CHAPTER

7

Systems of First Order Linear Equations

7.1

1. Introduce the variables $x_1 = u$ and $x_2 = u'$. It follows that $x_1' = x_2$ and

$$x_2' = u'' = -2u - 0.5\,u'.$$

In terms of the new variables, we obtain the system of two first order ODEs

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -2x_1 - 0.5\,x_2\,. \end{aligned}$$

3. First divide both sides of the equation by t^2, and write

$$u'' = -\frac{1}{t}\,u' - (1 - \frac{1}{4t^2})u\,.$$

Set $x_1 = u$ and $x_2 = u'$. It follows that $x_1' = x_2$ and

$$x_2' = u'' = -\frac{1}{t}\,u' - (1 - \frac{1}{4t^2})u\,.$$

We obtain the system of equations

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -(1 - \frac{1}{4t^2})x_1 - \frac{1}{t}\,x_2\,. \end{aligned}$$

5. Let $x_1 = u$ and $x_2 = u'$; then $u'' = x_2'$. In terms of the new variables, we have

$$x_2' + 0.25\,x_2 + 4\,x_1 = 2\,\cos 3t$$

with the initial conditions $x_1(0) = 1$ and $x_2(0) = -2$. The equivalent first order system is

$$x_1' = x_2$$
$$x_2' = -4\,x_1 - 0.25\,x_2 + 2\,\cos 3t$$

with the above initial conditions.

7.(a) Solving the first equation for x_2, we have $x_2 = x_1' + 2x_1$. Substitution into the second equation results in $(x_1' + 2x_1)' = x_1 - 2(x_1' + 2x_1)$. That is, $x_1'' + 4\,x_1' + 3\,x_1 = 0$. The resulting equation is a second order differential equation with constant coefficients. The general solution is $x_1(t) = c_1e^{-t} + c_2e^{-3t}$. With x_2 given in terms of x_1, it follows that $x_2(t) = c_1e^{-t} - c_2e^{-3t}$.

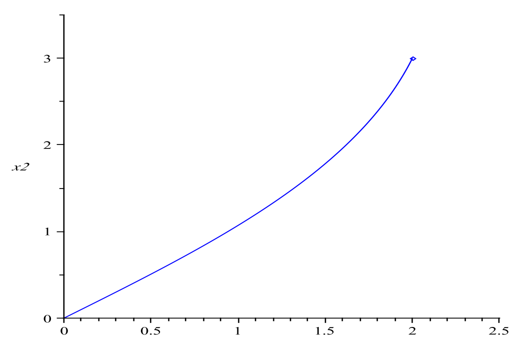
(b) Imposing the specified initial conditions, we obtain

$$c_1 + c_2 = 2, \qquad c_1 - c_2 = 3,$$

with solution $c_1 = 5/2$ and $c_2 = -1/2$. Hence

$$x_1(t) = \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t} \text{ and } x_2(t) = \frac{5}{2}e^{-t} + \frac{1}{2}e^{-3t}\,.$$

(c)

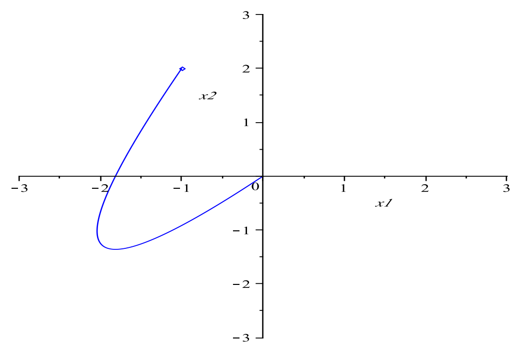


10.(a) Solving the first equation for x_2, we obtain $x_2 = (x_1 - x_1')/2$. Substitution into the second equation results in $(x_1 - x_1')'/2 = 3\,x_1 - 2(x_1 - x_1')$. Rearranging the terms, the single differential equation for x_1 is $x_1'' + 3\,x_1' + 2\,x_1 = 0$.

(b) The general solution is $x_1(t) = c_1e^{-t} + c_2e^{-2t}$. With x_2 given in terms of x_1, it follows that $x_2(t) = c_1e^{-t} + 3c_2e^{-2t}\,/2$. Invoking the specified initial conditions, $c_1 = -7$ and $c_2 = 6$. Hence

$$x_1(t) = -7e^{-t} + 6\,e^{-2t} \text{ and } x_2(t) = -7e^{-t} + 9\,e^{-2t}\,.$$

(c)

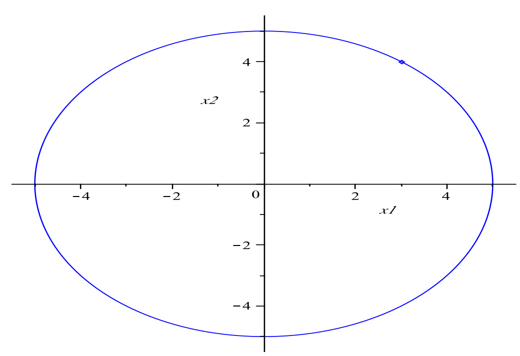


11.(a) Solving the first equation for x_2, we have $x_2 = x_1'/2$. Substitution into the second equation results in $x_1''/2 = -2\,x_1$. The resulting equation is $x_1'' + 4\,x_1 = 0$.

(b) The general solution is $x_1(t) = c_1 \cos 2t + c_2 \sin 2t$. With x_2 given in terms of x_1, it follows that $x_2(t) = -c_1 \sin 2t + c_2 \cos 2t$. Imposing the specified initial conditions, we obtain $c_1 = 3$ and $c_2 = 4$. Hence

$$x_1(t) = 3\cos 2t + 4\sin 2t \text{ and } x_2(t) = -3\sin 2t + 4\cos 2t\,.$$

(c)



13. Solving the first equation for V, we obtain $V = L \cdot I'$. Substitution into the second equation results in

$$L \cdot I'' = -\frac{I}{C} - \frac{L}{RC} I'\,.$$

Rearranging the terms, the single differential equation for I is

$$LRC \cdot I'' + L \cdot I' + R \cdot I = 0\,.$$

15. Let $x = c_1 x_1(t) + c_2 x_2(t)$ and $y = c_1 y_1(t) + c_2 y_2(t)$. Then

$$x' = c_1 x_1'(t) + c_2 x_2'(t)$$
$$y' = c_1 y_1'(t) + c_2 y_2'(t).$$

Since $x_1(t)$, $y_1(t)$ and $x_2(t)$, $y_2(t)$ are solutions for the original system,

$$x' = c_1(p_{11}x_1(t) + p_{12}y_1(t)) + c_2(p_{11}x_2(t) + p_{12}y_2(t))$$
$$y' = c_1(p_{21}x_1(t) + p_{22}y_1(t)) + c_2(p_{21}x_2(t) + p_{22}y_2(t)).$$

Rearranging terms gives

$$x' = p_{11}(c_1x_1(t) + c_2x_2(t)) + p_{12}(c_1y_1(t) + c_2y_2(t))$$
$$y' = p_{21}(c_1x_1(t) + c_2x_2(t)) + p_{22}(c_1y_1(t) + c_2y_2(t)),$$

and so x and y solve the original system.

16. Based on the hypothesis,

$$x_1'(t) = p_{11}(t)x_1(t) + p_{12}(t)y_1(t) + g_1(t)$$
$$x_2'(t) = p_{11}(t)x_2(t) + p_{12}(t)y_2(t) + g_1(t)\,.$$

Subtracting the two equations,

$$x_1'(t) - x_2'(t) = p_{11}(t)\left[x_1'(t) - x_2'(t)\right] + p_{12}(t)\left[y_1'(t) - y_2'(t)\right]\,.$$

Similarly,

$$y_1'(t) - y_2'(t) = p_{21}(t)\left[x_1'(t) - x_2'(t)\right] + p_{22}(t)\left[y_1'(t) - y_2'(t)\right]\,.$$

Hence the difference of the two solutions satisfies the homogeneous ODE.

17. For rectilinear motion in one dimension, Newton's second law can be stated as

$$\sum F = m\,x''.$$

The resisting force exerted by a linear spring is given by $F_s = k\,\delta$, in which δ is the displacement of the end of a spring from its equilibrium configuration. Hence, with $0 < x_1 < x_2$, the first two springs are in tension, and the last spring is in compression. The sum of the spring forces on m_1 is

$$F_s^1 = -k_1x_1 - k_2(x_2 - x_1)\,.$$

The total force on m_1 is

$$\sum F^1 = -k_1x_1 + k_2(x_2 - x_1) + F_1(t)\,.$$

Similarly, the total force on m_2 is

$$\sum F^2 = -k_2(x_2 - x_1) - k_3x_2 + F_2(t)\,.$$

18. One of the ways to transform the system is to assign the variables

$$y_1 = x_1, \qquad y_2 = x_2, \qquad y_3 = x_1', \qquad y_4 = x_2'.$$

Before proceeding, note that

$$x_1'' = \frac{1}{m_1}\left[-(k_1 + k_2)x_1 + k_2x_2 + F_1(t)\right]$$
$$x_2'' = \frac{1}{m_2}\left[k_2x_1 - (k_2 + k_3)x_2 + F_2(t)\right]\,.$$

Differentiating the new variables, we obtain the system of four first order equations

$$\begin{aligned} y_1' &= y_3 \\ y_2' &= y_4 \\ y_3' &= \frac{1}{m_1}(-(k_1+k_2)y_1 + k_2 y_2 + F_1(t)) \\ y_4' &= \frac{1}{m_2}(k_2 y_1 - (k_2+k_3)y_2 + F_2(t)) . \end{aligned}$$

19.(a) Taking a clockwise loop around each of the paths, it is easy to see that voltage drops are given by $V_1 - V_2 = 0$, and $V_2 - V_3 = 0$.

(b) Consider the right node. The current in is given by $I_1 + I_2$. The current leaving the node is $-I_3$. Hence the current passing through the node is $(I_1 + I_2) - (-I_3)$. Based on Kirchhoff's first law, $I_1 + I_2 + I_3 = 0$.

(c) In the capacitor,

$$C\,V_1{}' = I_1 .$$

In the resistor,

$$V_2 = R\,I_2 .$$

In the inductor,

$$L\,I_3' = V_3 .$$

(d) Based on part (a), $V_3 = V_2 = V_1$. Based on part (b),

$$C\,V_1{}' + \frac{1}{R}V_2 + I_3 = 0 .$$

It follows that

$$C\,V_1{}' = -\frac{1}{R}V_1 - I_3 \quad \text{and} \quad L\,I_3' = V_1 .$$

21. Let I_1, I_2, I_3,and I_4 be the current through the resistors, inductor, and capacitor, respectively. Assign V_1, V_2, V_3,and V_4 as the respective voltage drops. Based on Kirchhoff's second law, the net voltage drops, around each loop, satisfy

$$V_1 + V_3 + V_4 = 0, \quad V_1 + V_3 + V_2 = 0 \quad \text{and} \quad V_4 - V_2 = 0 .$$

Applying Kirchhoff's first law to the upper-right node,

$$I_3 - (I_2 + I_4) = 0 .$$

Likewise, in the remaining nodes,

$$I_1 - I_3 = 0 \quad \text{and} \quad I_2 + I_4 - I_1 = 0 .$$

That is,

$$V_4 - V_2 = 0, \quad V_1 + V_3 + V_4 = 0 \quad \text{and} \quad I_2 + I_4 - I_3 = 0 .$$

Using the current-voltage relations,

$$V_1 = R_1 I_1, \quad V_2 = R_2 I_2, \quad L\,I_3' = V_3, \quad C\,V_4{}' = I_4 .$$

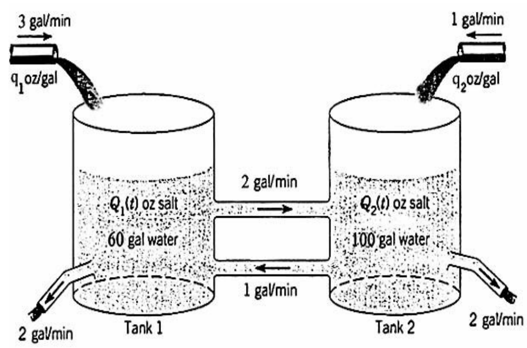
Combining these equations,

$$R_1 I_3 + L\,I_3' + V_4 = 0 \quad \text{and} \quad C\,V_4{}' = I_3 - \frac{V_4}{R_2}.$$

Now set $I_3 = I$ and $V_4 = V$, to obtain the system of equations

$$L\,I' = -R_1 I - V \quad \text{and} \quad C\,V' = I - \frac{V}{R_2}.$$

23.(a)



Let $Q_1(t)$ and $Q_2(t)$ be the amount of salt in the respective tanks at time t. Note that the volume of each tank remains constant. Based on conservation of mass, the rate of increase of salt, in any given tank, is given by

$$\textit{rate of increase} = \textit{rate in} - \textit{rate out.}$$

The rate of salt flowing into Tank 1 is

$$r_{in} = \left[q_1 \frac{\text{oz}}{\text{gal}}\right]\left[3 \frac{\text{gal}}{\text{min}}\right] + \left[\frac{Q_2}{100}\frac{\text{oz}}{\text{gal}}\right]\left[1 \frac{\text{gal}}{\text{min}}\right] = 3\,q_1 + \frac{Q_2}{100}\;\frac{\text{oz}}{\text{min}}.$$

The rate at which salt flows out of Tank 1 is

$$r_{out} = \left[\frac{Q_1}{60}\frac{\text{oz}}{\text{gal}}\right]\left[4 \frac{\text{gal}}{\text{min}}\right] = \frac{Q_1}{15}\frac{\text{oz}}{\text{min}}.$$

Hence

$$\frac{dQ_1}{dt} = 3\,q_1 + \frac{Q_2}{100} - \frac{Q_1}{15}.$$

Similarly, for Tank 2,

$$\frac{dQ_2}{dt} = q_2 + \frac{Q_1}{30} - \frac{3Q_2}{100}.$$

The process is modeled by the system of equations

$$\begin{aligned} Q_1' &= -\frac{Q_1}{15} + \frac{Q_2}{100} + 3\,q_1 \\ Q_2' &= \frac{Q_1}{30} - \frac{3Q_2}{100} + q_2\,. \end{aligned}$$

The initial conditions are $Q_1(0) = Q_1^0$ and $Q_2(0) = Q_2^0$.

(b) The equilibrium values are obtained by solving the system

$$-\frac{Q_1}{15}+\frac{Q_2}{100}+3\,q_1=0$$
$$\frac{Q_1}{30}-\frac{3Q_2}{100}+q_2=0\,.$$

Its solution leads to $Q_1^E=54\,q_1+6\,q_2$ and $Q_2^E=60\,q_1+40\,q_2$.

(c) The question refers to a possible solution of the system

$$54\,q_1+6\,q_2=60$$
$$60\,q_1+40\,q_2=50\,.$$

It is possible to formally solve the system of equations, but the unique solution gives

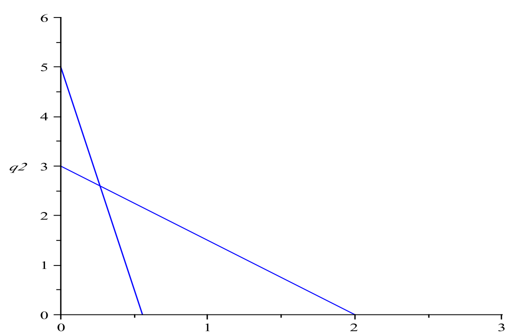
$$q_1=\frac{7}{6}\,\frac{\text{oz}}{\text{gal}}\qquad\text{and}\qquad q_2=-\frac{1}{2}\,\frac{\text{oz}}{\text{gal}},$$

which is not physically possible.

(d) We can write

$$q_2=-9\,q_1+\frac{Q_1^E}{6}$$
$$q_2=-\frac{3}{2}\,q_1+\frac{Q_2^E}{40}\,,$$

which are the equations of two lines in the q_1-q_2-plane:



The intercepts of the first line are $Q_1^E/54$ and $Q_1^E/6$. The intercepts of the second line are $Q_2^E/60$ and $Q_2^E/40$. Therefore the system will have a unique solution, in the first quadrant, as long as $Q_1^E/54\le Q_2^E/60$ or $Q_2^E/40\le Q_1^E/6$. That is,

$$\frac{10}{9}\le\frac{Q_2^E}{Q_1^E}\le\frac{20}{3}\,.$$

7.2

2.(a)

$$\mathbf{A}-2\,\mathbf{B}=\begin{pmatrix}1+i-2i & -1+2i-6\\ 3+2i-4 & 2-i+4i\end{pmatrix}=\begin{pmatrix}1-i & -7+2i\\ -1+2i & 2+3i\end{pmatrix}.$$

(b)

$$3\,\mathbf{A}+\,\mathbf{B}=\begin{pmatrix}3+3i+i & -3+6i+3\\ 9+6i+2 & 6-3i-2i\end{pmatrix}=\begin{pmatrix}3+4i & 6i\\ 11+6i & 6-5i\end{pmatrix}.$$

(c)

$$\begin{aligned}\mathbf{AB}\ &=\begin{pmatrix}(1+i)i+2(-1+2i) & 3(1+i)+(-1+2i)(-2i)\\ (3+2i)i+2(2-i) & 3(3+2i)+(2-i)(-2i)\end{pmatrix}\\ &=\begin{pmatrix}-3+5i & 7+5i\\ 2+i & 7+2i\end{pmatrix}.\end{aligned}$$

(d)

$$\begin{aligned}\mathbf{BA}\ &=\begin{pmatrix}(1+i)i+3(3+2i) & (-1+2i)i+3(2-i)\\ 2(1+i)+(-2i)(3+2i) & 2(-1+2i)+(-2i)(2-i)\end{pmatrix}\\ &=\begin{pmatrix}8+7i & 4-4i\\ 6-4i & -4\end{pmatrix}.\end{aligned}$$

3.(c,d)

$$\begin{aligned}\mathbf{A}^T+\mathbf{B}^T\ &=\begin{pmatrix}-2 & 1 & 2\\ 1 & 0 & -1\\ 2 & -3 & 1\end{pmatrix}+\begin{pmatrix}1 & 3 & -2\\ 2 & -1 & 1\\ 3 & -1 & 0\end{pmatrix}\\ &=\begin{pmatrix}-1 & 4 & 0\\ 3 & -1 & 0\\ 5 & -4 & 1\end{pmatrix}=(\mathbf{A}+\mathbf{B})^T.\end{aligned}$$

4.(b)

$$\overline{\mathbf{A}}=\begin{pmatrix}3+2i & 1-i\\ 2+i & -2-3i\end{pmatrix}.$$

(c) By definition,

$$\mathbf{A}^*=\overline{\mathbf{A}^T}=(\overline{\mathbf{A}})^T=\begin{pmatrix}3+2i & 2+i\\ 1-i & -2-3i\end{pmatrix}.$$

5.

$$2(\mathbf{A}+\mathbf{B})=2\begin{pmatrix}5 & 3 & -2\\ 0 & 2 & 5\\ 2 & 2 & 3\end{pmatrix}=\begin{pmatrix}10 & 6 & -4\\ 0 & 4 & 10\\ 4 & 4 & 6\end{pmatrix}.$$

7. Let $\mathbf{A}=(a_{ij})$ and $\mathbf{B}=(b_{ij})$. The given operations in (a)-(d) are performed elementwise. That is,

(a) $a_{ij}+b_{ij}=b_{ij}+a_{ij}$.

(b) $a_{ij} + (b_{ij} + c_{ij}) = (a_{ij} + b_{ij}) + c_{ij}$.

(c) $\alpha(a_{ij} + b_{ij}) = \alpha\, a_{ij} + \alpha\, b_{ij}$.

(d) $(\alpha + \beta)\, a_{ij} = \alpha\, a_{ij} + \beta\, a_{ij}$.

In the following, let $\mathbf{A}= (a_{ij})$, $\mathbf{B}= (b_{ij})$ and $\mathbf{C}= (c_{ij})$.

(e) Calculating the generic element,

$$(\mathbf{BC})_{ij} = \sum_{k=1}^{n} b_{ik}\, c_{kj} \,.$$

Therefore

$$[\mathbf{A}(\mathbf{BC})]_{ij} = \sum_{r=1}^{n} a_{ir}(\sum_{k=1}^{n} b_{rk}\, c_{kj}) = \sum_{r=1}^{n}\sum_{k=1}^{n} a_{ir}\, b_{rk}\, c_{kj} = \sum_{k=1}^{n}(\sum_{r=1}^{n} a_{ir}\, b_{rk})\, c_{kj}.$$

The inner summation is recognized as

$$\sum_{r=1}^{n} a_{ir}\, b_{rk} = (\mathbf{AB})_{ik} \,,$$

which is the ik-th element of the matrix $\mathbf{AB}$. Thus $[\mathbf{A}(\mathbf{BC})]_{ij} = [(\mathbf{AB})\mathbf{C}]_{ij}$.

(f) Likewise,

$$[\mathbf{A}(\mathbf{B}+\mathbf{C})]_{ij} = \sum_{k=1}^{n} a_{ik}(\, b_{kj} + \, c_{kj}) = \sum_{k=1}^{n} a_{ik}\, b_{kj} + \sum_{k=1}^{n} a_{ik}\, c_{kj} = (\mathbf{AB})_{ij} + (\mathbf{AC})_{ij} \,.$$

8.(a) $\mathbf{x}^T\mathbf{y}= 2(-1+i) + 2(3i) + (1-i)(3-i) = 4i$.

(b) $\mathbf{y}^T\mathbf{y}= (-1+i)^2 + 2^2 + (3-i)^2 = 12 - 8i$.

(c) $(\mathbf{x}, \mathbf{y}) = 2(-1-i) + 2(3i) + (1-i)(3+i) = 2 + 2i$.

(d) $(\mathbf{y}, \mathbf{y}) = (-1+i)(-1-i) + 2^2 + (3-i)(3+i) = 16$.

