20})^2 + \frac{399}{400}}\,.$$

It follows that

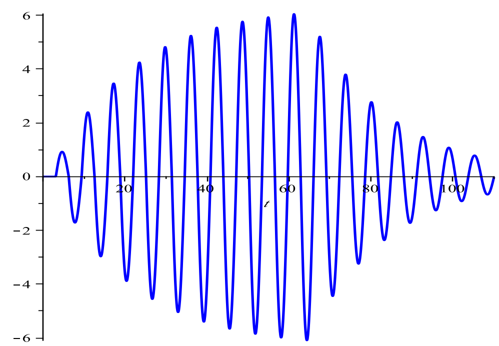
$$\mathcal{L}^{-1}\left[\frac{1}{s^2+0.1s+1}\right] = \frac{20}{\sqrt{399}} e^{-t/20} \sin(\frac{\sqrt{399}}{20}\, t).$$

Applying Theorem 6.3.1, term-by-term,

$$y(t) = \sum_{k=1}^{20} (-1)^{k+1} h(t - k\pi)\, u_{k\pi}(t)\,,$$

in which

$$h(t) = \frac{20}{\sqrt{399}} e^{-t/20} \sin(\frac{\sqrt{399}}{20}\, t).$$



(c) For $t > 20\pi$, each term in the sum for $y(t)$ will contain a decaying exponential term multiplied by a bounded term. Thus $y(t) \to 0$.

24.(b) Taking the initial conditions into consideration, the transform of the ODE is

$$s^2\, Y(s) + 0.1s\, Y(s) + \, Y(s) = \sum_{k=1}^{15} e^{-(2k-1)\pi s}.$$

Solving for the transform of the solution,

$$Y(s) = \sum_{k=1}^{15} \frac{e^{-(2k-1)\pi s}}{s^2 + 0.1s + 1}.$$

As shown in Problem 23,

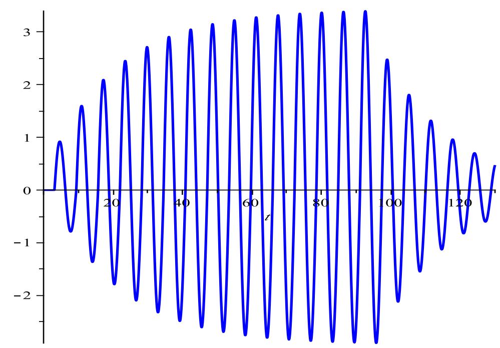
$$\mathcal{L}^{-1}\left[\frac{1}{s^2 + 0.1s + 1}\right] = \frac{20}{\sqrt{399}} e^{-t/20} \sin(\frac{\sqrt{399}}{20}\, t).$$

Applying Theorem 6.3.1, term-by-term,

$$y(t) = \sum_{k=1}^{15} h\,[t - (2k-1)\pi]\; u_{(2k-1)\pi}(t)\,,$$

in which

$$h(t) = \frac{20}{\sqrt{399}} e^{-t/20} \sin(\frac{\sqrt{399}}{20}\, t).$$



(c) For $t > 29\pi$, each term in the sum for $y(t)$ will contain a decaying exponential term multiplied by a bounded term. Thus $y(t) \to 0$.

6.6

2. Let $f(t) = e^t$. Then

$$(f * 1)(t) = \int_0^t e^{t-\tau} \cdot 1\, d\tau = e^t \int_0^t e^{-\tau} d\tau = e^t - 1\,.$$

3. It follows directly that

$$(f * f)(t) = \int_0^t \sin(t-\tau)\,\sin(\tau)\,d\tau = \frac{1}{2}\int_0^t [\cos(t-2\tau) - \cos(t)]d\tau = \frac{1}{2}(\sin t - t\cos t).$$

The range of the resulting function is $\mathbb{R}$.

5. We have $\mathcal{L}\left[e^{-t}\right] = 1/(s+1)$ and $\mathcal{L}\left[\sin t\right] = 1/(s^2+1)$. Based on Theorem 6.6.1,

$$\mathcal{L}\left[\int_0^t e^{-(t-\tau)}\sin(\tau)\,d\tau\right] = \frac{1}{s+1}\cdot\frac{1}{s^2+1} = \frac{1}{(s+1)(s^2+1)}\,.$$

6. Let $g(t) = t$ and $h(t) = e^t$. Then $f(t) = (g * h)(t)$. Applying Theorem 6.6.1,

$$\mathcal{L}\left[\int_0^t g(t-\tau)h(\tau)\,d\tau\right] = \frac{1}{s^2}\cdot\frac{1}{s-1} = \frac{1}{s^2(s-1)}\,.$$