9. Indeed,

$$5 + 3i = \mathbf{x}^T\mathbf{y} = \sum_{j=1}^{n} x_j\, y_j = \mathbf{y}^T\mathbf{x} \,,$$

and

$$3 - 5i = (\mathbf{x}, \mathbf{y}) = \sum_{j=1}^{n} x_j\, \overline{y}_j = \sum_{j=1}^{n} \overline{y}_j\, x_j = \overline{\sum_{j=1}^{n} y_j\, \overline{x}_j} = \overline{(\mathbf{y}, \mathbf{x})}.$$

11. First augment the given matrix by the identity matrix:

$$[\mathbf{A}\,|\,\mathbf{I}\,] = \begin{pmatrix} 3 & -1 & 1 & 0 \\ 6 & 2 & 0 & 1 \end{pmatrix}.$$

Divide the first row by 3, to obtain

$$\begin{pmatrix} 1 & -1/3 & 1/3 & 0 \\ 6 & 2 & 0 & 1 \end{pmatrix}.$$

Adding -6 times the first row to the second row results in

$$\begin{pmatrix} 1 & -1/3 & 1/3 & 0 \\ 0 & 4 & -2 & 1 \end{pmatrix}.$$

Divide the second row by 4, to obtain

$$\begin{pmatrix} 1 & -1/3 & 1/3 & 0 \\ 0 & 1 & -1/2 & 1/4 \end{pmatrix}.$$

Finally, adding $1/3$ times the second row to the first row results in

$$\begin{pmatrix} 1 & 0 & 1/6 & 1/12 \\ 0 & 1 & -1/2 & 1/4 \end{pmatrix}.$$

Hence

$$\begin{pmatrix} 3 & -1 \\ 6 & 2 \end{pmatrix}^{-1} = \frac{1}{12}\begin{pmatrix} 2 & 1 \\ -6 & 3 \end{pmatrix}.$$

13. The augmented matrix is

$$\begin{pmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{pmatrix}.$$

Combining the elements of the first row with the elements of the second and third rows results in

$$\begin{pmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -3 & 3 & -2 & 1 & 0 \\ 0 & 0 & 3 & -1 & 0 & 1 \end{pmatrix}.$$

Divide the elements of the second row by -3, and the elements of the third row by 3. Now subtracting the new second row from the first row yields

$$\begin{pmatrix} 1 & 0 & 0 & 1/3 & 1/3 & 0 \\ 0 & 1 & -1 & 2/3 & -1/3 & 0 \\ 0 & 0 & 1 & -1/3 & 0 & 1/3 \end{pmatrix}.$$

Finally, combine the third row with the second row to obtain

$$\begin{pmatrix} 1 & 0 & 0 & 1/3 & 1/3 & 0 \\ 0 & 1 & 0 & 1/3 & -1/3 & 1/3 \\ 0 & 0 & 1 & -1/3 & 0 & 1/3 \end{pmatrix}.$$

Hence

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}.$$

15. Elementary row operations yield

$$\begin{pmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{pmatrix} \to$$
$$\begin{pmatrix} 1 & 0 & -1/4 & 1/2 & -1/4 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & -1/4 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1/4 & 1/2 & -1/4 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & -1/4 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{pmatrix}.$$

Finally, combining the first and third rows results in

$$\begin{pmatrix} 1 & 0 & 0 & 1/2 & -1/4 & 1/8 \\ 0 & 1 & 0 & 0 & 1/2 & -1/4 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{pmatrix}, \text{ so } A^{-1} = \begin{pmatrix} 1/2 & -1/4 & 1/8 \\ 0 & 1/2 & -1/4 \\ 0 & 0 & 1/2 \end{pmatrix}.$$

16. Elementary row operations yield

$$\begin{pmatrix} 1 & -1 & -1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & -2 & 1 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 3 & 2 & -2 & 1 & 0 \\ 0 & 1 & 4 & -3 & 0 & 1 \end{pmatrix} \to$$
$$\begin{pmatrix} 1 & 0 & -1/3 & 1/3 & 1/3 & 0 \\ 0 & 1 & 2/3 & -2/3 & 1/3 & 0 \\ 0 & 0 & 10/3 & -7/3 & -1/3 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1/10 & 3/10 & 1/10 \\ 0 & 1 & 0 & -1/5 & 2/5 & -1/5 \\ 0 & 0 & 10/3 & -7/3 & -1/3 & 1 \end{pmatrix}.$$

Finally, normalizing the last row results in

$$\begin{pmatrix} 1 & 0 & 0 & 1/10 & 3/10 & 1/10 \\ 0 & 1 & 0 & -1/5 & 2/5 & -1/5 \\ 0 & 0 & 1 & -7/10 & -1/10 & 3/10 \end{pmatrix}, \text{ so } A^{-1} = \begin{pmatrix} 1/10 & 3/10 & 1/10 \\ -1/5 & 2/5 & -1/5 \\ -7/10 & -1/10 & 3/10 \end{pmatrix}.$$

17. Elementary row operations on the augmented matrix yield the row-reduced form of the augmented matrix

$$\begin{pmatrix} 1 & 0 & -1/7 & 0 & 1/7 & 2/7 \\ 0 & 1 & 3/7 & 0 & 4/7 & 1/7 \\ 0 & 0 & 0 & 1 & -2 & -1 \end{pmatrix}.$$

The left submatrix cannot be converted to the identity matrix. Hence the given matrix is singular.

18. Elementary row operations on the augmented matrix yield

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix},$$

so

$$A^{-1} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

19. Elementary row operations on the augmented matrix yield

$$\begin{pmatrix} 1 & -1 & 2 & 0 & 1 & 0 & 0 & 0 \\ -1 & 2 & -4 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 3 & 0 & 0 & 1 & 0 \\ -2 & 2 & 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 3 & -1 & 0 & 1 & 0 \\ 0 & 0 & 4 & -1 & 2 & 0 & 0 & 1 \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & 0 & 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 4 & -1 & 2 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 4 & -3 & -1 & 2 & 0 \\ 0 & 0 & 1 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 0 & -5 & 10 & 4 & -4 & 1 \end{pmatrix}.$$

Normalizing the last row and combining it with the others results in

$$\begin{pmatrix} 1 & 0 & 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 4 & -3 & -1 & 2 & 0 \\ 0 & 0 & 1 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & -4/5 & 4/5 & -1/5 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 & 6 & 13/5 & -8/5 & 2/5 \\ 0 & 1 & 0 & 0 & 5 & 11/5 & -6/5 & 4/5 \\ 0 & 0 & 1 & 0 & 0 & -1/5 & 1/5 & 1/5 \\ 0 & 0 & 0 & 1 & -2 & -4/5 & 4/5 & -1/5 \end{pmatrix},$$

so

$$A^{-1} = \begin{pmatrix} 6 & 13/5 & -8/5 & 2/5 \\ 5 & 11/5 & -6/5 & 4/5 \\ 0 & -1/5 & 1/5 & 1/5 \\ -2 & -4/5 & 4/5 & -1/5 \end{pmatrix}.$$

20. Suppose that there exist matrices $\mathbf{B}$ and $\mathbf{C}$, such that $\mathbf{AB} = \mathbf{I}$ and $\mathbf{CA} = \mathbf{I}$. Then $\mathbf{CAB} = \mathbf{IB} = \mathbf{B}$, also, $\mathbf{CAB} = \mathbf{CI} = \mathbf{C}$. This shows that $\mathbf{B} = \mathbf{C}$.

23. First note that

$$\mathbf{x}' = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} (e^t + t\,e^t) = \begin{pmatrix} 3e^t + 2t\,e^t \\ 2e^t + 2t\,e^t \end{pmatrix}.$$

We also have

$$\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} (t\, e^t)$$
$$= \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^t + \begin{pmatrix} 2 \\ 2 \end{pmatrix} (t\, e^t) = \begin{pmatrix} 2e^t + 2t\, e^t \\ 3e^t + 2t\, e^t \end{pmatrix}.$$

It follows that

$$\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t = \begin{pmatrix} 3e^t + 2t\, e^t \\ 2e^t + 2t\, e^t \end{pmatrix}.$$

24. It is easy to see that

$$\mathbf{x}' = \begin{pmatrix} -6 \\ 8 \\ 4 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ 4 \\ -4 \end{pmatrix} e^{2t} = \begin{pmatrix} -6e^{-t} \\ 8e^{-t} + 4e^{2t} \\ 4e^{-t} - 4e^{-2t} \end{pmatrix}.$$

On the other hand,

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ -8 \\ -4 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} e^{2t}$$
$$= \begin{pmatrix} -6 \\ 8 \\ 4 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ 4 \\ -4 \end{pmatrix} e^{2t}.$$

26. Differentiation, elementwise, results in

$$\mathbf{\Psi}' = \begin{pmatrix} e^t & -2e^{-2t} & 3e^{3t} \\ -4e^t & 2e^{-2t} & 6e^{3t} \\ -e^t & 2e^{-2t} & 3e^{3t} \end{pmatrix}.$$

On the other hand,

$$\begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \mathbf{\Psi} = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} e^t & e^{-2t} & e^{3t} \\ -4e^t & -e^{-2t} & 2e^{3t} \\ -e^t & -e^{-2t} & e^{3t} \end{pmatrix}$$
$$= \begin{pmatrix} e^t & -2e^{-2t} & 3e^{3t} \\ -4e^t & 2e^{-2t} & 6e^{3t} \\ -e^t & 2e^{-2t} & 3e^{3t} \end{pmatrix}.$$

7.3

4. The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & -1 & 2 & 0 \end{array}\right).$$

Adding -2 times the first row to the second row and subtracting the first row from the third row results in

$$\begin{pmatrix} 1 & 2 & -1 & | & 0 \\ 0 & -3 & 3 & | & 0 \\ 0 & -3 & 3 & | & 0 \end{pmatrix}.$$

Adding the negative of the second row to the third row results in

$$\begin{pmatrix} 1 & 2 & -1 & | & 0 \\ 0 & -3 & 3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

We evidently end up with an equivalent system of equations

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ -x_2 + x_3 &= 0\,. \end{aligned}$$

Since there is no unique solution, let $x_3 = \alpha$, where α is arbitrary. It follows that $x_2 = \alpha$, and $x_1 = -\alpha$. Hence all solutions have the form

$$x = \alpha \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

5. The augmented matrix is

$$\begin{pmatrix} 1 & 0 & -1 & | & 0 \\ 3 & 1 & 1 & | & 0 \\ -1 & 1 & 2 & | & 0 \end{pmatrix}.$$

Adding -3 times the first row to the second row and adding the first row to the last row yields

$$\begin{pmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & 3 & | & 0 \\ 0 & 1 & 1 & | & 0 \end{pmatrix}.$$

Now add the negative of the second row to the third row to obtain

$$\begin{pmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & 3 & | & 0 \\ 0 & 0 & -2 & | & 0 \end{pmatrix}.$$

We end up with an equivalent linear system

$$\begin{aligned} x_1 - x_3 &= 0 \\ x_2 + 3\,x_3 &= 0 \\ x_3 &= 0\,. \end{aligned}$$

Hence the unique solution of the given system of equations is $x_1 = x_2 = x_3 = 0$.

6. The augmented matrix is

$$\begin{pmatrix} 1 & 2 & -1 & | & -2 \\ -2 & -4 & 2 & | & 4 \\ 2 & 4 & -2 & | & -4 \end{pmatrix}.$$

Adding 2 times the first row to the second row and subtracting 2 times the first row from the third row results in

$$\begin{pmatrix} 1 & 2 & -1 & | & -2 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

We evidently end up with an equivalent system of equations

$$x_1 + 2x_2 - x_3 = -2.$$

Since there is no unique solution, let $x_2 = \alpha$, and $x_3 = \beta$, where α, β are arbitrary. It follows that $x_1 = -2 - 2\alpha + \beta$. Hence all solutions have the form

$$\mathbf{x} = \begin{pmatrix} -2 - 2\alpha + \beta \\ \alpha \\ \beta \end{pmatrix}.$$

8. Write the given vectors as columns of the matrix

$$\mathbf{X} = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is evident that $\det(\mathbf{X}) = 0$. Hence the vectors are linearly dependent. In order to find a linear relationship between them, write $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)} = \mathbf{0}$. The latter equation is equivalent to

$$\begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Performing elementary row operations,

$$\begin{pmatrix} 2 & 0 & -1 & | & 0 \\ 1 & 1 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1/2 & | & 0 \\ 0 & 1 & 5/2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

We obtain the system of equations

$$c_1 - c_3/2 = 0$$
$$c_2 + 5c_3/2 = 0.$$

Setting $c_3 = 2$, it follows that $c_1 = 1$ and $c_3 = -5$. Hence

$$\mathbf{x}^{(1)} - 5\mathbf{x}^{(2)} + 2\mathbf{x}^{(3)} = \mathbf{0}.$$

10. The matrix containing the given vectors as columns is

$$\mathbf{X} = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 3 & 0 & -1 \\ -1 & 1 & 2 & 1 \\ 0 & -1 & 2 & 3 \end{pmatrix}.$$

We find that $\det(\mathbf{X}) = -70$. Hence the given vectors are linearly independent.

11. Write the given vectors as columns of the matrix

$$\mathbf{X} = \begin{pmatrix} 1 & 3 & 2 & 4 \\ 2 & 1 & -1 & 3 \\ -2 & 0 & 1 & -2 \end{pmatrix}.$$

The four vectors are necessarily linearly dependent. Hence there are nonzero scalars such that $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)} + c_4\mathbf{x}^{(4)} = \mathbf{0}$. The latter equation is equivalent to

$$\begin{pmatrix} 1 & 3 & 2 & 4 \\ 2 & 1 & -1 & 3 \\ -2 & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Performing elementary row operations,

$$\begin{pmatrix} 1 & 3 & 2 & 4 & | & 0 \\ 2 & 1 & -1 & 3 & | & 0 \\ -2 & 0 & 1 & -2 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & | & 0 \\ 0 & 1 & 0 & 1 & | & 0 \\ 0 & 0 & 1 & 0 & | & 0 \end{pmatrix}.$$

We end up with an equivalent linear system

$$\begin{aligned} c_1 + c_4 &= 0 \\ c_2 + c_4 &= 0 \\ c_3 &= 0 \,. \end{aligned}$$

Let $c_4 = -1$. Then $c_1 = 1$ and $c_2 = 1$. Therefore we find that

$$\mathbf{x}^{(1)} + \mathbf{x}^{(2)} - \mathbf{x}^{(4)} = \mathbf{0}\,.$$

12. The matrix containing the given vectors as columns, $\mathbf{X}$, is of size $n \times m$. Since $n < m$, we can augment the matrix with $m - n$ rows of zeros. The resulting matrix, $\tilde{\mathbf{X}}$, is of size $m \times m$. Since $\tilde{\mathbf{X}}$ is a square matrix, with at least one row of zeros, it follows that $\det(\tilde{\mathbf{X}}) = 0$. Hence the column vectors of $\tilde{\mathbf{X}}$ are linearly dependent. That is, there is a nonzero vector, $\mathbf{c}$, such that $\tilde{\mathbf{X}}\mathbf{c}= \mathbf{0}_{m\times 1}$. If we write only the first n rows of the latter equation, we have $\mathbf{X}\mathbf{c}= \mathbf{0}_{n\times 1}$. Therefore the column vectors of $\mathbf{X}$ are linearly dependent.

13. By inspection, we find that

$$\mathbf{x}^{(1)}(t) - 2\mathbf{x}^{(2)}(t) = \begin{pmatrix} -e^{-t} \\ 0 \end{pmatrix}.$$

Hence $3\,\mathbf{x}^{(1)}(t) - 6\,\mathbf{x}^{(2)}(t) + \mathbf{x}^{(3)}(t) = \mathbf{0}$, and the vectors are linearly dependent.

17. The eigenvalues λ and eigenvectors $\mathbf{x}$ satisfy the equation

$$\begin{pmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $(3-\lambda)(-1-\lambda) + 8 = 0$, that is,

$$\lambda^2 - 2\lambda + 5 = 0\,.$$

The eigenvalues are $\lambda_1 = 1 - 2\,i$ and $\lambda_2 = 1 + 2\,i$. The components of the eigenvector $\mathbf{x}^{(1)}$ are solutions of the system

$$\begin{pmatrix} 2+2\,i & -2 \\ 4 & -2+2\,i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The two equations reduce to $(1+i)x_1 = x_2$. Hence $\mathbf{x}^{(1)} = (1\,, 1+i)^T$. Now setting $\lambda = \lambda_2 = 1 + 2\,i$, we have

$$\begin{pmatrix} 2-2\,i & -2 \\ 4 & -2-2\,i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

with solution given by $\mathbf{x}^{(2)} = (1\,, 1-i)^T$.

18. The eigenvalues λ and eigenvectors $\mathbf{x}$ satisfy the equation

$$\begin{pmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $(-2-\lambda)(-2-\lambda) - 1 = 0$, that is,

$$\lambda^2 + 4\lambda + 3 = 0\,.$$

The eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = -1$. For $\lambda_1 = -3$, the system of equations becomes

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which reduces to $x_1 + x_2 = 0$. A solution vector is given by $\mathbf{x}^{(1)} = (1\,, -1)^T$. Substituting $\lambda = \lambda_2 = -1$, we have

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The equations reduce to $x_1 = x_2$. Hence a solution vector is given by $\mathbf{x}^{(2)} = (1\,, 1)^T$.

20. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} 1-\lambda & \sqrt{3} \\ \sqrt{3} & -1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, the determinant of the coefficient matrix must be zero. That is,

$$\lambda^2 - 4 = 0\,.$$

Hence the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 2$. Substituting the first eigenvalue, $\lambda = -2$, yields

$$\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The system is equivalent to the equation $\sqrt{3}\,x_1 + x_2 = 0$. A solution vector is given by $\mathbf{x}^{(1)} = (1\,, -\sqrt{3}\,)^T$. Substitution of $\lambda = 2$ results in

$$\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which reduces to $x_1 = \sqrt{3}\, x_2$. A corresponding solution vector is $\mathbf{x}^{(2)} = (\sqrt{3}\,, 1)^T$.

21. The eigenvalues λ and eigenvectors x satisfy the equation

$$\begin{pmatrix} -3-\lambda & 3/4 \\ -5 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $(-3-\lambda)(1-\lambda) + 15/4 = 0$, that is,

$$\lambda^2 + 2\lambda + 3/4 = 0\,.$$

Hence the eigenvalues are $\lambda_1 = -3/2$ and $\lambda_2 = -1/2$. In order to determine the eigenvector corresponding to λ_1 , set $\lambda = -3/2$. The system of equations becomes

$$\begin{pmatrix} -3/2 & 3/4 \\ -5 & 5/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which reduces to $-2\,x_1 + x_2 = 0$. A solution vector is given by $\mathbf{x}^{(1)} = (1\,, 2)^T$. Substitution of $\lambda = \lambda_2 = -1/2$ results in

$$\begin{pmatrix} -5/2 & 3/4 \\ -5 & 3/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which reduces to $10\,x_1 = 3\,x_2$. A corresponding solution vector is $\mathbf{x}^{(2)} = (3\,, 10)^T$.

23. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} 3-\lambda & 2 & 2 \\ 1 & 4-\lambda & 1 \\ -2 & -4 & -1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$, with roots $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$. Setting $\lambda = \lambda_1 = 1$, we have

$$\begin{pmatrix} 2 & 2 & 2 \\ 1 & 3 & 1 \\ -2 & -4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system is reduces to the equations

$$\begin{aligned} x_1 + x_3 &= 0 \\ x_2 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\mathbf{x}^{(1)} = (1\,, 0\,, -1)^T$. Setting $\lambda = \lambda_2 = 2$, the reduced system of equations is

$$\begin{aligned} x_1 + 2\,x_2 &= 0 \\ x_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\mathbf{x}^{(2)} = (-2\,, 1\,, 0)^T$. Finally, setting $\lambda = \lambda_3 = 3$, the reduced system of equations is

$$\begin{aligned} x_1 &= 0 \\ x_2 + x_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\mathbf{x}^{(3)} = (0\,, 1\,, -1)^T$.

24. For computational purposes, note that if λ is an eigenvalue of $\mathbf{B}$, then $c\lambda$ is an eigenvalue of the matrix $\mathbf{A}= c\mathbf{B}$. Eigenvectors are unaffected, since they are only determined up to a scalar multiple. So with

$$\mathbf{B} = \begin{pmatrix} 11 & -2 & 8 \\ -2 & 2 & 10 \\ 8 & 10 & 5 \end{pmatrix},$$

the associated characteristic equation is $\mu^3 - 18\mu^2 - 81\mu + 1458 = 0$, with roots $\mu_1 = -9$, $\mu_2 = 9$ and $\mu_3 = 18$. Hence the eigenvalues of the given matrix, $\mathbf{A}$, are $\lambda_1 = -1$, $\lambda_2 = 1$ and $\lambda_3 = 2$. Setting $\lambda = \lambda_1 = -1$, (which corresponds to using $\mu_1 = -9$ in the modified problem) the reduced system of equations is

$$\begin{aligned} 2\,x_1 + x_3 &= 0 \\ x_2 + x_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\mathbf{x}^{(1)} = (1\,,2\,,-2)^T$. Setting $\lambda = \lambda_2 = 1$, the reduced system of equations is

$$\begin{aligned} x_1 + 2\,x_3 &= 0 \\ x_2 - 2\,x_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\mathbf{x}^{(2)} = (2\,,-2\,,-1)^T$. Finally, setting $\lambda = \lambda_2 = 1$, the reduced system of equations is

$$\begin{aligned} x_1 - x_3 &= 0 \\ 2\,x_2 - x_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\mathbf{x}^{(3)} = (2\,,1\,,2)^T$.

26.(b) By definition,

$$(\mathbf{Ax}\,,\mathbf{y}) = \sum_{i=0}^{n} (\mathbf{Ax})_i\,\overline{y_i} = \sum_{i=0}^{n}\sum_{j=0}^{n} a_{ij}\,x_j\,\overline{y_i}\,.$$

Let $b_{ij} = \overline{a_{ji}}$, so that $a_{ij} = \overline{b_{ji}}$. Now interchanging the order or summation,

$$(\mathbf{Ax}\,,\mathbf{y}) = \sum_{j=0}^{n} x_j \sum_{i=0}^{n} a_{ij}\,\overline{y_i} = \sum_{j=0}^{n} x_j \sum_{i=0}^{n} \overline{b_{ji}}\,\overline{y_i}\,.$$

Now note that

$$\sum_{i=0}^{n} \overline{b_{ji}}\,\overline{y_i} = \overline{\sum_{i=0}^{n} b_{ji}\,y_i} = \overline{(\mathbf{A}^*\mathbf{y})}_j\,.$$

Therefore

$$(\mathbf{Ax}\,,\mathbf{y}) = \sum_{j=0}^{n} x_j\,\overline{(\mathbf{A}^*\mathbf{y})}_j = (\mathbf{x}\,,\mathbf{A}^*\mathbf{y})\,.$$

(c) By definition of a Hermitian matrix, $\mathbf{A}=\mathbf{A}^*$.