7. We have $f(t) = (g * h)(t)$, in which $g(t) = \sin\, t$ and $h(t) = \cos\, t$. The transform of the convolution integral is

$$\mathcal{L}\left[\int_0^t g(t-\tau)h(\tau)\,d\tau\right] = \frac{1}{s^2+1}\cdot\frac{s}{s^2+1} = \frac{s}{(s^2+1)^2}\,.$$

9. It is easy to see that

$$\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] = e^{-t} \quad\text{and}\quad \mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] = \cos\, 2t\,.$$

Applying Theorem 6.6.1,

$$\mathcal{L}^{-1}\left[\frac{s}{(s+1)(s^2+4)}\right] = \int_0^t e^{-(t-\tau)}\cos 2\tau\, d\tau\,.$$

10. We first note that

$$\mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right] = t\,e^{-t} \quad \text{and} \quad \mathcal{L}^{-1}\left[\frac{1}{s^2+4}\right] = \frac{1}{2}\sin 2t\,.$$

Based on the convolution theorem,

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{1}{(s+1)^2(s^2+4)}\right] &= \frac{1}{2}\int_0^t (t-\tau)e^{-(t-\tau)}\sin 2\tau\, d\tau \\ &= \frac{1}{2}\int_0^t \tau\, e^{-\tau}\sin(2t-2\tau)\, d\tau\,.\end{aligned}$$

11. Let $g(t) = \mathcal{L}^{-1}[G(s)]$. Since $\mathcal{L}^{-1}\left[1/(s^2+1)\right] = \sin t$, the inverse transform of the product is

$$\mathcal{L}^{-1}\left[\frac{G(s)}{s^2+1}\right] = \int_0^t g(t-\tau)\,\sin\tau\, d\tau = \int_0^t \sin(t-\tau)\,g(\tau)\, d\tau\,.$$

12.(a) By definition,

$$f*g = \int_0^t (t-\tau)^m\tau^n\, d\tau\,.$$

Set $\tau = t - t\,u$, $d\tau = -t\,du$, so that

$$\int_0^t (t-\tau)^m\tau^n\, d\tau = -\int_1^0 (tu)^m(t-tu)^n\, t\, du = t^{m+n+1}\int_0^1 u^m(1-u)^n\, du\,.$$

(b) The Convolution Theorem states that $\mathcal{L}[f*g] = \mathcal{L}[f]\cdot\mathcal{L}[g]$. Noting that

$$\mathcal{L}\left[t^k\right] = \frac{k\,!}{s^{k+1}}\,,$$

it follows that

$$\frac{m\,!}{s^{m+1}}\,\frac{n\,!}{s^{m+1}} = \frac{(m+n+1)\,!}{s^{m+n+2}}\int_0^1 u^m(1-u)^n\, du\,.$$

Therefore

$$\int_0^1 u^m(1-u)^n\, du = \frac{m\,!\,n\,!}{(m+n+1)\,!}\,.$$

(c) If k is not an integer, we can write

$$\mathcal{L}[t^k] = \frac{\Gamma(k+1)}{s^{k+1}}$$

and

$$\frac{\Gamma(m+1)}{s^{m+1}}\cdot\frac{\Gamma(n+1)}{s^{n+1}} = \frac{\Gamma(m+1)}{s^{m+n+2}}\int_0^1 u^m(1-u)^n\, du,$$

so

$$\int_0^1 u^m(1-u)^n\,du = \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)}.$$

13. Taking the initial conditions into consideration, the transform of the ODE is

$$s^2\,Y(s) - 1 + \omega^2\,Y(s) = G(s).$$

Solving for the transform of the solution,

$$Y(s) = \frac{1}{s^2+\omega^2} + \frac{G(s)}{s^2+\omega^2}.$$

As shown in a related situation, Problem 11,

$$\mathcal{L}^{-1}\left[\frac{G(s)}{s^2+\omega^2}\right] = \frac{1}{\omega}\int_0^t \sin(\omega(t-\tau))\,g(\tau)\,d\tau\,.$$

Hence the solution of the IVP is

$$y(t) = \frac{1}{\omega}\,\sin(\omega t) + \frac{1}{\omega}\int_0^t \sin(\omega(t-\tau))\,g(\tau)\,d\tau\,.$$