27. Suppose that $\mathbf{A}\mathbf{x}=\mathbf{0}$, but that $\mathbf{x}\neq\mathbf{0}$. Let $\mathbf{A}=(a_{ij})$. Using elementary row operations, it is possible to transform the matrix into one that is not upper triangular. If it were upper triangular, backsubstitution would imply that $\mathbf{x}=\mathbf{0}$. Hence a linear combination of all the rows results in a row containing only zeros. That is, there are n scalars, β_i, one for each row and not all zero, such that for each for column j,

$$\sum_{i=1}^{n} \beta_i\, a_{ij} = 0\,.$$

Now consider $\mathbf{A}^* = (b_{ij})$. By definition, $b_{ij} = \overline{a_{ji}}$, or $a_{ij} = \overline{b_{ji}}$. It follows that for each j,

$$\sum_{i=1}^{n} \beta_i\, \overline{b_{ji}} = \sum_{k=1}^{n} \overline{b_{jk}}\, \beta_k = \sum_{k=1}^{n} b_{jk}\, \overline{\beta_k} = 0\,.$$

Let $\mathbf{y}=(\overline{\beta_1},\overline{\beta_2},\cdots,\overline{\beta_n})^T$. Hence we have a nonzero vector, $\mathbf{y}$, such that $\mathbf{A}^*\mathbf{y}=\mathbf{0}$.

29. By linearity,

$$\mathbf{A}(\mathbf{x}^{(0)} + \alpha\,\boldsymbol{\xi}) = \mathbf{A}\mathbf{x}^{(0)} + \alpha\,\mathbf{A}\boldsymbol{\xi} = \mathbf{b} + \mathbf{0} = \mathbf{b}\,.$$

30. Let $c_{ij} = \overline{a_{ji}}$. By the hypothesis, there is a nonzero vector, $\mathbf{y}$, such that

$$\sum_{j=1}^{n} c_{ij}\, y_j = \sum_{j=1}^{n} \overline{a_{ji}}\; y_j = 0\,,\; i = 1, 2, \cdots, n\,.$$

Taking the conjugate of both sides, and interchanging the indices, we have

$$\sum_{i=1}^{n} a_{ij}\; \overline{y_i} = 0\,.$$

This implies that a linear combination of each row of $\mathbf{A}$ is equal to zero. Now consider the augmented matrix $[\mathbf{A}\,|\mathbf{B}]$. Replace the last row by

$$\sum_{i=1}^{n} \overline{y_i}\; [a_{i1}\,, a_{i2}\,, \cdots, a_{in}\,, b_i] = \left[0\,, 0\,, \cdots, 0\,, \sum_{i=1}^{n} \overline{y_i}\, b_i\right].$$

We find that if $(\mathbf{B}\,,\mathbf{y}) = 0$, then the last row of the augmented matrix contains only zeros. Hence there are $n-1$ remaining equations. We can now set $x_n = \alpha$, some parameter, and solve for the other variables in terms of α. Therefore the system of equations $\mathbf{A}\mathbf{x}=\mathbf{b}$ has a solution.

31. If $\lambda = 0$ is an eigenvalue of $\mathbf{A}$, then there is a nonzero vector, $\mathbf{x}$, such that

$$\mathbf{A}\mathbf{x} = \lambda\,\mathbf{x} = \mathbf{0}\,.$$

That is, $\mathbf{A}\mathbf{x}=\mathbf{0}$ has a nonzero solution. This implies that the mapping defined by $\mathbf{A}$ is not 1-to-1, and hence not invertible. On the other hand, if $\mathbf{A}$ is singular, then $\det(\mathbf{A}) = 0$. Thus, $\mathbf{A}\mathbf{x}=\mathbf{0}$ has a nonzero solution. The latter equation can be written as $\mathbf{A}\mathbf{x}= 0\,\mathbf{x}$.

32.(a) Based on Problem 26, $(\mathbf{Ax}\,,\mathbf{x}) = (\mathbf{x}\,,\mathbf{Ax})$.

(b) Let $\mathbf{x}$ be an eigenvector corresponding to an eigenvalue λ. It then follows that $(\mathbf{Ax}\,,\mathbf{x}) = (\lambda\mathbf{x}\,,\mathbf{x})$ and $(\mathbf{x}\,,\mathbf{Ax}) = (\mathbf{x}\,,\lambda\mathbf{x})$. Based on the properties of the inner product, $(\lambda\mathbf{x}\,,\mathbf{x}) = \lambda(\mathbf{x}\,,\mathbf{x})$ and $(\mathbf{x}\,,\lambda\mathbf{x}) = \overline{\lambda}(\mathbf{x}\,,\mathbf{x})$. Then from part (a),

$$\lambda(\mathbf{x}\,,\mathbf{x}) = \overline{\lambda}(\mathbf{x}\,,\mathbf{x})\,.$$

(c) From part (b),

$$(\lambda - \overline{\lambda})(\mathbf{x}\,,\mathbf{x}) = 0\,.$$

Based on the definition of an eigenvector, $(\mathbf{x}\,,\mathbf{x}) = \|\mathbf{x}\|^2 > 0$. Hence we must have $\lambda - \overline{\lambda} = 0$, which implies that λ is real.

33. From Problem 26(c),

$$(\mathbf{Ax}^{(1)}\,,\mathbf{x}^{(2)}) = (\mathbf{x}^{(1)}\,,\mathbf{Ax}^{(2)})\,.$$

Hence

$$\lambda_1(\mathbf{x}^{(1)}\,,\mathbf{x}^{(2)}) = \overline{\lambda_2}(\mathbf{x}^{(1)}\,,\mathbf{x}^{(2)}) = \lambda_2(\mathbf{x}^{(1)}\,,\mathbf{x}^{(2)})\,,$$

since the eigenvalues are real. Therefore

$$(\lambda_1 - \lambda_2)(\mathbf{x}^{(1)}\,,\mathbf{x}^{(2)}) = 0\,.$$

Given that $\lambda_1 \neq \lambda_2$, we must have $(\mathbf{x}^{(1)}\,,\mathbf{x}^{(2)}) = 0$.

7.4

3. Equation (14) states that the Wronskian satisfies the first order linear ODE

$$\frac{dW}{dt} = (p_{11} + p_{22} + \cdots + p_{nn})W.$$

The general solution of this is given by Equation (15):

$$W(t) = C\, e^{\int (p_{11}+p_{22}+\cdots+p_{nn})dt}\,,$$

in which C is an arbitrary constant. Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be matrices representing two sets of fundamental solutions. It follows that

$$\det(\mathbf{X}_1) = W_1(t) = C_1 e^{\int (p_{11}+p_{22}+\cdots+p_{nn})dt}$$
$$\det(\mathbf{X}_2) = W_2(t) = C_2 e^{\int (p_{11}+p_{22}+\cdots+p_{nn})dt}\,.$$

Hence $\det(\mathbf{X}_1)/\det(\mathbf{X}_2) = C_1/C_2$. Note that $C_2 \neq 0$.

4. First note that $p_{11} + p_{22} = -p(t)$. As shown in Problem 3,

$$W\left[\mathbf{x}^{(1)}\,,\mathbf{x}^{(2)}\right] = c\, e^{-\int p(t)dt}.$$

For second order linear ODE, the Wronskian (as defined in Chapter 3) satisfies the first order differential equation $W' + p(t)W = 0$. It follows that

$$W\left[\mathbf{y}^{(1)}, \mathbf{y}^{(2)}\right] = c_1\, e^{-\int p(t)dt}.$$

Alternatively, based on the hypothesis,

$$\begin{aligned}\mathbf{y}^{(1)} &= \alpha_{11}\, x_{11} + \alpha_{12}\, x_{12} \\ \mathbf{y}^{(2)} &= \alpha_{21}\, x_{11} + \alpha_{22}\, x_{12}\,.\end{aligned}$$

Direct calculation shows that

$$\begin{aligned}W\left[\mathbf{y}^{(1)}, \mathbf{y}^{(2)}\right] &= \begin{vmatrix}\alpha_{11}\, x_{11} + \alpha_{12}\, x_{12} & \alpha_{21}\, x_{11} + \alpha_{22}\, x_{12} \\ \alpha_{11}\, x_{11}' + \alpha_{12}\, x_{12}' & \alpha_{21}\, x_{11}' + \alpha_{22}\, x_{12}'\end{vmatrix} \\ &= (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})x_{11}x_{12}' - (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})x_{12}x_{11}' \\ &= (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})x_{11}x_{22} - (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})x_{12}x_{21}\,.\end{aligned}$$

Here we used the fact that $\mathbf{x}_1' = \mathbf{x}_2$. Hence

$$W\left[\mathbf{y}^{(1)}, \mathbf{y}^{(2)}\right] = (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})W\left[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right]\,.$$

5. The particular solution satisfies the ODE $(\mathbf{x}^{(p)})' = \mathbf{P}(t)\mathbf{x}^{(p)} + \mathbf{g}(t)$. Now let $\mathbf{x}$ be any solution of the homogeneous equation, $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$. We know that $\mathbf{x} = \mathbf{x}^{(c)}$, in which $\mathbf{x}^{(c)}$ is a linear combination of some fundamental solution. By linearity of the differential equation, it follows that $\mathbf{x} = \mathbf{x}^{(p)} + \mathbf{x}^{(c)}$ is a solution of the ODE. Based on the uniqueness theorem, all solutions must have this form.

7.(a) By definition,

$$W\left[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right] = \begin{vmatrix} t^2 & e^t \\ 2t & e^t \end{vmatrix} = (t^2 - 2t)e^t.$$

(b) The Wronskian vanishes at $t_0 = 0$ and $t_0 = 2$. Hence the vectors are linearly independent on $\mathcal{D} = (-\infty, 0) \cup (0, 2) \cup (2, \infty)$.

(c) It follows from Theorem 7.4.3 that one or more of the coefficients of the ODE must be discontinuous at $t_0 = 0$ and $t_0 = 2$. If not, the Wronskian would not vanish.

(d) Let

$$\mathbf{x} = c_1\begin{pmatrix} t^2 \\ 2t \end{pmatrix} + c_2\begin{pmatrix} e^t \\ e^t \end{pmatrix}.$$

Then

$$\mathbf{x}' = c_1\begin{pmatrix} 2t \\ 2 \end{pmatrix} + c_2\begin{pmatrix} e^t \\ e^t \end{pmatrix}.$$

On the other hand,

$$\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \mathbf{x} = c_1 \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} t^2 \\ 2t \end{pmatrix} + c_2 \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} e^t \\ e^t \end{pmatrix}$$
$$= \begin{pmatrix} c_1 \left[p_{11}t^2 + 2p_{12}t\right] + c_2 \left[p_{11} + p_{12}\right] e^t \\ c_1 \left[p_{21}t^2 + 2p_{22}t\right] + c_2 \left[p_{21} + p_{22}\right] e^t \end{pmatrix}.$$

Comparing coefficients, we find that

$$\begin{aligned} p_{11}t^2 + 2p_{12}t &= 2t \\ p_{11} + p_{12} &= 1 \\ p_{21}t^2 + 2p_{22}t &= 2 \\ p_{21} + p_{22} &= 1\,. \end{aligned}$$

Solution of this system of equations results in

$$p_{11}(t) = 0, \quad p_{12}(t) = 1, \quad p_{21}(t) = \frac{2-2t}{t^2-2t}, \quad p_{22}(t) = \frac{t^2-2}{t^2-2t}\,.$$

Hence the vectors are solutions of the ODE

$$\mathbf{x}' = \frac{1}{t^2-2t} \begin{pmatrix} 0 & t^2-2t \\ 2-2t & t^2-2 \end{pmatrix} \mathbf{x}\,.$$

8. Suppose that the solutions $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$,$\cdots$, $\mathbf{x}^{(m)}$ are linearly dependent at $t = t_0$. Then there are constants $c_1, c_2, \cdots, c_m$ (not all zero) such that

$$c_1\mathbf{x}^{(1)}(t_0) + c_2\mathbf{x}^{(2)}(t_0) + \cdots + c_m\mathbf{x}^{(m)}(t_0) = \mathbf{0}\,.$$

Now let $\mathbf{z}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \cdots + c_m\mathbf{x}^{(m)}(t)$. Then clearly, $\mathbf{z}(t)$ is a solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, with $\mathbf{z}(t_0) = 0$. Furthermore, $\mathbf{y}(t) \equiv \mathbf{0}$ is also a solution, with $\mathbf{y}(t_0) = 0$. By the uniqueness theorem, $\mathbf{z}(t) = \mathbf{y}(t) = \mathbf{0}$. Hence

$$c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \cdots + c_m\mathbf{x}^{(m)}(t) = \mathbf{0}$$

on the entire interval $\alpha < t < \beta$. Going in the other direction is trivial.

9.(a) Let $\mathbf{y}(t)$ be any solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$. It follows that

$$\mathbf{z}(t) + \mathbf{y}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t) + \mathbf{y}(t)$$

is also a solution. Now let $t_0 \in (\alpha, \beta)$. Then the collection of vectors

$$\mathbf{x}^{(1)}(t_0), \mathbf{x}^{(2)}(t_0), \ldots, \mathbf{x}^{(n)}(t_0),\ \mathbf{y}(t_0)$$

constitutes $n+1$ vectors, each with n components. Based on the assertion in Problem 12, Section 7.3, these vectors are necessarily linearly dependent. That is, there are $n+1$ constants $b_1, b_2, \ldots, b_n, b_{n+1}$ (not all zero) such that

$$b_1\mathbf{x}^{(1)}(t_0) + b_2\mathbf{x}^{(2)}(t_0) + \cdots + b_n\mathbf{x}^{(n)}(t_0) + b_{n+1}\,\mathbf{y}(t_0) = \mathbf{0}\,.$$

From Problem 8, we have

$$b_1\mathbf{x}^{(1)}(t) + b_2\mathbf{x}^{(2)}(t) + \cdots + b_n\mathbf{x}^{(n)}(t) + b_{n+1}\,\mathbf{y}(t) = \mathbf{0}$$

for all $t \in (\alpha\,,\beta)$. Now $b_{n+1} \neq 0$, otherwise that would contradict the fact that the first n vectors are linearly independent. Hence

$$\mathbf{y}(t) = -\frac{1}{b_{n+1}}(b_1\mathbf{x}^{(1)}(t) + b_2\mathbf{x}^{(2)}(t) + \cdots + b_n\mathbf{x}^{(n)}(t)),$$

and the assertion is true.

(b) Consider $\mathbf{z}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\ \mathbf{x}^{(2)}(t) + \cdots + c_n\ \mathbf{x}^{(n)}(t)$, and suppose that we also have

$$\mathbf{z}(t) = k_1\mathbf{x}^{(1)}(t) + k_2\mathbf{x}^{(2)}(t) + \cdots + k_n\mathbf{x}^{(n)}(t)\,.$$

Based on the assumption,

$$(k_1 - c_1)\mathbf{x}^{(1)}(t) + (k_2 - c_2)\mathbf{x}^{(2)}(t) + \cdots + (k_n - c_n)\mathbf{x}^{(n)}(t) = \mathbf{0}\,.$$

The collection of vectors

$$\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t), \ldots, \mathbf{x}^{(n)}(t)$$

is linearly independent on $\alpha < t < \beta$. It follows that $k_i - c_i = 0$, for $i = 1, 2, \cdots, n$.

7.5

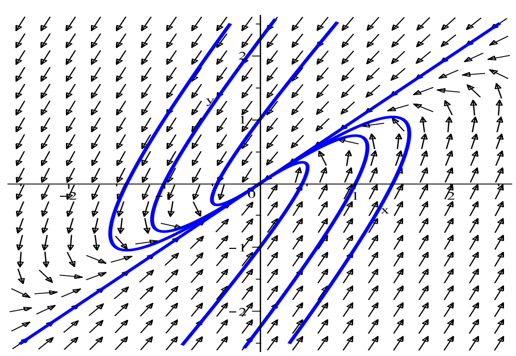
2.(a) Setting $\mathbf{x}= \boldsymbol{\xi}\, e^{rt}$, and substituting into the ODE, we obtain the algebraic equations

$$\begin{pmatrix} 1-r & -2 \\ 3 & -4-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 + 3r + 2 = 0$. The roots of the characteristic equation are $r_1 = -1$ and $r_2 = -2$. For $r = -1$, the two equations reduce to $\xi_1 = \xi_2$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1, 1)^T$. Substitution of $r = -2$ results in the single equation $3\xi_1 = 2\xi_2$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (2, 3)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t} + c_2\begin{pmatrix} 2 \\ 3 \end{pmatrix}e^{-2t}.$$

(b)



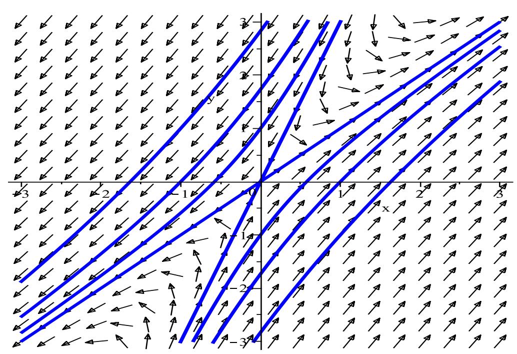
3.(a) Setting $\mathbf{x} = \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 2-r & -1 \\ 3 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 - 1 = 0$. The roots of the characteristic equation are $r_1 = 1$ and $r_2 = -1$. For $r = 1$, the system of equations reduces to $\xi_1 = \xi_2$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1,1)^T$. Substitution of $r = -1$ results in the single equation $3\xi_1 = \xi_2$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (1,3)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}.$$

(b)



The system has an unstable eigendirection along $\boldsymbol{\xi}^{(1)} = (1,1)^T$. Unless $c_1 = 0$, all solutions will diverge.

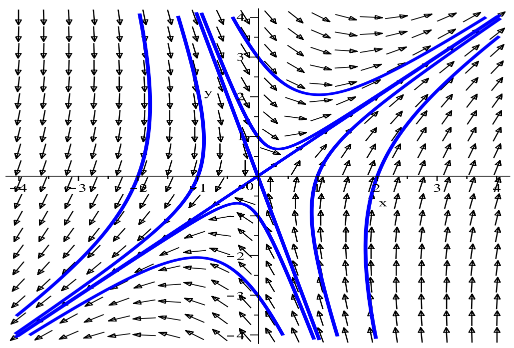
4.(a) Solution of the ODE requires analysis of the algebraic equations

$$\begin{pmatrix} 1-r & 1 \\ 4 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 + r - 6 = 0$. The roots of the characteristic equation are $r_1 = 2$ and $r_2 = -3$. For $r = 2$, the system of equations reduces to $\xi_1 = \xi_2$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1,1)^T$. Substitution of $r = -3$ results in the single equation $4\xi_1 + \xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (1,-4)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}.$$

(b)



The system has an unstable eigendirection along $\boldsymbol{\xi}^{(1)} = (1, 1)^T$. Unless $c_1 = 0$, all solutions will diverge.

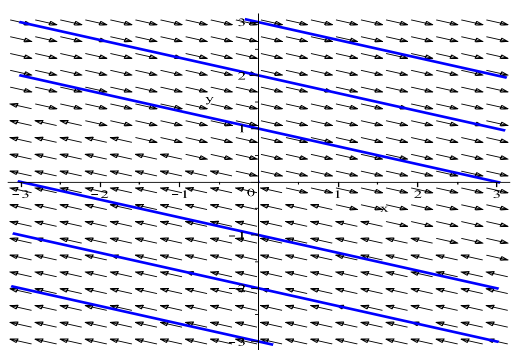
8.(a) Setting $\mathbf{x} = \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 3-r & 6 \\ -1 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 - r = 0$. The roots of the characteristic equation are $r_1 = 1$ and $r_2 = 0$. With $r = 1$, the system of equations reduces to $\xi_1 + 3\xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (3, -1)^T$. For the case $r = 0$, the system is equivalent to the equation $\xi_1 + 2\xi_2 = 0$. An eigenvector is $\boldsymbol{\xi}^{(2)} = (2, -1)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 3 \\ -1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

(b)



The entire line along the eigendirection $\boldsymbol{\xi}^{(2)} = (2, -1)^T$ consists of equilibrium points. All other solutions diverge. The direction field changes across the line $x_1 + 2x_2 = 0$. Eliminating the exponential terms in the solution, the trajectories are given by $x_1 + 3\,x_2 = -c_2$.

10. The characteristic equation is given by

$$\begin{vmatrix} 2-r & 2+i \\ -1 & -1-i-r \end{vmatrix} = r^2 - (1-i)r - i = 0\,.$$

The equation has complex roots $r_1 = 1$ and $r_2 = -i$. For $r = 1$, the components of the solution vector must satisfy $\xi_1 + (2+i)\xi_2 = 0$. Thus the corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (2+i\,, -1)^T$. Substitution of $r = -i$ results in the single equation $\xi_1 + \xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (1\,, -1)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 2+i \\ -1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-it}.$$

11. Setting $\mathbf{x} = \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 1-r & 1 & 2 \\ 1 & 2-r & 1 \\ 2 & 1 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^3 - 4r^2 - r + 4 = 0$. The roots of the characteristic equation are $r_1 = 4$, $r_2 = 1$ and $r_3 = -1$. Setting $r = 4$, we have

$$\begin{pmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system is reduces to the equations

$$\begin{aligned} \xi_1 - \xi_3 &= 0 \\ \xi_2 - \xi_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(1)} = (1\,, 1\,, 1)^T$. Setting $\lambda = 1$, the reduced system of equations is

$$\begin{aligned} \xi_1 - \xi_3 &= 0 \\ \xi_2 + 2\,\xi_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(2)} = (1\,, -2\,, 1)^T$. Finally, setting $\lambda = -1$, the reduced system of equations is

$$\begin{aligned} \xi_1 + \xi_3 &= 0 \\ \xi_2 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(3)} = (1\,, 0\,, -1)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{t} + c_3 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t}.$$

12. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} 3-r & 2 & 4 \\ 2 & -r & 2 \\ 4 & 2 & 3-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $r^3 - 6r^2 - 15r - 8 = 0$, with roots $r_1 = 8$, $r_2 = -1$ and $r_3 = -1$. Setting $r = r_1 = 8$, we have

$$\begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system is reduced to the equations

$$\begin{aligned} \xi_1 - \xi_3 &= 0 \\ 2\,\xi_2 - \xi_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(1)} = (2\,,1\,,2)^T$. Setting $r = -1$, the system of equations is reduced to the single equation

$$2\,\xi_1 + \xi_2 + 2\,\xi_3 = 0\,.$$

Two independent solutions are obtained as

$$\boldsymbol{\xi}^{(2)} = (1\,,-2\,,0)^T \text{ and } \boldsymbol{\xi}^{(3)} = (0\,,-2\,,1)^T.$$

Hence the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} e^{8t} + c_2 \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} e^{-t}.$$

13. Setting $\mathbf{x}= \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 1-r & 1 & 1 \\ 2 & 1-r & -1 \\ -8 & -5 & -3-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^3 + r^2 - 4r - 4 = 0$. The roots of the characteristic equation are $r_1 = 2$, $r_2 = -2$ and $r_3 = -1$. Setting $r = 2$, we have

$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -8 & -5 & -5 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system is reduces to the equations

$$\begin{aligned} \xi_1 &= 0 \\ \xi_2 + \xi_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(1)} = (0\,,1\,,-1)^T$. Setting $\lambda = -1$, the reduced system of equations is

$$\begin{aligned} 2\,\xi_1 + 3\,\xi_3 &= 0 \\ \xi_2 - 2\,\xi_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(2)} = (3\,,-4\,,-2)^T$. Finally, setting $\lambda = -2$, the reduced system of equations is

$$7\,\xi_1 + 4\,\xi_3 = 0$$
$$7\xi_2 - 5\,\xi_3 = 0\,.$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(3)} = (4\,,-5\,,-7)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 3 \\ -4 \\ -2 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 4 \\ -5 \\ -7 \end{pmatrix} e^{-2t}.$$

15. Setting $\mathbf{x}= \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 5-r & -1 \\ 3 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 - 6r + 8 = 0$. The roots of the characteristic equation are $r_1 = 4$ and $r_2 = 2$. With $r = 4$, the system of equations reduces to $\xi_1 - \xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1\,,1)^T$. For the case $r = 2$, the system is equivalent to the equation $3\,\xi_1 - \xi_2 = 0$. An eigenvector is $\boldsymbol{\xi}^{(2)} = (1\,,3)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}.$$

Invoking the initial conditions, we obtain the system of equations

$$c_1 + c_2 = 2$$
$$c_1 + 3\,c_2 = -1\,.$$

Hence $c_1 = 7/2$ and $c_2 = -3/2$, and the solution of the IVP is

$$\mathbf{x} = \frac{7}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} - \frac{3}{2} \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}.$$

17. Setting $\mathbf{x}= \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 1-r & 1 & 2 \\ 0 & 2-r & 2 \\ -1 & 1 & 3-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^3 - 6r^2 + 11r - 6 = 0$. The roots of the characteristic equation are $r_1 = 1$, $r_2 = 2$ and $r_3 = 3$. Setting $r = 1$, we have

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system reduces to the equations

$$\xi_1 = 0$$
$$\xi_2 + 2\,\xi_3 = 0\,.$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(1)} = (0\,,-2\,,1)^T$. Setting $\lambda = 2$, the reduced system of equations is

$$\begin{aligned} \xi_1 - \xi_2 &= 0 \\ \xi_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(2)} = (1\,,1\,,0)^T$. Finally, upon setting $\lambda = 3$, the reduced system of equations is

$$\begin{aligned} \xi_1 - 2\,\xi_3 &= 0 \\ \xi_2 - 2\,\xi_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(3)} = (2\,,2\,,1)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} e^{3t}.$$

Invoking the initial conditions, the coefficients must satisfy the equations

$$\begin{aligned} c_2 + 2\,c_3 &= 2 \\ -2\,c_1 + c_2 + 2\,c_3 &= 0 \\ c_1 + c_3 &= 1\,. \end{aligned}$$

It follows that $c_1 = 1$, $c_2 = 2$ and $c_3 = 0$. Hence the solution of the IVP is

$$\mathbf{x} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} e^t + 2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t}.$$

18. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} -r & 0 & -1 \\ 2 & -r & 0 \\ -1 & 2 & 4-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $r^3 - 4r^2 - r + 4 = 0$, with roots $r_1 = -1$, $r_2 = 1$ and $r_3 = 4$. Setting $r = r_1 = -1$, we have

$$\begin{pmatrix} -1 & 0 & -1 \\ 2 & -1 & 0 \\ -1 & 2 & 3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system is reduced to the equations

$$\begin{aligned} \xi_1 - \xi_3 &= 0 \\ \xi_2 + 2\,\xi_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(1)} = (1\,,-2\,,1)^T$. Setting $r = 1$, the system reduces to the equations

$$\begin{aligned} \xi_1 + \xi_3 &= 0 \\ \xi_2 + 2\,\xi_3 &= 0\,. \end{aligned}$$

The corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (1\,,2\,,-1)^T$. Finally, upon setting $r = 4$, the system is equivalent to the equations

$$\begin{aligned} 4\,\xi_1 + \xi_3 &= 0 \\ 8\,\xi_2 + \xi_3 &= 0\,. \end{aligned}$$

The corresponding eigenvector is $\boldsymbol{\xi}^{(3)} = (2\,,1\,,-8)^T$. Hence the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} e^{t} + c_3 \begin{pmatrix} 2 \\ 1 \\ -8 \end{pmatrix} e^{4t}.$$

Invoking the initial conditions,

$$\begin{aligned} c_1 + c_2 + 2\,c_3 &= 7 \\ -2\,c_1 + 2\,c_2 + c_3 &= 5 \\ c_1 - c_2 - 8\,c_3 &= 5\,. \end{aligned}$$

It follows that $c_1 = 3$, $c_2 = 6$ and $c_3 = -1$. Hence the solution of the IVP is

$$\mathbf{x} = 3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{-t} + 6 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} e^{t} - \begin{pmatrix} 2 \\ 1 \\ -8 \end{pmatrix} e^{4t}.$$

19. Set $\mathbf{x}= \boldsymbol{\xi}\, t^r$. Substitution into the system of differential equations results in

$$t \cdot r t^{r-1} \boldsymbol{\xi} \;\; = \mathbf{A}\,\boldsymbol{\xi}\, t^r,$$

which upon simplification yields is, $\mathbf{A}\,\boldsymbol{\xi} - r\boldsymbol{\xi} = \mathbf{0}$. Hence the vector $\boldsymbol{\xi}$ and constant r must satisfy $(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$.

21. Setting $\mathbf{x}= \boldsymbol{\xi}\, t^r$ results in the algebraic equations

$$\begin{pmatrix} 5-r & -1 \\ 3 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 - 6r + 8 = 0$. The roots of the characteristic equation are $r_1 = 4$ and $r_2 = 2$. With $r = 4$, the system of equations reduces to $\xi_1 - \xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1\,,1)^T$. For the case $r = 2$, the system is equivalent to the equation $3\,\xi_1 - \xi_2 = 0$. An eigenvector is $\boldsymbol{\xi}^{(2)} = (1\,,3)^T$. It follows that

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t^4 \text{ and } \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} t^2.$$

The Wronskian of this solution set is $W\left[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right] = 2t^6$. Thus the solutions are linearly independent for $t > 0$. Hence the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} t^4 + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} t^2.$$

22. As shown in Problem 19, solution of the ODE requires analysis of the equations

$$\begin{pmatrix} 4-r & -3 \\ 8 & -6-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 + 2r = 0$. The roots of the characteristic equation are $r_1 = 0$ and $r_2 = -2$. For $r = 0$, the system of equations reduces to $4\,\xi_1 = 3\,\xi_2$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (3\,,4)^T$. Setting $r = -2$ results in the single equation $2\,\xi_1 - \xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (1\,,2)^T$. It follows that

$$\mathbf{x}^{(1)} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \text{ and } \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t^{-2}.$$

The Wronskian of this solution set is $W\left[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right] = 2\,t^{-2}$. These solutions are linearly independent for $t > 0$. Hence the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 3 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} t^{-2}.$$

23. Setting $\mathbf{x} = \boldsymbol{\xi}\, t^r$ results in the algebraic equations

$$\begin{pmatrix} 3 - r & -2 \\ 2 & -2 - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 - r - 2 = 0$. The roots of the characteristic equation are $r_1 = 2$ and $r_2 = -1$. Setting $r = 2$, the system of equations reduces to $\xi_1 - 2\xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (2, 1)^T$. With $r = -1$, the system is equivalent to the equation $2\xi_1 - \xi_2 = 0$. An eigenvector is $\boldsymbol{\xi}^{(2)} = (1, 2)^T$. It follows that

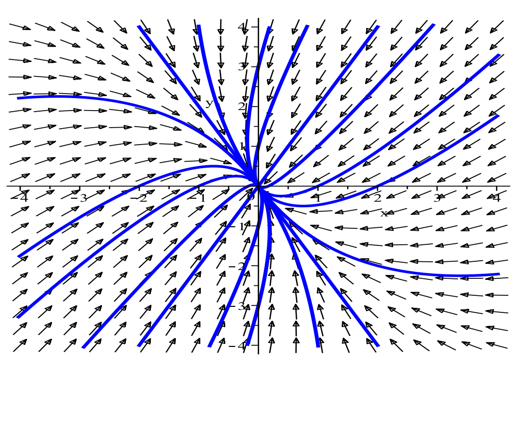
$$\mathbf{x}^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} t^2 \text{ and } \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t^{-1}.$$

The Wronskian of this solution set is $W\left[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right] = 3\,t$. Thus the solutions are linearly independent for $t > 0$. Hence the general solution is

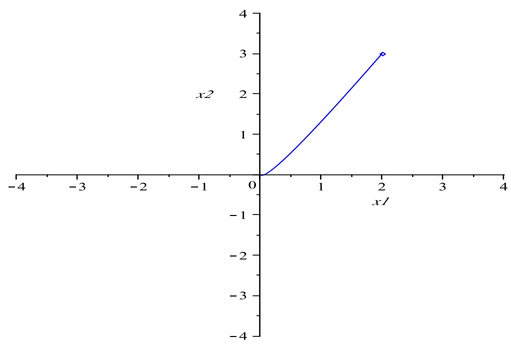
$$\mathbf{x} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} t^2 + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} t^{-1}.$$

24.(a) The general solution is

$$x = c_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t}.$$



(b)



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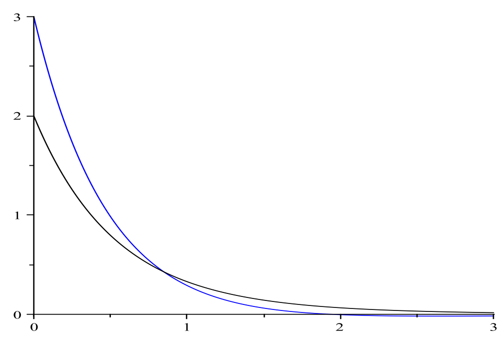
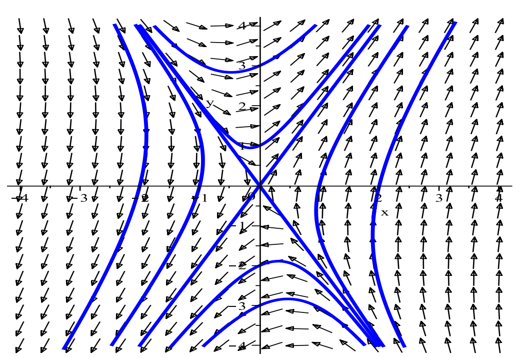


26.(a) The general solution is

$$x = c_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}.$$



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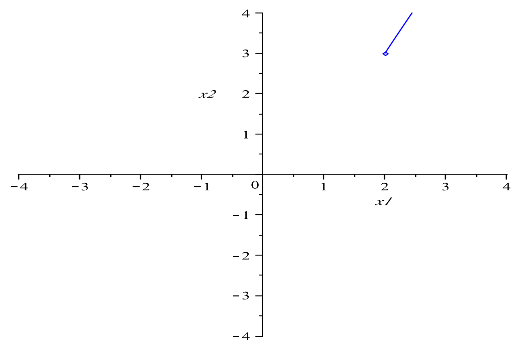

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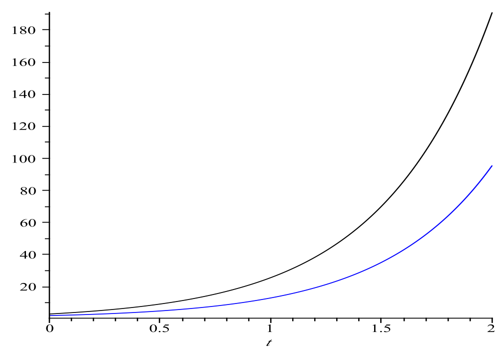


28.(a) We note that $(\mathbf{A}-r_i\mathbf{I})\boldsymbol{\xi}^{(i)}=\mathbf{0}$, for $i=1,2$.

(b) It follows that $(\mathbf{A}-r_2\mathbf{I})\boldsymbol{\xi}^{(1)}=\mathbf{A}\,\boldsymbol{\xi}^{(1)}-r_2\boldsymbol{\xi}^{(1)}=r_1\boldsymbol{\xi}^{(1)}-r_2\boldsymbol{\xi}^{(1)}$.

(c) Suppose that $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ are linearly dependent. Then there exist constants c_1 and c_2, not both zero, such that $c_1\boldsymbol{\xi}^{(1)}+c_2\boldsymbol{\xi}^{(2)}=\mathbf{0}$. Assume that $c_1\neq 0$. It is clear that $(\mathbf{A}-r_2\mathbf{I})(c_1\boldsymbol{\xi}^{(1)}+c_2\,\boldsymbol{\xi}^{(2)})=\mathbf{0}$. On the other hand,

$$(\mathbf{A}-r_2\mathbf{I})(c_1\boldsymbol{\xi}^{(1)}+c_2\,\boldsymbol{\xi}^{(2)})=c_1(r_1-r_2)\boldsymbol{\xi}^{(1)}+\mathbf{0}=c_1(r_1-r_2)\boldsymbol{\xi}^{(1)}.$$

Since $r_1\neq r_2$, we must have $c_1=0$, which leads to a contradiction.

(d) Note that $(\mathbf{A}-r_1\mathbf{I})\boldsymbol{\xi}^{(2)}=(r_2-r_1)\boldsymbol{\xi}^{(2)}$.

(e) Let $n=3$, with $r_1\neq r_2\neq r_3$. Suppose that $\boldsymbol{\xi}^{(1)}$, $\boldsymbol{\xi}^{(2)}$ and $\boldsymbol{\xi}^{(3)}$ are indeed linearly dependent. Then there exist constants c_1, c_2 and c_3, not all zero, such that

$$c_1\boldsymbol{\xi}^{(1)}+c_2\boldsymbol{\xi}^{(2)}+c_3\boldsymbol{\xi}^{(3)}=\mathbf{0}.$$

Assume that $c_1\neq 0$. It is clear that $(\mathbf{A}-r_2\mathbf{I})(c_1\boldsymbol{\xi}^{(1)}+c_2\,\boldsymbol{\xi}^{(2)}+c_3\boldsymbol{\xi}^{(3)})=\mathbf{0}$. On the other hand,

$$(\mathbf{A}-r_2\mathbf{I})(c_1\boldsymbol{\xi}^{(1)}+c_2\,\boldsymbol{\xi}^{(2)}+c_3\boldsymbol{\xi}^{(3)})=c_1(r_1-r_2)\boldsymbol{\xi}^{(1)}+c_3(r_3-r_2)\boldsymbol{\xi}^{(3)}.$$

It follows that $c_1(r_1-r_2)\boldsymbol{\xi}^{(1)}+c_3(r_3-r_2)\boldsymbol{\xi}^{(3)}=\mathbf{0}$. Based on the result of part (a), which is actually not dependent on the value of n, the vectors $\boldsymbol{\xi}^{(1)}$and $\boldsymbol{\xi}^{(3)}$ are linearly independent. Hence we must have $c_1(r_1-r_2)=c_3(r_3-r_2)=0$, which leads to a contradiction.

29.(a) Let $x_1=y$ and $x_2=y'$. It follows that $x_1'=x_2$ and

$$x_2'=y''=-\frac{1}{a}(c\,y+b\,y').$$

In terms of the new variables, we obtain the system of two first order ODEs

$$\begin{aligned}x_1'&=x_2\\ x_2'&=-\frac{1}{a}(c\,x_1+b\,x_2)\,.\end{aligned}$$

(b) The coefficient matrix is given by

$$\mathbf{A}=\begin{pmatrix}0 & 1\\ -\frac{c}{a} & -\frac{b}{a}\end{pmatrix}.$$

Setting $\mathbf{x}=\boldsymbol{\xi}\,e^{rt}$ results in the algebraic equations

$$\begin{pmatrix}-r & 1\\ -\frac{c}{a} & -\frac{b}{a}-r\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

For a nonzero solution, we must have

$$\det(\mathbf{A}-r\mathbf{I})=r^2+\frac{b}{a}r+\frac{c}{a}=0\,.$$

Multiplying both sides of the equation by a, we obtain $a\,r^2+b\,r+c=0$.

30.(a) Solution of the ODE requires analysis of the algebraic equations

$$\begin{pmatrix}-1/10-r & 3/40\\ 1/10 & -1/5-r\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A}-r\mathbf{I})=0$. The characteristic equation is $80\,r^2+24\,r+1=0$, with roots $r_1=-1/4$ and $r_2=-1/20$. With $r=-1/4$, the system of equations reduces to $2\,\xi_1+\xi_2=0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)}=(1\,,-2)^T$. Substitution of $r=-1/20$ results in the equation $2\,\xi_1-3\,\xi_2=0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)}=(3\,,2)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x}=c_1\begin{pmatrix}1\\-2\end{pmatrix}e^{-t/4}+c_2\begin{pmatrix}3\\2\end{pmatrix}e^{-t/20}.$$

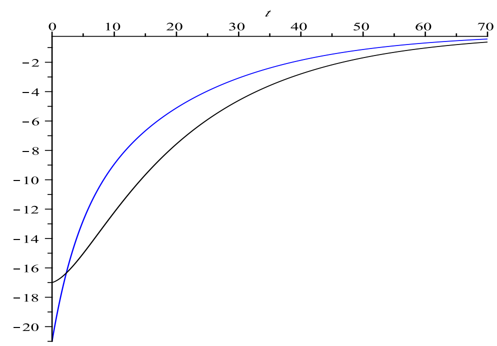
Invoking the initial conditions, we obtain the system of equations

$$\begin{aligned}c_1+3\,c_2&=-17\\ -2\,c_1+2\,c_2&=-21\,.\end{aligned}$$

Hence $c_1 = 29/8$ and $c_2 = -55/8$, and the solution of the IVP is

$$\mathbf{x} = \frac{29}{8}\begin{pmatrix}1\\-2\end{pmatrix}e^{-t/4} - \frac{55}{8}\begin{pmatrix}3\\2\end{pmatrix}e^{-t/20}.$$

(b)



(c) Both functions are monotone increasing. It is easy to show that $-0.5 \le x_1(t) < 0$ and $-0.5 \le x_2(t) < 0$ provided that $t > T \approx 74.39$.

32.(a) The system of differential equations is

$$\frac{d}{dt}\begin{pmatrix}I\\V\end{pmatrix} = \begin{pmatrix}-1/2 & -1/2\\3/2 & -5/2\end{pmatrix}\begin{pmatrix}I\\V\end{pmatrix}.$$

Solution of the system requires analysis of the eigenvalue problem

$$\begin{pmatrix}-1/2-r & -1/2\\3/2 & -5/2-r\end{pmatrix}\begin{pmatrix}\xi_1\\\xi_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

The characteristic equation is $r^2 + 3r + 2 = 0$, with roots $r_1 = -1$ and $r_2 = -2$. With $r = -1$, the equations reduce to $\xi_1 - \xi_2 = 0$. A corresponding eigenvector is given by $\boldsymbol{\xi}^{(1)} = (1,1)^T$. Setting $r = -2$, the system reduces to the equation $3\xi_1 - \xi_2 = 0$. An eigenvector is $\boldsymbol{\xi}^{(2)} = (1,3)^T$. Hence the general solution is

$$\begin{pmatrix}I\\V\end{pmatrix} = c_1\begin{pmatrix}1\\1\end{pmatrix}e^{-t} + c_2\begin{pmatrix}1\\3\end{pmatrix}e^{-2t}.$$

(b) The eigenvalues are distinct and both negative. We find that the equilibrium point $(0,0)$ is a stable node. Hence all solutions converge to $(0,0)$.

33.(a) Solution of the ODE requires analysis of the algebraic equations

$$\begin{pmatrix}-\frac{R_1}{L}-r & -\frac{1}{L}\\\frac{1}{C} & -\frac{1}{CR_2}-r\end{pmatrix}\begin{pmatrix}\xi_1\\\xi_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

The characteristic equation is

$$r^2 + (\frac{L + CR_1R_2}{LCR_2})r + \frac{R_1 + R_2}{LCR_2} = 0.$$

The eigenvectors are real and distinct, provided that the discriminant is positive. That is,

$$(\frac{L + CR_1R_2}{LCR_2})^2 - 4(\frac{R_1 + R_2}{LCR_2}) > 0,$$

which simplifies to the condition

$$(\frac{1}{CR_2} - \frac{R_1}{L})^2 - \frac{4}{LC} > 0\,.$$

(b) The parameters in the ODE are all positive. Observe that the sum of the roots is

$$-\frac{L + CR_1R_2}{LCR_2} < 0\,.$$

Also, the product of the roots is

$$\frac{R_1 + R_2}{LCR_2} > 0\,.$$

It follows that both roots are negative. Hence the equilibrium solution $I = 0$, $V = 0$ represents a stable node, which attracts all solutions.