15. The transform of the ODE (given the specified initial conditions) is

$$4s^2\,Y(s) + 4s\,Y(s) + \,17\,Y(s) = G(s).$$

Solving for the transform of the solution,

$$Y(s) = \quad \frac{G(s)}{4s^2+4s+\,17}.$$

First write

$$\frac{1}{4s^2+4s+\,17} = \frac{\frac{1}{4}}{(s+\frac{1}{2})^2+4}\,.$$

Based on the elementary properties of the Laplace transform,

$$\mathcal{L}^{-1}\left[\frac{1}{4s^2+4s+\,17}\right] = \frac{1}{8}e^{-t/2}\sin\,2t\,.$$

Applying the convolution theorem, the solution of the IVP is

$$y(t) = \frac{1}{8}\int_0^t e^{-(t-\tau)/2}\,\sin\,2(t-\tau)\,g(\tau)\,d\tau\,.$$

17. Taking the initial conditions into consideration, the transform of the ODE is

$$s^2\,Y(s) - 2s + 3 + 4\,[s\,Y(s) - 2] + 4\,Y(s) = G(s).$$

Solving for the transform of the solution,

$$Y(s) = \frac{2s+5}{(s+2)^2} + \frac{G(s)}{(s+2)^2}.$$

We can write

$$\frac{2s+5}{(s+2)^2} = \frac{2}{s+2} + \frac{1}{(s+2)^2}\,.$$

It follows that

$$\mathcal{L}^{-1}\left[\frac{2}{s+2}\right] = 2e^{-2t} \quad \text{and} \quad \mathcal{L}^{-1}\left[\frac{1}{(s+2)^2}\right] = t\,e^{-2t}.$$

Based on the convolution theorem, the solution of the IVP is

$$y(t) = 2e^{-2t} + t\,e^{-2t} + \int_0^t (t-\tau)e^{-2(t-\tau)}g(\tau)\,d\tau.$$

19. The transform of the ODE (given the specified initial conditions) is

$$s^4\,Y(s) - Y(s) = G(s).$$

Solving for the transform of the solution,

$$Y(s) = \frac{G(s)}{s^4-1}.$$

First write

$$\frac{1}{s^4-1} = \frac{1}{2}\left[\frac{1}{s^2-1} - \frac{1}{s^2+1}\right].$$

It follows that

$$\mathcal{L}^{-1}\left[\frac{1}{s^4-1}\right] = \frac{1}{2}\left[\sinh t - \sin t\right].$$

Based on the convolution theorem, the solution of the IVP is

$$y(t) = \frac{1}{2}\int_0^t \left[\sinh(t-\tau) - \sin(t-\tau)\right] g(\tau)\,d\tau.$$

20. Taking the initial conditions into consideration, the transform of the ODE is

$$s^4\,Y(s) - s^3 + 5s^2\,Y(s) - 5s + 4\,Y(s) = G(s).$$

Solving for the transform of the solution,

$$Y(s) = \frac{s^3+5s}{(s^2+1)(s^2+4)} + \frac{G(s)}{(s^2+1)(s^2+4)}.$$

Using partial fractions, we find that

$$\frac{s^3+5s}{(s^2+1)(s^2+4)} = \frac{1}{3}\left[\frac{4s}{s^2+1} - \frac{s}{s^2+4}\right],$$

and

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{1}{3}\left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right].$$

It follows that

$$\mathcal{L}^{-1}\left[\frac{s(s^2+5)}{(s^2+1)(s^2+4)}\right] = \frac{4}{3}\cos t - \frac{1}{3}\cos 2t,$$

and

$$\mathcal{L}^{-1}\left[\frac{1}{(s^2+1)(s^2+4)}\right] = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t.$$

Based on the convolution theorem, the solution of the IVP is

$$y(t) = \frac{4}{3}\cos t - \frac{1}{3}\cos 2t + \frac{1}{6}\int_0^t [2\sin(t-\tau) - \sin 2(t-\tau)]\, g(\tau)\, d\tau\,.$$

22.(a) Taking the Laplace transform of the integral equation, with $\Phi(s) = \mathcal{L}[\phi(t)]$,

$$\Phi(s) + \frac{1}{s^2}\cdot\Phi(s) = \frac{2}{s^2+4}\,.$$

Note that the convolution theorem was applied. Solving for the transform $\Phi(s)$,

$$\Phi(s) = \frac{2s^2}{(s^2+1)(s^2+4)}\,.$$

Using partial fractions, we can write

$$\frac{2s^2}{(s^2+1)(s^2+4)} = \frac{2}{3}\left[\frac{4}{s^2+4} - \frac{1}{s^2+1}\right].$$

Therefore the solution of the integral equation is

$$\phi(t) = \frac{4}{3}\sin 2t - \frac{2}{3}\sin t\,.$$

(b) Differentiate both sides of the equation, we get

$$\phi'(t) + (t-t)\phi(t) + \int_0^t \phi(\xi)d\xi = 2\cos 2t.$$

Clearly, $t - t = 0$, so differentiating this equation again we obtain

$$\phi''(t) + \phi(t) = -4\sin 2t.$$

Plugging $t = 0$ into the original equation gives us $\phi(0) = 0$. Also, $t = 0$ in the first equation here in part (b) gives $\phi'(0) = 2$.

(c) Taking the Laplace transform of the ODE, with $\Phi(s) = \mathcal{L}[\phi(t)]$,

$$s^2\Phi(s) - 2 + \Phi(s) = -\frac{8}{s^2+4}\,.$$

Solving for the transform of the solution,

$$\Phi(s) = \frac{2s^2}{(s^2+1)(s^2+4)}\,.$$

This is identical to the Laplace transform we obtained in part (a), so the solution will be the same.

23.(a) Taking the Laplace transform of both sides of the integral equation (using the Convolution Theorem)

$$\Phi(s) + \frac{1}{s^2}\Phi(s) = \frac{1}{s}\,.$$

It follows that

$$\Phi(s) = \frac{s}{s^2+1}$$

and

$$\phi(t) = \mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] = \cos\, t\ .$$

(b) Differentiating both sides of the equation twice, we get

$$\phi'(t) + \int_0^t \phi(\xi)\, d\xi = 0,$$

and then $\phi''(t) + \phi(t) = 0$. Plugging $t = 0$ into the original equation and the first equation above gives $\phi(0) = 1$ and $\phi'(0) = 0$.

(c) The function $\phi(t) = \cos t$ clearly solves the initial value problem in part (b).

25.(a) The Laplace transform of both sides of the integral equation (using the Convolution Theorem) is

$$\Phi(s) + \frac{2\,s}{s^2+1}\,\Phi(s) = \frac{1}{s+1}\ .$$

Solving for $\Phi(s)$:

$$\Phi(s) = \frac{s^2+1}{(s+1)^3}\ .$$

Rewriting,

$$\Phi(s) = \frac{(s+1)^2 - 2(s+1) + 2}{(s+1)^3} = \frac{1}{s+1} - \frac{2}{(s+1)^2} + \frac{2}{(s+1)^3}\ .$$

The solution of the integral equation is

$$\phi(t) = \mathcal{L}^{-1}\left[\frac{1}{s+1} - \frac{2}{(s+1)^2} + \frac{2}{(s+1)^3}\right] = e^{-t} - 2\,t\,e^{-t} + t^2\,e^{-t}\,.$$

(b) Differentiating both sides of the equation twice, we get

$$\phi'(t) + 2\phi(t) - 2\int_0^t \sin(t-\xi)\phi(\xi)\, d\xi = -e^{-t},$$

and then

$$\phi''(t) + 2\phi'(t) - 2\int_0^t \cos(t-\xi)\phi(\xi)\, d\xi = e^{-t}.$$

Using the original equation, we can convert the second equation to

$$\phi''(t) + 2\phi'(t) + \phi(t) = 2e^{-t}.$$

Plugging $t = 0$ into the original equation and the first equation above gives $\phi(0) = 1$ and $\phi'(0) = -3$.

(c) It is easily confirmed that the function $\phi(t) = e^{-t} - 2te^{-t} + t^2e^{-t}$ solves the initial value problem in part (b).

27.(a) Taking the Laplace transform of both sides of the integro-differential equation

$$s\,\Phi(s) - 1 - \frac{1}{s^3}\,\Phi(s) = -\frac{1}{s^2}\ .$$

Solving for $\Phi(s)$:

$$\Phi(s) = \frac{s}{s^2+1}\,.$$

Taking the inverse Laplace transform,

$$\phi(t) = \cos\, t\,.$$

(b) Differentiating both sides of the equation three times, we get

$$\phi''(t) - \int_0^t (t-\xi)\phi(\xi)\,d\xi = -1,$$

then

$$\phi'''(t) - \int_0^t \phi(\xi)\,d\xi = 0,$$

and then $\phi^{(4)}(t) - \phi(t) = 0$. Plugging $t = 0$ into the original equation and the first two equations above gives $\phi'(0) = 0$, $\phi''(0) = -1$ and $\phi'''(0) = 0$.