(c) If the condition in part (a) is not satisfied, that is,

$$(\frac{1}{CR_2} - \frac{R_1}{L})^2 - \frac{4}{LC} \le 0\,,$$

then the real part of the eigenvalues is

$$\mathrm{Re}(r_{1,2}) = -\frac{L + CR_1R_2}{2\,LCR_2}\,.$$

As long as the parameters are all positive, then the solutions will still converge to the equilibrium point $(0\,,0)$.

7.6

2.(a) Setting $\mathbf{x} = \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} -1-r & -4 \\ 1 & -1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we require that $\det(\mathbf{A} - r\mathbf{I}) = r^2 + 2r + 5 = 0$. The roots of the characteristic equation are $r = -1 \pm 2i$. Substituting $r = -1 - 2i$, the two equations reduce to $\xi_1 + 2i\,\xi_2 = 0$. The two eigenvectors are $\boldsymbol{\xi}^{(1)} = (-2i\,,1)^T$ and $\boldsymbol{\xi}^{(2)} = (2i\,,1)^T$. Hence one of the complex-valued solutions is given by

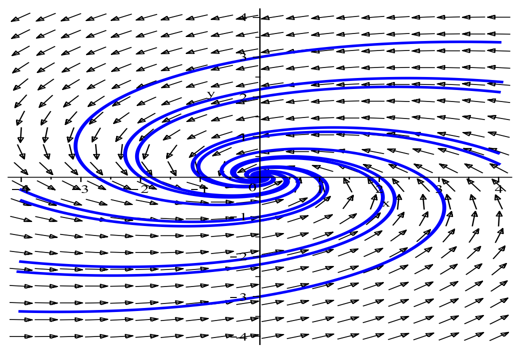
$$\mathbf{x}^{(1)} = \begin{pmatrix} -2i \\ 1 \end{pmatrix} e^{-(1+2i)t} = \begin{pmatrix} -2i \\ 1 \end{pmatrix} e^{-t}(\cos\,2t - i\,\sin\,2t) =$$

$$= e^{-t}\begin{pmatrix} -2\,\sin\,2t \\ \cos\,2t \end{pmatrix} + i\,e^{-t}\begin{pmatrix} -2\,\cos\,2t \\ -\sin\,2t \end{pmatrix}.$$

Based on the real and imaginary parts of this solution, the general solution is

$$\mathbf{x} = c_1\, e^{-t}\begin{pmatrix} -2\,\sin\,2t \\ \cos\,2t \end{pmatrix} + c_2\, e^{-t}\begin{pmatrix} 2\,\cos\,2t \\ \sin\,2t \end{pmatrix}.$$

(b)



3.(a) Solution of the ODEs is based on the analysis of the algebraic equations

$$\begin{pmatrix} 2-r & -5 \\ 1 & -2-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we require that $\det(\mathbf{A} - r\mathbf{I}) = r^2 + 1 = 0$. The roots of the characteristic equation are $r = \pm i$. Setting $r = i$, the equations are equivalent to $\xi_1 - (2+i)\xi_2 = 0$. The eigenvectors are $\boldsymbol{\xi}^{(1)} = (2+i\,, 1)^T$ and $\boldsymbol{\xi}^{(2)} = (2-i\,, 1)^T$. Hence one of the complex-valued solutions is given by

$$\mathbf{x}^{(1)} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{it} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}(\cos\,t + i\,\sin\,t) =$$

$$= \begin{pmatrix} 2\,\cos\,t - \sin\,t \\ \cos\,t \end{pmatrix} + i\begin{pmatrix} \cos\,t + 2\,\sin\,t \\ \sin\,t \end{pmatrix}.$$

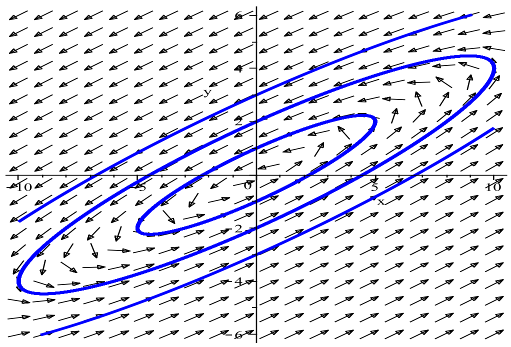
Therefore the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} 2\,\cos\,t - \sin\,t \\ \cos\,t \end{pmatrix} + c_2\begin{pmatrix} \cos\,t + 2\,\sin\,t \\ \sin\,t \end{pmatrix}.$$

The solution may also be written as

$$\mathbf{x} = c_1\begin{pmatrix} 5\,\cos\,t \\ 2\,\cos\,t + \sin\,t \end{pmatrix} + c_2\begin{pmatrix} 5\,\sin\,t \\ -\cos\,t + 2\,\sin\,t \end{pmatrix}.$$

(b)



4.(a) Setting $\mathbf{x}=\boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 2-r & -5/2 \\ 9/5 & -1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we require that $\det(\mathbf{A}-r\mathbf{I}) = r^2 - r + \frac{5}{2} = 0$. The roots of the characteristic equation are $r = (1 \pm 3i)/2$. With $r = (1+3\,i)/2$, the equations reduce to the single equation $(3-3i)\xi_1 - 5\,\xi_2 = 0$. The corresponding eigenvector is given by $\boldsymbol{\xi}^{(1)} = (5\,, 3-3\,i)^T$. Hence one of the complex-valued solutions is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 5 \\ 3-3i \end{pmatrix} e^{(1+3i)t/2} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{t/2} (\cos\, \frac{3}{2}t + i\, \sin\, \frac{3}{2}t) =$$

$$= e^{t/2} \begin{pmatrix} 2\, \cos\, \frac{3}{2}t - \sin\, \frac{3}{2}t \\ \cos\, \frac{3}{2}t \end{pmatrix} + i e^{t/2} \begin{pmatrix} \cos\, \frac{3}{2}t + 2\, \sin\, \frac{3}{2}t \\ \sin\, \frac{3}{2}t \end{pmatrix}.$$

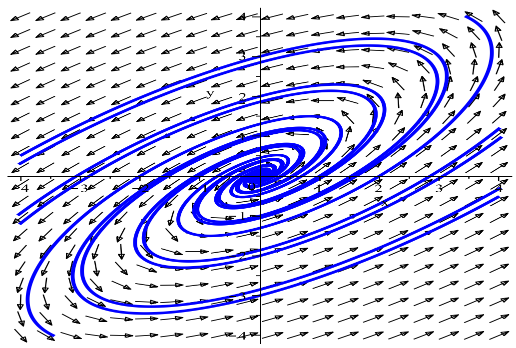
The general solution is

$$\mathbf{x} = c_1\, e^{t/2} \begin{pmatrix} 2\, \cos\, \frac{3}{2}t - \sin\, \frac{3}{2}t \\ \cos\, \frac{3}{2}t \end{pmatrix} + c_2\, e^{t/2} \begin{pmatrix} \cos\, \frac{3}{2}t + 2\, \sin\, \frac{3}{2}t \\ \sin\, \frac{3}{2}t \end{pmatrix}.$$

The solution may also be written as

$$\mathbf{x} = c_1\, e^{t/2} \begin{pmatrix} 5\, \cos\, \frac{3}{2}t \\ 3\, \cos\, \frac{3}{2}t + 3 \sin\, \frac{3}{2}t \end{pmatrix} + c_2\, e^{t/2} \begin{pmatrix} 5\, \sin\, \frac{3}{2}t \\ -3\, \cos\, \frac{3}{2}t + 3 \sin\, \frac{3}{2}t \end{pmatrix}.$$

(b)



5.(a) Setting $\mathbf{x}= \boldsymbol{\xi}\, t^r$ results in the algebraic equations

$$\begin{pmatrix} 1-r & -1 \\ 5 & -3-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 + 2\,r + 2 = 0$, with roots $r = -1 \pm i$. Substituting $r = -1 - i$ reduces the system of equations to $(2+i)\xi_1 - \xi_2 = 0$. The eigenvectors are $\boldsymbol{\xi}^{(1)} = (1\,, 2+i)^T$ and $\boldsymbol{\xi}^{(2)} = (1\,, 2-i)^T$. Hence one of the complex-valued solutions is given by

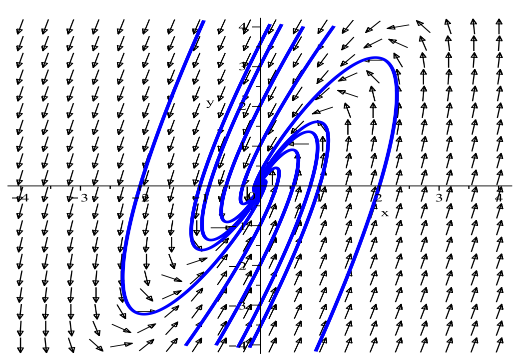
$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 2+i \end{pmatrix} e^{-(1+i)t} = \begin{pmatrix} 1 \\ 2+i \end{pmatrix} e^{-t}(\cos\, t - i\, \sin\, t) =$$

$$= e^{-t}\begin{pmatrix} \cos\, t \\ 2\,\cos\, t + \sin\, t \end{pmatrix} + ie^{-t}\begin{pmatrix} -\sin\, t \\ \cos\, t - 2\sin\, t \end{pmatrix}.$$

The general solution is

$$\mathbf{x} = c_1\, e^{-t}\begin{pmatrix} \cos\, t \\ 2\,\cos\, t + \sin\, t \end{pmatrix} + c_2\, e^{-t}\begin{pmatrix} \sin\, t \\ -\cos\, t + 2\sin\, t \end{pmatrix}.$$

(b)



6.(a) Solution of the ODEs is based on the analysis of the algebraic equations

$$\begin{pmatrix} 1-r & 2 \\ -5 & -1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we require that $\det(\mathbf{A} - r\mathbf{I}) = r^2 + 9 = 0$. The roots of the characteristic equation are $r = \pm 3\,i$. Setting $r = 3\,i$, the two equations reduce to $(1 - 3\,i)\xi_1 + 2\xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (-2\,, 1 - 3i)^T$. Hence one of the complex-valued solutions is given by

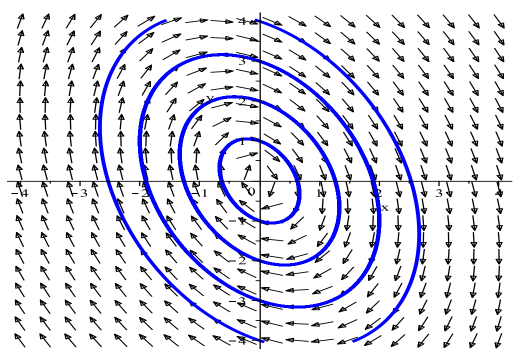
$$\mathbf{x}^{(1)} = \begin{pmatrix} -2 \\ 1-3i \end{pmatrix} e^{3it} = \begin{pmatrix} -2 \\ 1-3i \end{pmatrix}(\cos 3t + i \sin 3t) =$$

$$= \begin{pmatrix} -2\cos 3t \\ \cos 3t + 3\sin 3t \end{pmatrix} + i\begin{pmatrix} -2\sin 3t \\ -3\cos 3t + \sin 3t \end{pmatrix}.$$

The general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} -2\cos 3t \\ \cos 3t + 3\sin 3t \end{pmatrix} + c_2 \begin{pmatrix} 2\sin 3t \\ 3\cos 3t - \sin 3t \end{pmatrix}.$$

(b)



8. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} -3-r & 0 & 2 \\ 1 & -1-r & 0 \\ -2 & -1 & -r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $r^3 + 4r^2 + 7r + 6 = 0$, with roots $r_1 = -2$, $r_2 = -1 - \sqrt{2}\,i$ and $r_3 = -1 + \sqrt{2}\,i$. Setting $r = -2$, the equations reduce to

$$-\xi_1 + 2\,\xi_3 = 0$$
$$\xi_1 + \xi_2 = 0\,.$$

The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (2\,, -2\,, 1)^T$. With $r = -1 - \sqrt{2}\,i$, the system of equations is equivalent to

$$(2 - i\sqrt{2})\xi_1 - 2\,\xi_3 = 0$$
$$\xi_1 + i\sqrt{2}\,\xi_2 = 0\,.$$

An eigenvector is given by $\boldsymbol{\xi}^{(2)} = (-i\sqrt{2}\,, 1\,, -1 - i\sqrt{2}\,)^T$. Hence one of the complex-valued solutions is given by

$$\mathbf{x}^{(2)} = \begin{pmatrix} -i\sqrt{2} \\ 1 \\ -1 - i\sqrt{2} \end{pmatrix} e^{-(1+i\sqrt{2})t} = \begin{pmatrix} -i\sqrt{2} \\ 1 \\ -1 - i\sqrt{2} \end{pmatrix} e^{-t}(\cos\sqrt{2}\,t - i\,\sin\sqrt{2}\,t) =$$

$$= e^{-t}\begin{pmatrix} -\sqrt{2}\,\sin\sqrt{2}\,t \\ \cos\sqrt{2}\,t \\ -\cos\sqrt{2}\,t - \sqrt{2}\,\sin\sqrt{2}\,t \end{pmatrix} + ie^{-t}\begin{pmatrix} -\sqrt{2}\,\cos\sqrt{2}\,t \\ -\sin\sqrt{2}\,t \\ -\sqrt{2}\,\cos\sqrt{2}\,t + \sin\sqrt{2}\,t \end{pmatrix}.$$

The other complex-valued solution is $\mathbf{x}^{(3)} = \overline{\boldsymbol{\xi}^{(2)}}\, e^{r_3 t}$. The general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} e^{-2t} +$$

$$+ c_2\, e^{-t}\begin{pmatrix} \sqrt{2}\,\sin\sqrt{2}\,t \\ -\cos\sqrt{2}\,t \\ \cos\sqrt{2}\,t + \sqrt{2}\,\sin\sqrt{2}\,t \end{pmatrix} + c_3 e^{-t}\begin{pmatrix} \sqrt{2}\,\cos\sqrt{2}\,t \\ \sin\sqrt{2}\,t \\ \sqrt{2}\,\cos\sqrt{2}\,t - \sin\sqrt{2}\,t \end{pmatrix}.$$

It is easy to see that all solutions converge to the equilibrium point $(0\,,0\,,0)$.

10. Solution of the system of ODEs requires that

$$\begin{pmatrix} -3 - r & 2 \\ -1 & -1 - r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 + 4\,r + 5 = 0$, with roots $r = -2 \pm i$. Substituting $r = -2 + i$, the equations are equivalent to $\xi_1 - (1 - i)\xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1 - i\,, 1)^T$. One of the complex-valued solutions is given by

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 - i \\ 1 \end{pmatrix} e^{(-2+i)t} = \begin{pmatrix} 1 - i \\ 1 \end{pmatrix} e^{-2t}(\cos t + i\,\sin t) =$$

$$= e^{-2t}\begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + ie^{-2t}\begin{pmatrix} -\cos t + \sin t \\ \sin t \end{pmatrix}.$$

Hence the general solution is

$$\mathbf{x} = c_1\, e^{-2t}\begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + c_2\, e^{-2t}\begin{pmatrix} -\cos t + \sin t \\ \sin t \end{pmatrix}.$$

Invoking the initial conditions, we obtain the system of equations

$$\begin{aligned} c_1 - c_2 &= 1 \\ c_1 &= -2\,. \end{aligned}$$

Solving for the coefficients, the solution of the initial value problem is

$$\begin{aligned} \mathbf{x} &= -2\, e^{-2t}\begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} - 3\, e^{-2t}\begin{pmatrix} -\cos t + \sin t \\ \sin t \end{pmatrix} \\ &= e^{-2t}\begin{pmatrix} \cos t - 5\,\sin t \\ -2\,\cos t - 3\,\sin t \end{pmatrix}. \end{aligned}$$

The solution converges to $(0,0)$ as $t \to \infty$.

12. Solution of the ODEs is based on the analysis of the algebraic equations

$$\begin{pmatrix} -\frac{4}{5} - r & 2 \\ -1 & \frac{6}{5} - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $25\,r^2 - 10\,r + 26 = 0$, with roots $r = 1/5 \pm i$. Setting $r = 1/5 + i$, the two equations reduce to $\xi_1 - (1-i)\xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1-i\,,1)^T$. One of the complex-valued solutions is given by

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1-i \\ 1 \end{pmatrix} e^{(\frac{1}{5}+i)t} = \begin{pmatrix} 1-i \\ 1 \end{pmatrix} e^{t/5}(\cos\,t + i\,\sin\,t) =$$

$$= e^{t/5}\begin{pmatrix} \cos\,t + \sin\,t \\ \cos\,t \end{pmatrix} + i e^{t/5}\begin{pmatrix} -\cos\,t + \sin\,t \\ \sin\,t \end{pmatrix}.$$

Hence the general solution is

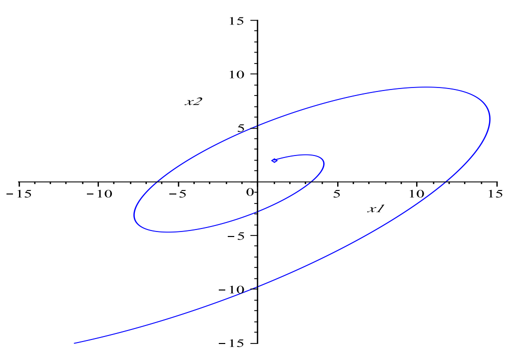
$$\mathbf{x} = c_1 e^{t/5}\begin{pmatrix} \cos\,t + \sin\,t \\ \cos\,t \end{pmatrix} + c_2 e^{t/5}\begin{pmatrix} -\cos\,t + \sin\,t \\ \sin\,t \end{pmatrix}.$$

(b) Let $\mathbf{x}(0) = (x_1^0\,, x_2^0)^T$. The solution of the initial value problem is

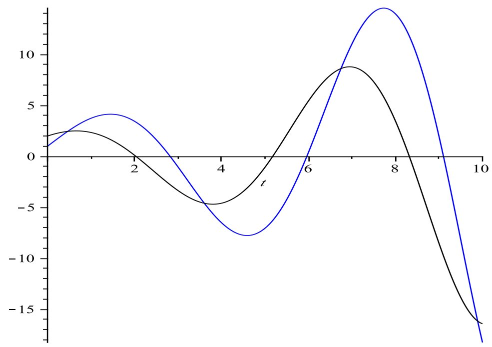
$$\begin{aligned}\mathbf{x} &= x_2^0\, e^{t/5}\begin{pmatrix} \cos\,t + \sin\,t \\ \cos\,t \end{pmatrix} + (x_2^0 - x_1^0)e^{t/5}\begin{pmatrix} -\cos\,t + \sin\,t \\ \sin\,t \end{pmatrix} \\ &= e^{t/5}\begin{pmatrix} x_1^0\,\cos\,t + (2\,x_2^0 - x_1^0)\sin\,t \\ x_2^0\,\cos\,t + (x_2^0 - x_1^0)\sin\,t \end{pmatrix}.\end{aligned}$$

With $\mathbf{x}(0) = (1\,,2)^T$, the solution is

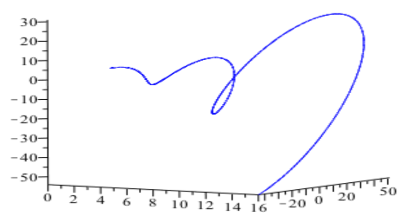
$$\mathbf{x} = e^{t/5}\begin{pmatrix} \cos\,t + 3\,\sin\,t \\ 2\,\cos\,t + \sin\,t \end{pmatrix}.$$



(c)



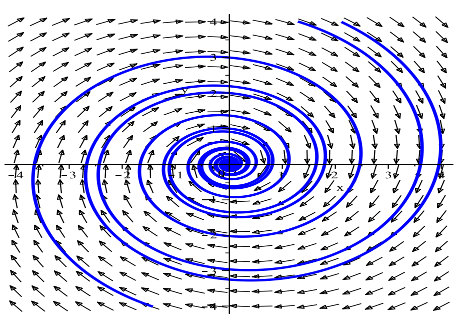
(d)



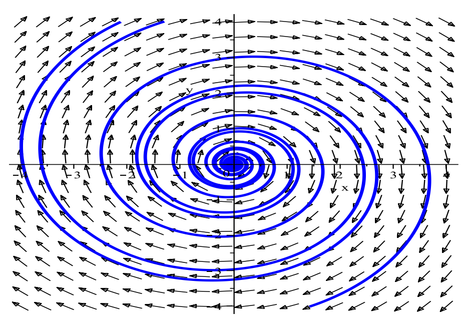
13.(a) The characteristic equation is $r^2 - 2\alpha r + 1 + \alpha^2 = 0$, with roots $r = \alpha \pm i$.

(b) When $\alpha < 0$ and $\alpha > 0$, the equilibrium point $(0,0)$ is a stable spiral and an unstable spiral, respectively. The equilibrium point is a center when $\alpha = 0$.

(c)



(a) $\alpha = -1/8$



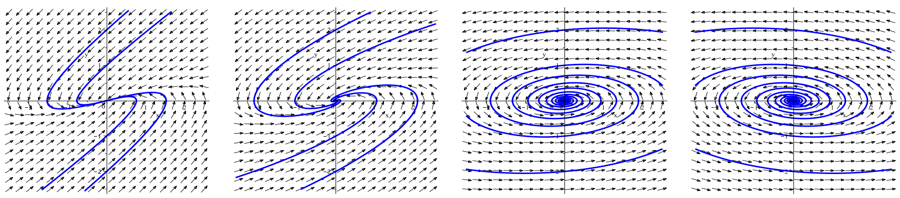
(b) $\alpha = 1/8$

14.(a) The roots of the characteristic equation, $r^2 - \alpha\, r + 5 = 0$, are

$$r_{1,2} = \frac{\alpha}{2} \pm \frac{1}{2}\sqrt{\alpha^2 - 20}\,.$$

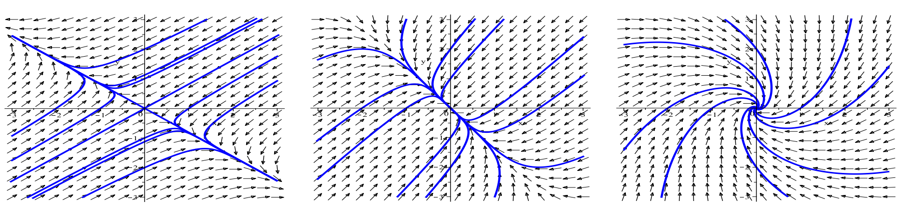
(b) Note that the roots are complex when $-\sqrt{20} < \alpha < \sqrt{20}$. For the case when $\alpha \in (-\sqrt{20}\,, 0)$, the equilibrium point $(0\,, 0)$ is a stable spiral. On the other hand, when $\alpha \in (0\,, \sqrt{20})$, the equilibrium point is an unstable spiral. For the case $\alpha = 0$, the roots are purely imaginary, so the equilibrium point is a center. When $\alpha^2 > 20$, the roots are real and distinct. The equilibrium point becomes a node, with its stability dependent on the sign of α. Finally, the case $\alpha^2 = 20$ marks the transition from spirals to nodes.

(c)



(a) $\alpha = -5$ (b) $\alpha = -3$ (c) $\alpha = -1/2$ (d) $\alpha = 1/2$

17. The characteristic equation of the coefficient matrix is $r^2 + 2r + 1 + \alpha = 0$, with roots given formally as $r_{1,2} = -1 \pm \sqrt{-\alpha}$. The roots are real provided that $\alpha \le 0$. First note that the sum of the roots is -2 and the product of the roots is $1 + \alpha$. For negative values of α, the roots are distinct, with one always negative. When $\alpha < -1$, the roots have opposite signs. Hence the equilibrium point is a saddle. For the case $-1 < \alpha < 0$, the roots are both negative, and the equilibrium point is a stable node. $\alpha = -1$ represents a transition from saddle to node. When $\alpha = 0$, both roots are equal. For the case $\alpha > 0$, the roots are complex conjugates, with negative real part. Hence the equilibrium point is a stable spiral.



(a) $\alpha = -3/2$ (b) $\alpha = -1/2$ (c) $\alpha = 1/2$

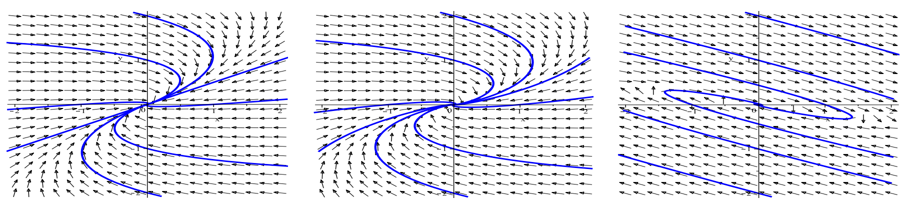
19. The characteristic equation for the system is given by

$$r^2 + (4 - \alpha)r + 10 - 4\alpha = 0.$$

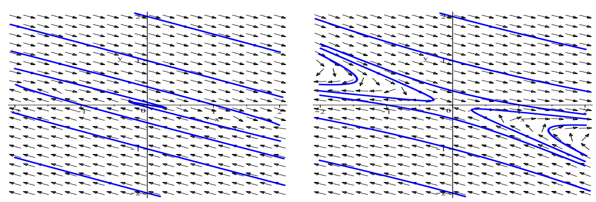
The roots are

$$r_{1,2} = -2 + \frac{\alpha}{2} \pm \sqrt{\alpha^2 + 8\alpha - 24}\,.$$

First note that the roots are complex when $-4 - 2\sqrt{10} < \alpha < -4 + 2\sqrt{10}$. We also find that when $-4 - 2\sqrt{10} < \alpha < -4 + 2\sqrt{10}$, the equilibrium point is a stable spiral. For $\alpha > -4 + 2\sqrt{10}$, the roots are real. When $\alpha > 2.5$, the roots have opposite signs, with the equilibrium point being a saddle. For the case $-4 + 2\sqrt{10} < \alpha < 2.5$, the roots are both negative, and the equilibrium point is a stable node. Finally, when $\alpha < -4 - 2\sqrt{10}$, both roots are negative, with the equilibrium point being a stable node.



(a) $\alpha = -11$ (b) $\alpha = -8$ (c) $\alpha = 2$

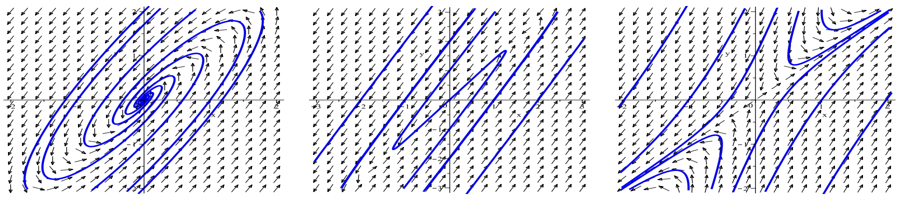


(d) $\alpha = 2.4$ (e) $\alpha = 4$

20. The characteristic equation is $r^2 + 2\,r - (24 + 8\alpha) = 0$, with roots

$$r_{1,2} = -1 \pm \sqrt{25 + 8\alpha}\,.$$

The roots are complex when $\alpha < -25/8$. Since the real part is negative, the origin is a stable spiral. Otherwise the roots are real. When $-25/8 < \alpha < -3$, both roots are negative, and hence the equilibrium point is a stable node. For $\alpha > -3$, the roots are of opposite sign and the origin is a saddle.



(a) $\alpha=-4$ (b) $\alpha=-3.05$ (c) $\alpha=-2$

22. Based on the method in Problem 19 of Section 7.5, setting $\mathbf{x}=\boldsymbol{\xi}\,t^r$ results in the algebraic equations

$$\begin{pmatrix}2-r & -5\\ 1 & -2-r\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

The characteristic equation for the system is $r^2+1=0$, with roots $r_{1,2}=\pm i$. With $r=i$, the equations reduce to the single equation $\xi_1-(2+i)\xi_2=0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(1)}=(2+i\,,1)^T$. One complex-valued solution is

$$\mathbf{x}^{(1)}=\begin{pmatrix}2+i\\1\end{pmatrix}t^i\,.$$

We can write $t^i=e^{i\ln t}$. Hence

$$\mathbf{x}^{(1)}=\begin{pmatrix}2+i\\1\end{pmatrix}e^{i\,\ln t}=\begin{pmatrix}2+i\\1\end{pmatrix}[\cos(\ln t)+i\,\sin(\ln t)]=$$

$$=\begin{pmatrix}2\,\cos(\ln t)-\sin(\ln t)\\ \cos(\ln t)\end{pmatrix}+i\begin{pmatrix}\cos(\ln t)+2\,\sin(\ln t)\\ \sin(\ln t)\end{pmatrix}.$$

Therefore the general solution is

$$\mathbf{x}=c_1\begin{pmatrix}2\,\cos(\ln t)-\sin(\ln t)\\ \cos(\ln t)\end{pmatrix}+c_2\begin{pmatrix}\cos(\ln t)+2\,\sin(\ln t)\\ \sin(\ln t)\end{pmatrix}.$$

Other combinations are also possible.

24.(a) The characteristic equation of the system is

$$r^3+\frac{2}{5}r^2+\frac{81}{80}r-\frac{17}{160}=0\,,$$

with eigenvalues $r_1=1/10$, and $r_{2,3}=-1/4\pm i$. For $r=1/10$, simple calculations reveal that a corresponding eigenvector is $\boldsymbol{\xi}^{(1)}=(0,0,1)^T$. Setting $r=-1/4-i$, we obtain the system of equations

$$\begin{aligned}\xi_1-i\,\xi_2&=0\\ \xi_3&=0\,.\end{aligned}$$

A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (i\,,1\,,0)^T$. Hence one solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{t/10}.$$

Another solution, which is complex-valued, is given by

$$\mathbf{x}^{(2)} = \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} e^{-(\frac{1}{4}+i)t} = \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} e^{-t/4}(\cos t - i \sin t) =$$

$$= e^{-t/4} \begin{pmatrix} \sin t \\ \cos t \\ 0 \end{pmatrix} + ie^{-t/4} \begin{pmatrix} \cos t \\ -\sin t \\ 0 \end{pmatrix}.$$

Using the real and imaginary parts of $\mathbf{x}^{(2)}$, the general solution is constructed as

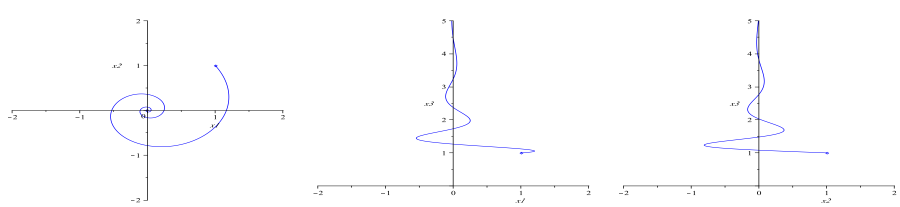
$$\mathbf{x} = c_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{t/10} + c_2\, e^{-t/4} \begin{pmatrix} \sin t \\ \cos t \\ 0 \end{pmatrix} + c_3\, e^{-t/4} \begin{pmatrix} \cos t \\ -\sin t \\ 0 \end{pmatrix}.$$

(b) Let $\mathbf{x}(0) = (x_1^0\,, x_2^0\,, x_3^0)$. The solution can be written as

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ x_3^0\ e^{t/10} \end{pmatrix} + e^{-t/4} \begin{pmatrix} x_2^0 \sin t + x_1^0 \cos t \\ x_2^0 \cos t - x_1^0 \sin t \\ 0 \end{pmatrix}.$$

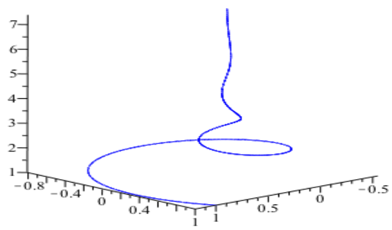
With $\mathbf{x}(0) = (1\,,1\,,1)$, the solution of the initial value problem is

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ e^{t/10} \end{pmatrix} + e^{-t/4} \begin{pmatrix} \sin t + \cos t \\ \cos t - \sin t \\ 0 \end{pmatrix}.$$



(a) $x_1 - x_2$ (b) $x_1 - x_3$ (c) $x_2 - x_3$

(c)



25.(a) Based on Problems 19-21 of Section 7.1, the system of differential equations is

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} -\frac{R_1}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{CR_2} \end{pmatrix}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{8} \\ 2 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} I \\ V \end{pmatrix},$$

since $R_1 = R_2 = 4$ ohms, $C = 1/2$ farads and $L = 8$ henrys.

(b) The eigenvalue problem is

$$\begin{pmatrix} -\frac{1}{2} - r & -\frac{1}{8} \\ 2 & -\frac{1}{2} - r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the system is $r^2 + r + \frac{1}{2} = 0$, with eigenvalues

$$r_{1,2} = -\frac{1}{2} \pm \frac{1}{2}i\,.$$

Setting $r = -1/2 + i/2$, the algebraic equations reduce to $4i\xi_1 + \xi_2 = 0$. It follows that $\boldsymbol{\xi}^{(1)} = (1\,, -4i)^T$. Hence one complex-valued solution is

$$\begin{pmatrix} I \\ V \end{pmatrix}^{(1)} = \begin{pmatrix} 1 \\ -4i \end{pmatrix} e^{(-1+i)t/2} = \begin{pmatrix} 1 \\ -4i \end{pmatrix} e^{-t/2}\left[\cos(t/2) + i\,\sin(t/2)\right] =$$

$$= e^{-t/2}\begin{pmatrix} \cos(t/2) \\ 4\,\sin(t/2) \end{pmatrix} + ie^{-t/2}\begin{pmatrix} \sin(t/2) \\ -4\,\cos(t/2) \end{pmatrix}.$$

Therefore the general solution is

$$\begin{pmatrix} I \\ V \end{pmatrix} = c_1 e^{-t/2}\begin{pmatrix} \cos(t/2) \\ 4\,\sin(t/2) \end{pmatrix} + c_2 e^{-t/2}\begin{pmatrix} \sin(t/2) \\ -4\,\cos(t/2) \end{pmatrix}.$$

(c) Imposing the initial conditions, we arrive at the equations $c_1 = 2$ and $c_2 = -3/4$, and

$$\begin{pmatrix} I \\ V \end{pmatrix} = e^{-t/2}\begin{pmatrix} 2\,\cos(t/2) - \frac{3}{4}\,\sin(t/2) \\ 8\,\sin(t/2) + 3\,\cos(t/2) \end{pmatrix}.$$

(d) Since the eigenvalues have negative real parts, all solutions converge to the origin.

26.(a) The characteristic equation of the system is

$$r^2 + \frac{1}{RC}r + \frac{1}{CL} = 0\,,$$

with eigenvalues

$$r_{1,2} = -\frac{1}{2RC} \pm \frac{1}{2RC}\sqrt{1 - \frac{4R^2C}{L}}\,.$$

The eigenvalues are real and different provided that

$$1 - \frac{4R^2C}{L} > 0\,.$$

The eigenvalues are complex conjugates as long as

$$1 - \frac{4R^2C}{L} < 0\,.$$

(b) With the specified values, the eigenvalues are $r_{1,2} = -1 \pm i$. The eigenvector corresponding to $r = -1 + i$ is $\boldsymbol{\xi}^{(1)} = (1\,, -4i)^T$. Hence one complex-valued solution is

$$\begin{pmatrix} I \\ V \end{pmatrix}^{(1)} = \begin{pmatrix} 1 \\ -1+i \end{pmatrix} e^{(-1+i)t} = \begin{pmatrix} 1 \\ -1+i \end{pmatrix} e^{-t}(\cos\, t + i\,\sin\, t) =$$

$$= e^{-t}\begin{pmatrix} \cos\, t \\ -\cos\, t - \sin\, t \end{pmatrix} + ie^{-t}\begin{pmatrix} \sin\, t \\ \cos\, t - \sin\, t \end{pmatrix}.$$

Therefore the general solution is

$$\begin{pmatrix} I \\ V \end{pmatrix} = c_1 e^{-t}\begin{pmatrix} \cos\, t \\ -\cos\, t - \sin\, t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} \sin\, t \\ \cos\, t - \sin\, t \end{pmatrix}.$$

(c) Imposing the initial conditions, we arrive at the equations

$$\begin{aligned} c_1 &= 2 \\ -c_1 + c_2 &= 1\,, \end{aligned}$$

with $c_1 = 2$ and $c_2 = 3$. Therefore the solution of the IVP is

$$\begin{pmatrix} I \\ V \end{pmatrix} = e^{-t}\begin{pmatrix} 2\,\cos\, t + 3\,\sin\, t \\ \cos\, t - 5\sin\, t \end{pmatrix}.$$

(d) Since $\mathrm{Re}(r_{1,2}) = -1$, all solutions converge to the origin.

27.(a) Suppose that $c_1\mathbf{a}+c_2\mathbf{b}=\mathbf{0}$. Since $\mathbf{a}$ and $\mathbf{b}$ are the real and imaginary parts of the vector $\boldsymbol{\xi}^{(1)}$, respectively, $\mathbf{a}= (\boldsymbol{\xi}^{(1)} + \overline{\boldsymbol{\xi}^{(1)}})/2$ and $\mathbf{b}= (\boldsymbol{\xi}^{(1)} - \overline{\boldsymbol{\xi}^{(1)}})/2i$. Hence

$$c_1(\boldsymbol{\xi}^{(1)} + \overline{\boldsymbol{\xi}^{(1)}}) - ic_2(\boldsymbol{\xi}^{(1)} - \overline{\boldsymbol{\xi}^{(1)}}) = \mathbf{0}\,,$$

which leads to

$$(c_1 - ic_2)\boldsymbol{\xi}^{(1)} + (c_1 + ic_2)\overline{\boldsymbol{\xi}^{(1)}} = \mathbf{0}\,.$$

(b) Now since $\boldsymbol{\xi}^{(1)}$ and $\overline{\boldsymbol{\xi}^{(1)}}$ are linearly independent, we must have

$$c_1 - ic_2 = 0$$
$$c_1 + ic_2 = 0\,.$$

It follows that $c_1 = c_2 = 0$.

(c) Recall that

$$\mathbf{u}(t) = e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t)$$
$$\mathbf{v}(t) = e^{\lambda t}(\mathbf{a}\cos\mu t + \mathbf{b}\sin\mu t)\,.$$

Consider the equation $c_1\mathbf{u}(t_0) + c_2\mathbf{v}(t_0) = \mathbf{0}$, for some t_0. We can then write

$$c_1 e^{\lambda t_0}(\mathbf{a}\cos\mu t_0 - \mathbf{b}\sin\mu t_0) + c_2 e^{\lambda t_0}(\mathbf{a}\cos\mu t_0 + \mathbf{b}\sin\mu t_0) = \mathbf{0}\,.\ (*)$$

Rearranging the terms, and dividing by the exponential,

$$(c_1 + c_2)\cos\mu t_0\,\mathbf{a} + (c_2 - c_1)\sin\mu t_0\,\mathbf{b} = \mathbf{0}\,.$$

From part (b), since **a** and **b** are linearly independent, it follows that

$$(c_1 + c_2)\cos\mu t_0 = (c_2 - c_1)\sin\mu t_0 = 0\,.$$

Without loss of generality, assume that the trigonometric factors are nonzero. Otherwise proceed again from Equation $(*)$, above. We then conclude that

$$c_1 + c_2 = 0 \text{ and } c_2 - c_1 = 0\,,$$

which leads to $c_1 = c_2 = 0$. Thus $\mathbf{u}(t_0)$ and $\mathbf{v}(t_0)$ are linearly independent for some t_0, and hence the functions are linearly independent at every point.

28.(a) Let $x_1 = u$ and $x_2 = u'$. It follows that $x_1' = x_2$ and

$$x_2' = u'' = -\frac{k}{m}\,u\,.$$

In terms of the new variables, we obtain the system of two first order ODEs

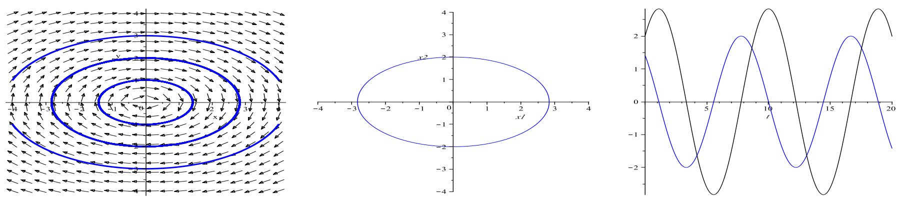
$$x_1' = x_2$$
$$x_2' = -\frac{k}{m}\,x_1\,.$$

(b) The associated eigenvalue problem is

$$\begin{pmatrix} -r & 1 \\ -k/m & -r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 + k/m = 0$, with roots $r_{1,2} = \pm i\sqrt{k/m}$.

(c) Since the eigenvalues are purely imaginary, the origin is a center. Hence the phase curves are ellipses, with a clockwise flow. For computational purposes, let $k = 1$ and $m = 2$.



(a) $k = 1,\, m = 2$ (b) $x_1 - x_2$ (c) x_1, x_2 vs t

(d) The general solution of the second order equation is

$$u(t) = c_1 \cos\sqrt{\frac{k}{m}}\,t + c_2 \sin\sqrt{\frac{k}{m}}\,t\,.$$

The general solution of the system of ODEs is given by

$$\mathbf{x} = c_1\begin{pmatrix}\sqrt{\frac{m}{k}}\,\sin\sqrt{\frac{k}{m}}\,t\\ \cos\sqrt{\frac{k}{m}}\,t\end{pmatrix} + c_2\begin{pmatrix}\sqrt{\frac{m}{k}}\,\cos\sqrt{\frac{k}{m}}\,t\\ -\sin\sqrt{\frac{k}{m}}\,t\end{pmatrix}.$$

It is evident that the natural frequency of the system is equal to $|r_1| = |r_2|$.

29.(a) Set $\mathbf{x} = (x_1\,, x_2\,)^T$. We can rewrite Equation (22) in the form

$$\begin{pmatrix}2 & 0\\ 0 & 9/4\end{pmatrix}\begin{pmatrix}\frac{d^2x_1}{dt^2}\\ \frac{d^2x_2}{dt^2}\end{pmatrix} = \begin{pmatrix}-4 & 3\\ 3 & -\frac{27}{4}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}.$$

Multiplying both sides of this equation by the inverse of the diagonal matrix, we obtain

$$\begin{pmatrix}\frac{d^2x_1}{dt^2}\\ \frac{d^2x_2}{dt^2}\end{pmatrix} = \begin{pmatrix}-2 & 3/2\\ 4/3 & -3\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}.$$

(b) Substituting $\mathbf{x} = \boldsymbol{\xi}\; e^{rt}$,

$$r^2\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}e^{rt} = \begin{pmatrix}-2 & 3/2\\ 4/3 & -3\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}e^{rt},$$

which can be written as

$$(\mathbf{A} - r^2\mathbf{I})\boldsymbol{\xi} = \mathbf{0}\;.$$

(c) The eigenvalues are $r_1^2 = -1$ and $r_2^2 = -4$, with corresponding eigenvectors

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix}3\\ 2\end{pmatrix} \text{ and } \boldsymbol{\xi}^{(2)} = \begin{pmatrix}3\\ -4\end{pmatrix}.$$

(d) The linearly independent solutions are

$$\mathbf{x}^{(1)} = \tilde{C}_1\begin{pmatrix}3\\ 2\end{pmatrix}e^{it} \text{ and } \mathbf{x}^{(2)} = \tilde{C}_2\begin{pmatrix}3\\ -4\end{pmatrix}e^{2it}\,.$$

in which $\tilde{C}_1$ and $\tilde{C}_2$ are arbitrary complex coefficients. In scalar form,

$$x_1 = 3c_1 \cos t + 3c_2 \sin t + 3c_3 \cos 2t + 3c_4 \sin 2t$$
$$x_2 = 2c_1 \cos t + 2c_2 \sin t - 4c_3 \cos 2t - 4c_4 \sin 2t$$

(e) Differentiating the above expressions,

$$x_1' = -3c_1 \sin t + 3c_2 \cos t - 6c_3 \sin 2t + 6c_4 \cos 2t$$
$$x_2' = -2c_1 \sin t + 2c_2 \cos t + 8c_3 \sin 2t - 8c_4 \cos 2t$$

It is evident that $\mathbf{y} = (x_1, x_2, x_1', x_2')^T$ as in Equation (31).