(c) It is easily confirmed that the function $\phi(t) = \cos t$ solves the initial value problem in part (b).

28.(a) The Laplace transform of both sides of the integro-differential equation is

$$s\,\Phi(s) - 1 + \Phi(s) = \frac{1}{s^2+1}\,\Phi(s)\ .$$

Solving for $\Phi(s)$:

$$\Phi(s) = \frac{s^2+1}{s(s^2+s+1)} = \frac{1}{s} - \frac{1}{(s^2+s+1)}\,.$$

Note further that

$$\frac{1}{(s^2+s+1)} = \frac{1}{(s+1/2)^2+3/4} = \frac{2}{\sqrt{3}}\,\frac{\sqrt{3}/2}{(s+1/2)^2+3/4}\,.$$

Taking the inverse Laplace transform,

$$\phi(t) = 1 - \frac{2}{\sqrt{3}}e^{-t/2}\,\sin(\frac{\sqrt{3}}{2}\,t)\,.$$

(b) Differentiating both sides of the equation twice, we get

$$\phi''(t) + \phi'(t) - \int_0^t \cos(t-\xi)\phi(\xi)\,d\xi = 0,$$

and then

$$\phi'''(t) + \phi''(t) - \phi(t) + \int_0^t \sin(t-\xi)\phi(\xi)\,d\xi = 0.$$

Using the original equation, we can convert the second equation to

$$\phi'''(t) + \phi''(t) + \phi'(t) = 0.$$

Plugging $t = 0$ into the original equation and the first equation above gives $\phi'(0) = -1$ and $\phi''(0) = 1$.

(c) It is easily confirmed that the function $\phi(t) = 1 - (2/\sqrt{3})e^{-t/2}\sin(\sqrt{3}\,t/2)$ solves the initial value problem in part (b).

29.(a) First note that

$$\int_0^b \frac{f(y)}{\sqrt{b-y}}\,dy = \left(\frac{1}{\sqrt{y}} * f\right)(b).$$

Take the Laplace transformation of both sides of the equation. Using the convolution theorem, with $F(s) = \mathcal{L}\,[f(y)]$,

$$\frac{T_0}{s} = \frac{1}{\sqrt{2g}}F(s)\cdot\mathcal{L}\left[\frac{1}{\sqrt{y}}\right].$$

It was shown in Problem 31(c), Section 6.1 , that

$$\mathcal{L}\left[\frac{1}{\sqrt{y}}\right] = \sqrt{\frac{\pi}{s}}\,.$$

Hence

$$\frac{T_0}{s} = \frac{1}{\sqrt{2g}}F(s)\cdot\sqrt{\frac{\pi}{s}}\,,$$

and

$$F(s) = \sqrt{\frac{2g}{\pi}}\cdot\frac{T_0}{\sqrt{s}}\,.$$

Taking the inverse transform, we obtain

$$f(y) = \frac{T_0}{\pi}\sqrt{\frac{2g}{y}}\,.$$

(b) Combining equations (i) and (iv),

$$\frac{2g\,T_0^2}{\pi^2\,y} = 1 + \left(\frac{dx}{dy}\right)^2.$$

Solving for the derivative dx/dy ,

$$\frac{dx}{dy} = \sqrt{\frac{2\alpha - y}{y}}\,,$$

in which $\alpha = gT_0^2/\pi^2$.

(c) Consider the change of variable $y = 2\alpha\,\sin^2(\theta/2)$. Using the chain rule,

$$\frac{dy}{dx} = 2\alpha\,\sin(\theta/2)\cos(\theta/2)\cdot\frac{d\theta}{dx}$$

and

$$\frac{dx}{dy} = \frac{1}{2\alpha\,\sin(\theta/2)\cos(\theta/2)} \cdot \frac{dx}{d\theta}\,.$$

It follows that

$$\frac{dx}{d\theta} = 2\alpha\,\sin(\theta/2)\cos(\theta/2)\sqrt{\frac{\cos^2(\theta/2)}{\sin^2(\theta/2)}} = 2\alpha\,\cos^2(\theta/2) = \alpha + \alpha\,\cos\,\theta\,.$$

Direct integration results in

$$x(\theta) = \alpha\,\theta + \alpha\,\sin\,\theta + C\,.$$

Since the curve passes through the origin, we require $y(0) = x(0) = 0$. Hence $C = 0$, and $x(\theta) = \alpha\,\theta + \alpha\,\sin\,\theta$. We also have

$$y(\theta) = 2\alpha\,\sin^2(\theta/2) = \alpha - \alpha\,\cos\,\theta\,.$$