31.(a) The second order system is given by

$$\frac{d^2x_1}{dt^2} = -2x_1 + x_2$$
$$\frac{d^2x_2}{dt^2} = x_1 - 2x_2$$

Let $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_1'$ and $y_4 = x_2'$. In terms of the new variables, we have

$$y_1' = y_3$$
$$y_2' = y_4$$
$$y_3' = -2y_1 + y_2$$
$$y_4' = y_1 - 2y_2$$

hence the coefficient matrix is

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \end{pmatrix}.$$

(b) The eigenvalues and corresponding eigenvectors of $\mathbf{A}$ are:

$$r_1 = i\,, \quad \boldsymbol{\xi}^{(1)} = (1, 1, i, i)^T$$
$$r_2 = -i\,, \quad \boldsymbol{\xi}^{(2)} = (1, 1, -i, -i)^T$$
$$r_3 = \sqrt{3}\, i\,, \quad \boldsymbol{\xi}^{(3)} = (1, -1, \sqrt{3}\, i, -\sqrt{3}\, i)^T$$
$$r_4 = -\sqrt{3}\, i\,, \quad \boldsymbol{\xi}^{(4)} = (1, -1, -\sqrt{3}\, i, \sqrt{3}\, i)^T$$

(c) Note that

$$\boldsymbol{\xi}^{(1)} e^{it} = \begin{pmatrix} 1 \\ 1 \\ i \\ i \end{pmatrix} (\cos t + i \sin t)$$

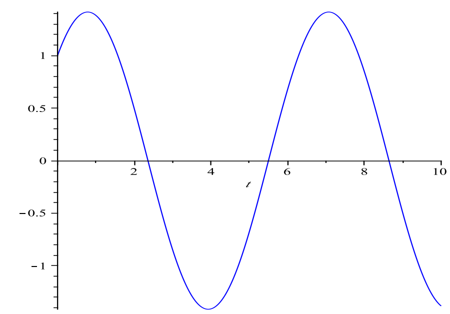
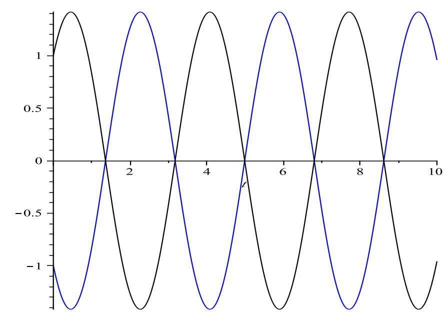
and

$$\boldsymbol{\xi}^{(3)} e^{\sqrt{3}\, it} = \begin{pmatrix} 1 \\ -1 \\ \sqrt{3}\, i \\ -\sqrt{3}\, i \end{pmatrix} (\cos \sqrt{3} t + i \sin \sqrt{3} t).$$

Hence the general solution is

$$\mathbf{y} = c_1 \begin{pmatrix} \cos t \\ \cos t \\ -\sin t \\ -\sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \sin t \\ \cos t \\ \cos t \end{pmatrix} + c_3 \begin{pmatrix} \cos \sqrt{3}t \\ -\cos \sqrt{3}t \\ -\sqrt{3}\sin \sqrt{3}t \\ \sqrt{3}\sin \sqrt{3}t \end{pmatrix} + c_4 \begin{pmatrix} \sin \sqrt{3}t \\ -\sin \sqrt{3}t \\ \sqrt{3}\cos \sqrt{3}t \\ -\sqrt{3}\cos \sqrt{3}t \end{pmatrix}.$$

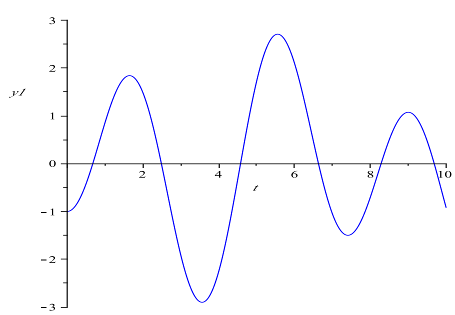
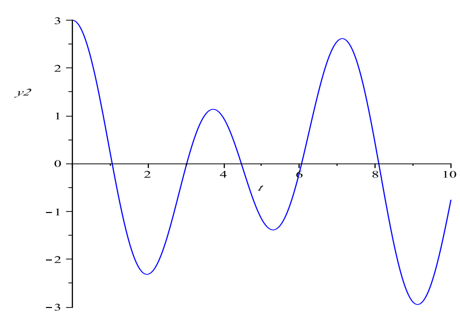
(d) The two modes have natural frequencies of $\omega_1 = 1$ rad/sec and $\omega_2 = \sqrt{3}$ rad/sec.

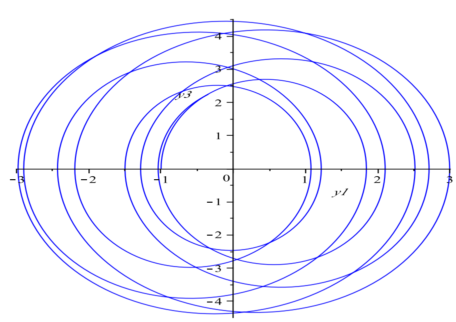
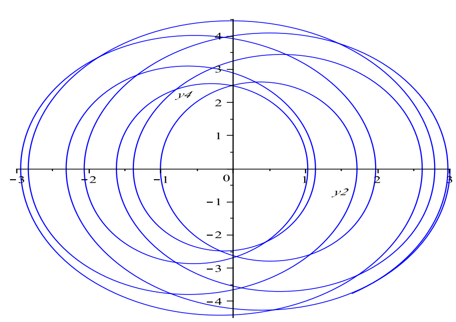
(e) For the initial condition $\mathbf{y}(0) = (-1, 3, 0, 0)^T$, it is necessary that

$$\begin{pmatrix} -1 \\ 3 \\ 0 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + c_4 \begin{pmatrix} 0 \\ 0 \\ \sqrt{3} \\ -\sqrt{3} \end{pmatrix},$$

resulting in the coefficients $c_1 = 1$, $c_2 = 0$, $c_3 = -2$ and $c_4 = 0$.

The solutions are not periodic, since the two natural frequencies are incommensurate.

7.7

1.(a) The eigenvalues and eigenvectors were found in Problem 1, Section 7.5.

$$r_1 = -1, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}; \quad r_2 = 2, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} e^{-t} \\ 2\,e^{-t} \end{pmatrix} + c_2 \begin{pmatrix} 2\,e^{2t} \\ e^{2t} \end{pmatrix}.$$

Hence a fundamental matrix is given by

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} e^{-t} & 2\,e^{2t} \\ 2\,e^{-t} & e^{2t} \end{pmatrix}.$$

(b) We now have

$$\boldsymbol{\Psi}(0) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \text{ and } \boldsymbol{\Psi}^{-1}(0) = \frac{1}{3}\begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix},$$

So that

$$\boldsymbol{\Phi}(t) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(0) = \frac{1}{3}\begin{pmatrix} -e^{-t} + 4e^{2t} & 2e^{-t} - 2e^{2t} \\ -2e^{-t} + 2e^{2t} & 4e^{-t} - e^{2t} \end{pmatrix}.$$

3.(a) The eigenvalues and eigenvectors were found in Problem 3, Section 7.5. The general solution of the system is

$$\mathbf{x} = c_1 \begin{pmatrix} e^{t} \\ e^{t} \end{pmatrix} + c_2 \begin{pmatrix} e^{-t} \\ 3e^{-t} \end{pmatrix}.$$

Hence a fundamental matrix is given by

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} e^{t} & e^{-t} \\ e^{t} & 3e^{-t} \end{pmatrix}.$$

(b) Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(1)}$, we solve the equations

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_1 + 3c_2 &= 0\,, \end{aligned}$$

to obtain $c_1 = 3/2\,,\; c_2 = -1/2\,.$ The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} \frac{3}{2}e^{t} - \frac{1}{2}e^{-t} \\ \frac{3}{2}e^{t} - \frac{3}{2}e^{-t} \end{pmatrix}.$$

Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(2)}$, we solve the equations

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 + 3c_2 &= 1\,, \end{aligned}$$

to obtain $c_1 = -1/2\,,\, c_2 = 1/2\,.$ The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} -\frac{1}{2}e^t + \frac{1}{2}e^{-t} \\ -\frac{1}{2}e^t + \frac{3}{2}e^{-t} \end{pmatrix}.$$

Therefore the fundamental matrix is

$$\mathbf{\Phi}(t) = \frac{1}{2}\begin{pmatrix} 3e^t - e^{-t} & -e^t + e^{-t} \\ 3e^t - 3e^{-t} & -e^t + 3e^{-t} \end{pmatrix}.$$

5.(a) The general solution, found in Problem 3, Section 7.6, is given by

$$\mathbf{x} = c_1\begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2\begin{pmatrix} 5\sin t \\ -\cos t + 2\sin t \end{pmatrix}.$$

Hence a fundamental matrix is given by

$$\mathbf{\Psi}(t) = \begin{pmatrix} 5\cos t & 5\sin t \\ 2\cos t + \sin t & -\cos t + 2\sin t \end{pmatrix}.$$

(b) Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(1)}$, we solve the equations

$$5c_1 = 1$$
$$2c_1 - c_2 = 0\,,$$

resulting in $c_1 = 1/5\,,\, c_2 = 2/5\,.$ The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}.$$

Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(2)}$, we solve the equations

$$5c_1 = 0$$
$$2c_1 - c_2 = 1\,,$$

resulting in $c_1 = 0\,,\, c_2 = -1\,.$ The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} -5\sin t \\ \cos t - 2\sin t \end{pmatrix}.$$

Therefore the fundamental matrix is

$$\mathbf{\Phi}(t) = \begin{pmatrix} \cos t + 2\sin t & -5\sin t \\ \sin t & \cos t - 2\sin t \end{pmatrix}.$$

7.(a) The general solution, found in Problem 15, Section 7.5, is given by

$$\mathbf{x} = c_1\begin{pmatrix} e^{2t} \\ 3e^{2t} \end{pmatrix} + c_2\begin{pmatrix} e^{4t} \\ e^{4t} \end{pmatrix}.$$

Hence a fundamental matrix is given by

$$\mathbf{\Psi}(t) = \begin{pmatrix} e^{2t} & e^{4t} \\ 3e^{2t} & e^{4t} \end{pmatrix}.$$

(b) Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(1)}$, we solve the equations

$$c_1 + c_2 = 1$$
$$3c_1 + c_2 = 0\,,$$

resulting in $c_1 = -1/2\,,\ c_2 = 3/2\,.$ The corresponding solution is

$$\mathbf{x} = \frac{1}{2}\begin{pmatrix} -e^{2t} + 3e^{4t} \\ -3e^{2t} + 3e^{4t} \end{pmatrix}.$$

The initial conditions $\mathbf{x}(0) = \mathbf{e}^{(2)}$ require that

$$c_1 + c_2 = 0$$
$$3c_1 + c_2 = 1\,,$$

resulting in $c_1 = 1/2\,,\ c_2 = -1/2\,.$ The corresponding solution is

$$\mathbf{x} = \frac{1}{2}\begin{pmatrix} e^{2t} - e^{4t} \\ 3e^{2t} - e^{4t} \end{pmatrix}.$$

Therefore the fundamental matrix is

$$\mathbf{\Phi}(t) = \frac{1}{2}\begin{pmatrix} -e^{2t} + 3e^{4t} & e^{2t} - e^{4t} \\ -3e^{2t} + 3e^{4t} & 3e^{2t} - e^{4t} \end{pmatrix}.$$

8.(a) The general solution, found in Problem 5, Section 7.6, is given by

$$\mathbf{x} = c_1 e^{-t}\begin{pmatrix} \cos\, t \\ 2\,\cos\, t + \sin\, t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} \sin\, t \\ -\cos\, t + 2\,\sin\, t \end{pmatrix}.$$

Hence a fundamental matrix is given by

$$\mathbf{\Psi}(t) = \begin{pmatrix} e^{-t}\cos\, t & e^{-t}\sin t \\ 2e^{-t}\cos\, t + e^{-t}\sin\, t & -e^{-t}\cos\, t + 2e^{-t}\,\sin\, t \end{pmatrix}.$$

(b) The specific solution corresponding to the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(1)}$ is

$$\mathbf{x} = e^{-t}\begin{pmatrix} \cos\, t + 2\,\sin\, t \\ 5\sin\, t \end{pmatrix}.$$

For the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(2)}$, the solution is

$$\mathbf{x} = e^{-t}\begin{pmatrix} -\sin\, t \\ \cos\, t - 2\sin\, t \end{pmatrix}.$$

Therefore the fundamental matrix is

$$\mathbf{\Phi}(t) = e^{-t}\begin{pmatrix} \cos\, t + 2\,\sin\, t & -\sin\, t \\ 5\sin\, t & \cos\, t - 2\sin\, t \end{pmatrix}.$$

9.(a) The general solution, found in Problem 13, Section 7.5, is given by

$$\mathbf{x} = c_1\begin{pmatrix} 4e^{-2t} \\ -5e^{-2t} \\ -7e^{-2t} \end{pmatrix} + c_2\begin{pmatrix} 3e^{-t} \\ -4e^{-t} \\ -2e^{-t} \end{pmatrix} + c_3\begin{pmatrix} 0 \\ e^{2t} \\ -e^{2t} \end{pmatrix}.$$

Hence a fundamental matrix is given by

$$\mathbf{\Psi}(t) = \begin{pmatrix} 4e^{-2t} & 3e^{-t} & 0 \\ -5e^{-2t} & -4e^{-t} & e^{2t} \\ -7e^{-2t} & -2e^{-t} & -e^{2t} \end{pmatrix}.$$

(b) Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(1)}$, we solve the equations

$$\begin{aligned} 4c_1 + 3c_2 &= 1 \\ -5c_1 - 4c_2 + c_3 &= 0 \\ -7c_1 - 2c_2 - c_3 &= 0\,, \end{aligned}$$

resulting in $c_1 = -1/2\,,\ c_2 = 1\,,\ c_3 = 3/2\,.$ The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} -2e^{-2t} + 3e^{-t} \\ 5e^{-2t}/2 - 4e^{-t} + 3e^{2t}/2 \\ 7e^{-2t}/2 - 2e^{-t} - 3e^{2t}/2 \end{pmatrix}.$$

The initial conditions $\mathbf{x}(0) = \mathbf{e}^{(2)}$, we solve the equations

$$\begin{aligned} 4c_1 + 3c_2 &= 0 \\ -5c_1 - 4c_2 + c_3 &= 1 \\ -7c_1 - 2c_2 - c_3 &= 0\,, \end{aligned}$$

resulting in $c_1 = -1/4\,,\ c_2 = 1/3\,,\ c_3 = 13/12\,.$ The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} -e^{-2t} + e^{-t} \\ 5e^{-2t}/4 - 4e^{-t}/3 + 13e^{2t}/12 \\ 7e^{-2t}/4 - 2e^{-t}/3 - 13e^{2t}/12 \end{pmatrix}.$$

The initial conditions $\mathbf{x}(0) = \mathbf{e}^{(3)}$, we solve the equations

$$\begin{aligned} 4c_1 + 3c_2 &= 0 \\ -5c_1 - 4c_2 + c_3 &= 0 \\ -7c_1 - 2c_2 - c_3 &= 1\,, \end{aligned}$$

resulting in $c_1 = -1/4\,,\ c_2 = 1/3\,,\ c_3 = 1/12\,.$ The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} -e^{-2t} + e^{-t} \\ 5e^{-2t}/4 - 4e^{-t}/3 + e^{2t}/12 \\ 7e^{-2t}/4 - 2e^{-t}/3 - e^{2t}/12 \end{pmatrix}.$$

Therefore the fundamental matrix is

$$\mathbf{\Phi}(t) = \frac{1}{12}\begin{pmatrix} -24e^{-2t} + 36e^{-t} & -12e^{-2t} + 12e^{-t} & -12e^{-2t} + 12e^{-t} \\ 30e^{-2t} - 48e^{-t} + 18e^{2t} & 15e^{-2t} - 16e^{-t} + 13e^{2t} & 15e^{-2t} - 16e^{-t} + e^{2t} \\ 42e^{-2t} - 24e^{-t} - 18e^{2t} & 21e^{-2t} - 8e^{-t} - 13e^{2t} & 21e^{-2t} - 8e^{-t} - e^{2t} \end{pmatrix}.$$

12. The solution of the initial value problem is given by

$$\mathbf{x} = \mathbf{\Phi}(t)\mathbf{x}(0) = \begin{pmatrix} e^{-t}\cos 2t & -2e^{-t}\sin 2t \\ \frac{1}{2}e^{-t}\sin 2t & e^{-t}\cos 2t \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} =$$

$$= e^{-t}\begin{pmatrix} 3\cos 2t - 2\sin 2t \\ \frac{3}{2}\sin 2t + \cos 2t \end{pmatrix}.$$

13. Let

$$\mathbf{\Psi}(t) = \begin{pmatrix} x_1^{(1)}(t) & \cdots & x_1^{(n)}(t) \\ \vdots & & \vdots \\ x_n^{(1)}(t) & \cdots & x_n^{(n)}(t) \end{pmatrix}.$$

It follows that

$$\mathbf{\Psi}(t_0) = \begin{pmatrix} x_1^{(1)}(t_0) & \cdots & x_1^{(n)}(t_0) \\ \vdots & & \vdots \\ x_n^{(1)}(t_0) & \cdots & x_n^{(n)}(t_0) \end{pmatrix}$$

is a scalar matrix, which is invertible, since the solutions are linearly independent. Let $\mathbf{\Psi}^{-1}(t_0) = (c_{ij})$. Then

$$\mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0) = \begin{pmatrix} x_1^{(1)}(t) & \cdots & x_1^{(n)}(t) \\ \vdots & & \vdots \\ x_n^{(1)}(t) & \cdots & x_n^{(n)}(t) \end{pmatrix} \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix}.$$

The j-th column of the product matrix is

$$\left[\mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0)\right]^{(j)} = \sum_{k=1}^{n} c_{kj}\,\mathbf{x}^{(k)},$$

which is a solution vector, since it is a linear combination of solutions. Furthermore, the columns are all linearly independent, since the vectors $\mathbf{x}^{(k)}$ are. Hence the product is a fundamental matrix. Finally, setting $t = t_0$, $\mathbf{\Psi}(t_0)\mathbf{\Psi}^{-1}(t_0) = \mathbf{I}$. This is precisely the definition of $\mathbf{\Phi}(t)$.

14. The fundamental matrix $\mathbf{\Phi}(t)$ for the system

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\mathbf{x}$$

is given by

$$\mathbf{\Phi}(t) = \frac{1}{4}\begin{pmatrix} 2e^{3t} + 2e^{-t} & e^{3t} - e^{-t} \\ 4e^{3t} - 4e^{-t} & 2e^{3t} + 2e^{-t} \end{pmatrix}.$$

Direct multiplication results in

$$\begin{aligned} \mathbf{\Phi}(t)\mathbf{\Phi}(s) &= \frac{1}{16}\begin{pmatrix} 2e^{3t} + 2e^{-t} & e^{3t} - e^{-t} \\ 4e^{3t} - 4e^{-t} & 2e^{3t} + 2e^{-t} \end{pmatrix}\begin{pmatrix} 2e^{3s} + 2e^{-s} & e^{3s} - e^{-s} \\ 4e^{3s} - 4e^{-s} & 2e^{3s} + 2e^{-s} \end{pmatrix} \\ &= \frac{1}{16}\begin{pmatrix} 8(e^{3t+3s} + e^{-t-s}) & 4(e^{3t+3s} - e^{-t-s}) \\ 16(e^{3t+3s} - e^{-t-s}) & 8(e^{3t+3s} + e^{-t-s}) \end{pmatrix}. \end{aligned}$$

Hence

$$\mathbf{\Phi}(t)\mathbf{\Phi}(s) = \frac{1}{4}\begin{pmatrix} 2e^{3(t+s)} + 2e^{-(t+s)} & e^{3(t+s)} - e^{-(t+s)} \\ 4e^{3(t+s)} - 4e^{-(t+s)} & 2e^{3(t+s)} + 2e^{-(t+s)} \end{pmatrix} = \mathbf{\Phi}(t+s).$$

15.(a) Let s be arbitrary, but fixed, and t variable. Similar to the argument in Problem 13, the columns of the matrix $\mathbf{\Phi}(t)\mathbf{\Phi}(s)$ are linear combinations of fundamental solutions. Hence the columns of $\mathbf{\Phi}(t)\mathbf{\Phi}(s)$ are also solution of the system of equations. Further, setting $t=0$, $\mathbf{\Phi}(0)\mathbf{\Phi}(s) = \mathbf{I}\,\mathbf{\Phi}(s) = \mathbf{\Phi}(s)$. That is, $\mathbf{\Phi}(t)\mathbf{\Phi}(s)$ is a solution of the initial value problem $\mathbf{Z}' = \mathbf{AZ}$, with $\mathbf{Z}(0) = \mathbf{\Phi}(s)$. Now consider the change of variable $\tau = t + s$. Let $\mathbf{W}(\tau) = \mathbf{Z}(\tau - s)$. The given initial value problem can be reformulated as

$$\frac{d}{d\tau}\mathbf{W} = \mathbf{AW}\,,\text{ with }\mathbf{W}(s) = \mathbf{\Phi}(s)\,.$$

Since $\mathbf{\Phi}(t)$ is a fundamental matrix satisfying $\mathbf{\Phi}' = \mathbf{A\Phi}$, with $\mathbf{\Phi}(0) = \mathbf{I}$, it follows that

$$\mathbf{W}(\tau) = \left[\mathbf{\Phi}(\tau)\mathbf{\Phi}^{-1}(s)\right]\mathbf{\Phi}(s) = \mathbf{\Phi}(\tau).$$

That is, $\mathbf{\Phi}(t+s) = \mathbf{\Phi}(\tau) = \mathbf{W}(\tau) = \mathbf{Z}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}(s)$.

(b) Based on part (a), $\mathbf{\Phi}(t)\mathbf{\Phi}(-t) = \mathbf{\Phi}(t+(-t)) = \mathbf{\Phi}(0) = \mathbf{I}$. Hence $\mathbf{\Phi}(-t) = \mathbf{\Phi}^{-1}(t)$.

(c) It also follows that $\mathbf{\Phi}(t-s) = \mathbf{\Phi}(t+(-s)) = \mathbf{\Phi}(t)\mathbf{\Phi}(-s) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(s)$.

16. Let $\mathbf{A}$ be a diagonal matrix, with $\mathbf{A} = \left[a_1\mathbf{e}^{(1)}, a_2\mathbf{e}^{(2)}, \cdots, a_n\mathbf{e}^{(n)}\right]$. Note that for any positive integer k,

$$\mathbf{A}^k = \left[a_1^k\,\mathbf{e}^{(1)}, a_2^k\,\mathbf{e}^{(2)}, \cdots, a_n^k\,\mathbf{e}^{(n)}\right].$$

It follows, from basic matrix algebra, that

$$\mathbf{I} + \sum_{k=1}^{m}\mathbf{A}^k\frac{t^k}{k!} = \begin{pmatrix} \sum_{k=0}^m a_1^k\frac{t^k}{k!} & 0 & \cdots & 0 \\ 0 & \sum_{k=0}^m a_2^k\frac{t^k}{k!} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sum_{k=0}^m a_n^k\frac{t^k}{k!} \end{pmatrix}.$$

It can be shown that the partial sums on the left hand side converge for all t. Taking the limit as $m \to \infty$ on both sides of the equation, we obtain

$$e^{\mathbf{A}t} = \begin{pmatrix} e^{a_1 t} & 0 & \cdots & 0 \\ 0 & e^{a_2 t} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & e^{a_n t} \end{pmatrix}.$$

Alternatively, consider the system $\mathbf{x}' = \mathbf{Ax}$. Since the ODEs are uncoupled, the vectors $\mathbf{x}^{(j)} = e^{a_j t}\,\mathbf{e}^{(j)}$, $j = 1, 2, \cdots n$, are a set of linearly independent solutions. Hence the matrix

$$\mathbf{x} = \left[e^{a_1 t}\,\mathbf{e}^{(1)}, e^{a_2 t}\,\mathbf{e}^{(2)}, \cdots, e^{a_n t}\,\mathbf{e}^{(n)}\right]$$

is a fundamental matrix. Finally, since $\mathbf{X}(0) = \mathbf{I}$, it follows that

$$\left[e^{a_1 t}\,\mathbf{e}^{(1)}, e^{a_2 t}\,\mathbf{e}^{(2)}, \cdots, e^{a_n t}\,\mathbf{e}^{(n)}\right] = \mathbf{\Phi}(t) = e^{\mathbf{A}t}.$$

17.(a) Let $x_1 = u$ and $x_2 = u'$; then $u'' = x_2'$. In terms of the new variables, we have

$$x_2' + \omega^2 x_1 = 0$$

with the initial conditions $x_1(0) = u_0$ and $x_2(0) = v_0$. The equivalent first order system is

$$x_1' = x_2$$
$$x_2' = -\omega^2 x_1$$

which can be expressed in the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}; \quad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}.$$

(b) Setting

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix},$$

it is easy to show that

$$\mathbf{A}^2 = -\omega^2 \mathbf{I},\ \mathbf{A}^3 = -\omega^2 \mathbf{A} \ \text{ and } \ \mathbf{A}^4 = \omega^4 \mathbf{I}.$$

It follows inductively that

$$\mathbf{A}^{2k} = (-1)^k \omega^{2k} \mathbf{I}$$

and

$$\mathbf{A}^{2k+1} = (-1)^k \omega^{2k} \mathbf{A}.$$

Hence

$$\begin{aligned} e^{\mathbf{A}t} &= \sum_{k=0}^{\infty} \left[(-1)^k \frac{\omega^{2k} t^{2k}}{(2k)!} \mathbf{I} + (-1)^k \frac{\omega^{2k} t^{2k+1}}{(2k+1)!} \mathbf{A} \right] \\ &= \left[\sum_{k=0}^{\infty} (-1)^k \frac{\omega^{2k} t^{2k}}{(2k)!} \right] \mathbf{I} + \frac{1}{\omega} \left[\sum_{k=0}^{\infty} (-1)^k \frac{\omega^{2k+1} t^{2k+1}}{(2k+1)!} \right] \mathbf{A} \end{aligned}$$

and therefore

$$e^{\mathbf{A}t} = \cos \omega t \mathbf{I} + \frac{1}{\omega} \sin \omega t \mathbf{A}.$$

(c) From Equation (28),

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \left[\cos \omega t \mathbf{I} + \frac{1}{\omega} \sin \omega t \mathbf{A} \right] \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \\ &= \cos \omega t \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \frac{1}{\omega} \sin \omega t \begin{pmatrix} v_0 \\ -\omega^2 u_0 \end{pmatrix}. \end{aligned}$$

18.(a) Assuming that $\mathbf{x} = \phi(t)$ is a solution, then $\phi' = \mathbf{A}\phi$, with $\phi(0) = \mathbf{x}^0$. Integrate both sides of the equation to obtain

$$\phi(t) - \phi(0) = \int_0^t \mathbf{A}\phi(s) ds.$$

Hence

$$\phi(t) = \mathbf{x}^0 + \int_0^t \mathbf{A}\phi(s)ds\,.$$

(b) Proceed with the iteration

$$\phi^{(i+1)}(t) = \mathbf{x}^0 + \int_0^t \mathbf{A}\phi^{(i)}(s)ds\,.$$

With $\phi^{(0)}(t) = \mathbf{x}^0$, and noting that $\mathbf{A}$ is a constant matrix,

$$\phi^{(1)}(t) = \mathbf{x}^0 + \int_0^t \mathbf{A}\mathbf{x}^0 ds = \mathbf{x}^0 + \mathbf{A}\mathbf{x}^0 t\,.$$

That is, $\phi^{(1)}(t) = (\mathbf{I} + \mathbf{A}t)\mathbf{x}^0$.

(c) We then have

$$\phi^{(2)}(t) = \mathbf{x}^0 + \int_0^t \mathbf{A}(\mathbf{I} + \mathbf{A}t)\mathbf{x}^0 ds = \mathbf{x}^0 + \mathbf{A}\mathbf{x}^0 t + \mathbf{A}^2\mathbf{x}^0\frac{t^2}{2} = (\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2})\mathbf{x}^0.$$

Now suppose that

$$\phi^{(n)}(t) = (\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2} + \cdots + \mathbf{A}^n\frac{t^n}{n!})\mathbf{x}^0.$$

It follows that

$$\begin{aligned}
&\int_0^t \mathbf{A}(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2} + \cdots + \mathbf{A}^n\frac{t^n}{n!})\mathbf{x}^0 ds = \\
&= \mathbf{A}(\mathbf{I}t + \mathbf{A}\frac{t^2}{2} + \mathbf{A}^2\frac{t^3}{3!} + \cdots + \mathbf{A}^n\frac{t^{n+1}}{(n+1)!})\mathbf{x}^0 \\
&= (\mathbf{A}t + \mathbf{A}^2\frac{t^2}{2} + \mathbf{A}^3\frac{t^3}{3!} + \cdots + \mathbf{A}^{n+1}\frac{t^n}{n!})\mathbf{x}^0.
\end{aligned}$$

Therefore

$$\phi^{(n+1)}(t) = (\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2} + \cdots + \mathbf{A}^{n+1}\frac{t^{n+1}}{(n+1)!})\mathbf{x}^0.$$

By induction, the asserted form of $\phi^{(n)}(t)$ is valid for all $n \geq 0$.

(d) Define $\phi^{(\infty)}(t) = \lim_{n\to\infty}\phi^{(n)}(t)$. It can be shown that the limit does exist. In fact,

$$\phi^{(\infty)}(t) = e^{\mathbf{A}t}\mathbf{x}^0.$$

Term-by-term differentiation results in

$$\begin{aligned}
\frac{d}{dt}\phi^{(\infty)}(t) &= \frac{d}{dt}(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2} + \cdots + \mathbf{A}^n\frac{t^n}{n!} + \cdots)\mathbf{x}^0 \\
&= (\mathbf{A} + \mathbf{A}^2 t + \cdots + \mathbf{A}^n\frac{t^{n-1}}{(n-1)!} + \cdots)\mathbf{x}^0 \\
&= \mathbf{A}(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2} + \cdots + \mathbf{A}^{n-1}\frac{t^{n-1}}{(n-1)!} + \cdots)\mathbf{x}^0.
\end{aligned}$$

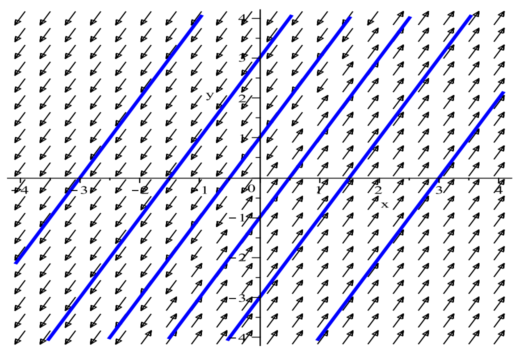
That is,

$$\frac{d}{dt}\phi^{(\infty)}(t) = \mathbf{A}\phi^{(\infty)}(t).$$

Furthermore, $\phi^{(\infty)}(0) = \mathbf{x}^0$. Based on uniqueness of solutions, $\phi(t) = \phi^{(\infty)}(t)$.

7.8

2.(a)



(b) All of the points on the line $x_2 = 2x_1$ are equilibrium points. Solutions starting at all other points become unbounded.

(c) Setting $\mathbf{x} = \boldsymbol{\xi}\, t^r$ results in the algebraic equations

$$\begin{pmatrix} 4-r & -2 \\ 8 & -4-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 = 0$, with the single root $r = 0$. Substituting $r = 0$ reduces the system of equations to $2\xi_1 - \xi_2 = 0$. Therefore the only eigenvector is $\boldsymbol{\xi} = (1\,, 2)^T$. One solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

which is a constant vector. In order to generate a second linearly independent solution, we must search for a generalized eigenvector. This leads to the system of equations

$$\begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

This system also reduces to a single equation, $2\eta_1 - \eta_2 = 1/2$. Setting $\eta_1 = k$, some arbitrary constant, we obtain $\eta_2 = 2k - 1/2$. A second solution is

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} k \\ 2k - 1/2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1/2 \end{pmatrix} + k\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

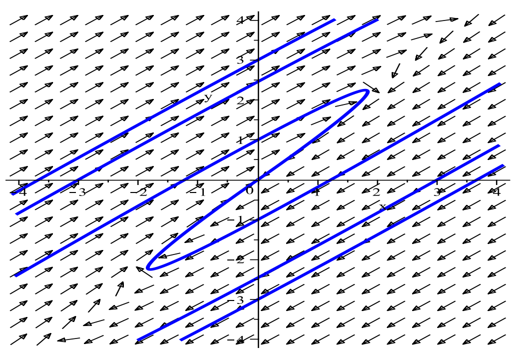
Note that the last term is a multiple of $\mathbf{x}^{(1)}$ and may be dropped. Hence

$$\mathbf{x}^{(2)} = \begin{pmatrix}1\\2\end{pmatrix}t + \begin{pmatrix}0\\-1/2\end{pmatrix}.$$

The general solution is

$$\mathbf{x} = c_1\begin{pmatrix}1\\2\end{pmatrix} + c_2\left[\begin{pmatrix}1\\2\end{pmatrix}t + \begin{pmatrix}0\\-1/2\end{pmatrix}\right].$$

4.(a)



(b) All trajectories converge to the origin.

(c) Solution of the ODE requires analysis of the algebraic equations

$$\begin{pmatrix}-3-r & \frac{5}{2}\\ -\frac{5}{2} & 2-r\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 + r + 1/4 = 0$. The only root is $r = -1/2$, which is an eigenvalue of multiplicity two. Setting $r = -1/2$ is the coefficient matrix reduces the system to the single equation $-\xi_1 + \xi_2 = 0$. Hence the corresponding eigenvector is $\boldsymbol{\xi} = (1\,,1)^T$. One solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix}1\\1\end{pmatrix}e^{-t/2}.$$

In order to obtain a second linearly independent solution, we find a solution of the system

$$\begin{pmatrix}-5/2 & 5/2\\ -5/2 & 5/2\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}.$$

There equations reduce to $-5\eta_1 + 5\eta_2 = 2$. Set $\eta_1 = k$, some arbitrary constant. Then $\eta_2 = k + 2/5$. A second solution is

$$\mathbf{x}^{(2)} = \begin{pmatrix}1\\1\end{pmatrix}te^{-t/2} + \begin{pmatrix}k\\k+2/5\end{pmatrix}e^{-t/2} = \begin{pmatrix}1\\1\end{pmatrix}te^{-t/2} + \begin{pmatrix}0\\2/5\end{pmatrix}e^{-t/2} + k\begin{pmatrix}1\\1\end{pmatrix}e^{-t/2}.$$

Dropping the last term, the general solution is

$$\mathbf{x} = c_1\begin{pmatrix}1\\1\end{pmatrix}e^{-t/2} + c_2\left[\begin{pmatrix}1\\1\end{pmatrix}te^{-t/2} + \begin{pmatrix}0\\2/5\end{pmatrix}e^{-t/2}\right].$$

6. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} -r & 1 & 1 \\ 1 & -r & 1 \\ 1 & 1 & -r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $r^3 - 3r - 2 = 0$, with roots $r_1 = 2$ and $r_{2,3} = -1$. Setting $r = 2$, we have

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system is reduced to the equations

$$\begin{aligned} \xi_1 - \xi_3 &= 0 \\ \xi_2 - \xi_3 &= 0\,. \end{aligned}$$

A corresponding eigenvector is given by $\boldsymbol{\xi}^{(1)} = (1\,,1\,,1)^T$. Setting $r = -1$, the system of equations is reduced to the single equation

$$\xi_1 + \xi_2 + \xi_3 = 0\,.$$

An eigenvector vector is given by $\boldsymbol{\xi}^{(2)} = (1\,,0\,,-1)^T$. Since the last equation has two free variables, a third linearly independent eigenvector (associated with $r = -1$) is $\boldsymbol{\xi}^{(3)} = (0\,,1\,,-1)^T$. Therefore the general solution may be written as

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t}.$$

7.(a) Solution of the ODE requires analysis of the algebraic equations

$$\begin{pmatrix} 1-r & -4 \\ 4 & -7-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 + 6r + 9 = 0$. The only root is $r = -3$, which is an eigenvalue of multiplicity two. Substituting $r = -3$ into the coefficient matrix, the system reduces to the single equation $\xi_1 - \xi_2 = 0$. Hence the corresponding eigenvector is $\boldsymbol{\xi} = (1\,,1)^T$. One solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}.$$

For a second linearly independent solution, we search for a generalized eigenvector. Its components satisfy

$$\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

that is, $4\eta_1 - 4\eta_2 = 1$. Let $\eta_2 = k$, some arbitrary constant. Then $\eta_1 = k + 1/4$. It follows that a second solution is given by

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} k + 1/4 \\ k \end{pmatrix} e^{-3t} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} e^{-3t} + k \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}.$$

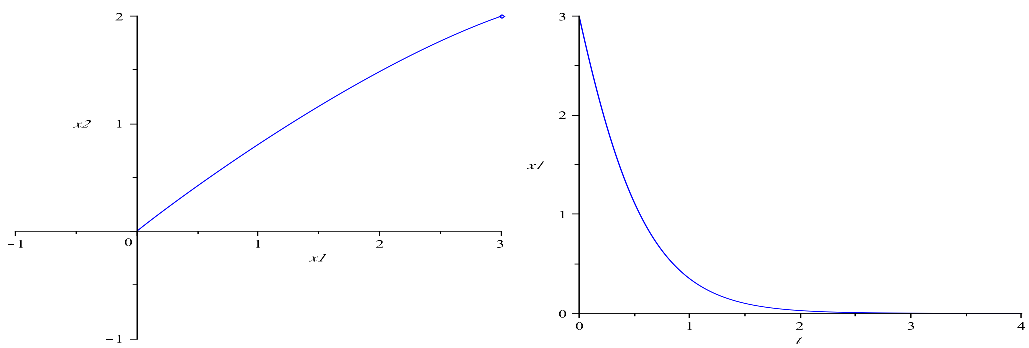
Dropping the last term, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} + c_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-3t} + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} e^{-3t} \right].$$

Imposing the initial conditions, we require that $c_1 + c_2/4 = 3$, $c_1 = 2$, which results in $c_1 = 2$ and $c_2 = 4$. Therefore the solution of the IVP is

$$\mathbf{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-3t} + \begin{pmatrix} 4 \\ 4 \end{pmatrix} t e^{-3t}.$$

(b)



8.(a) Solution of the ODEs is based on the analysis of the algebraic equations

$$\begin{pmatrix} -\frac{5}{2} - r & \frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 + 2r + 1 = 0$, with a single root $r = -1$. Setting $r = -1$, the two equations reduce to $-\xi_1 + \xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi} = (1, 1)^T$. One solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}.$$

A second linearly independent solution is obtained by solving the system

$$\begin{pmatrix} -3/2 & 3/2 \\ -3/2 & 3/2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The equations reduce to the single equation $-3\eta_1 + 3\eta_2 = 2$. Let $\eta_1 = k$. We obtain $\eta_2 = 2/3 + k$, and a second linearly independent solution is

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-t} + \begin{pmatrix} k \\ 2/3 + k \end{pmatrix} e^{-t} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-t} + \begin{pmatrix} 0 \\ 2/3 \end{pmatrix} e^{-t} + k \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}.$$

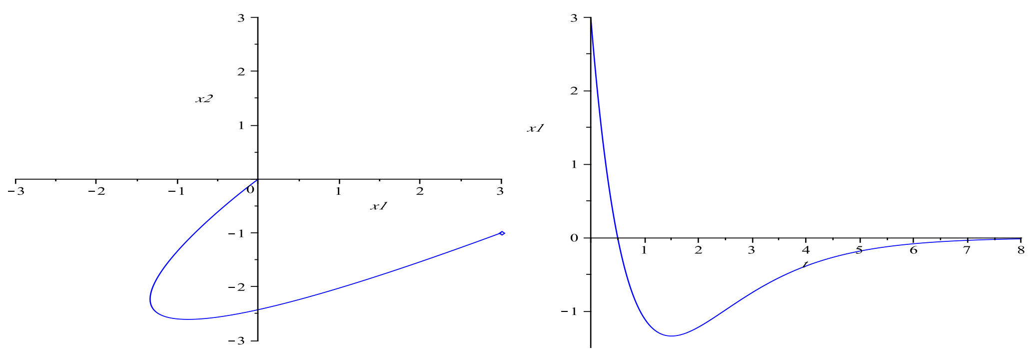
Dropping the last term, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-t} + \begin{pmatrix} 0 \\ 2/3 \end{pmatrix} e^{-t} \right].$$

Imposing the initial conditions, we find that $c_1 = 3$, $c_1 + 2c_2/3 = -1$, so that $c_1 = 3$ and $c_2 = -6$. Therefore the solution of the IVP is

$$\mathbf{x} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} e^{-t} - \begin{pmatrix} 6 \\ 6 \end{pmatrix} te^{-t}.$$

(b)



10.(a) The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} 3-r & 9 \\ -1 & -3-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 = 0$, with a single root $r = 0$. Setting $r = 0$, the two equations reduce to $\xi_1 + 3\xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi} = (-3\,, 1)^T$. Hence one solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} -3 \\ 1 \end{pmatrix},$$

which is a constant vector. A second linearly independent solution is obtained from the system

$$\begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

The equations reduce to the single equation $\eta_1 + 3\eta_2 = -1$. Let $\eta_2 = k$. We obtain $\eta_1 = -1 - 3k$, and a second linearly independent solution is

$$\mathbf{x}^{(2)} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} t + \begin{pmatrix} -1-3k \\ k \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} t + \begin{pmatrix} -1 \\ 0 \end{pmatrix} + k \begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

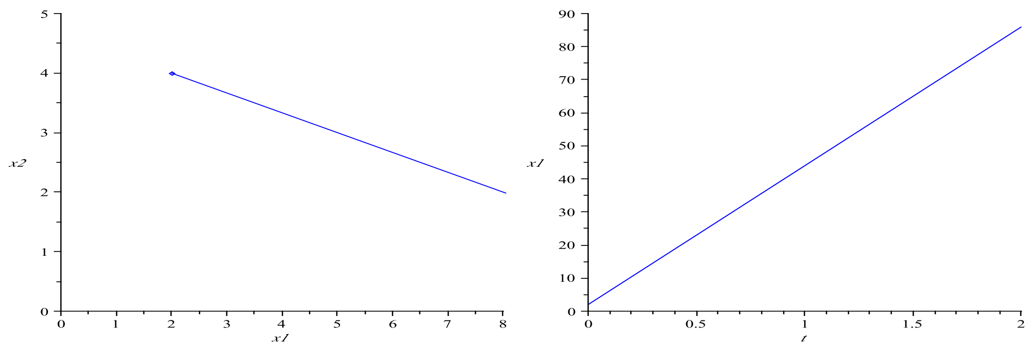
Dropping the last term, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_2 \left[\begin{pmatrix} -3 \\ 1 \end{pmatrix} t + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right].$$

Imposing the initial conditions, we require that $-3c_1 - c_2 = 2$, $c_1 = 4$, which results in $c_1 = 4$ and $c_2 = -14$. Therefore the solution of the IVP is

$$\mathbf{x} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} - 14 \begin{pmatrix} -3 \\ 1 \end{pmatrix} t.$$

(b)



13. Setting $\mathbf{x} = \boldsymbol{\xi}\, t^r$ results in the algebraic equations

$$\begin{pmatrix} 3-r & -4 \\ 1 & -1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 - 2r + 1 = 0$, with a single root of $r_{1,2} = 1$. With $r = 1$, the system reduces to a single equation $\xi_1 - 2\,\xi_2 = 0$. An eigenvector is given by $\boldsymbol{\xi} = (2\,,1)^T$. Hence one solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} t\,.$$

In order to find a second linearly independent solution, we search for a generalized eigenvector whose components satisfy

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

These equations reduce to $\eta_1 - 2\,\eta_2 = 1$. Let $\eta_2 = k$, some arbitrary constant. Then $\eta_1 = 1 + 2k$. (Before proceeding, note that if we set $u = \ln t$, the original equation is transformed into a constant coefficient equation with independent variable u. Recall that a second solution is obtained by multiplication of the first solution by the factor u. This implies that we must multiply first solution by a factor of $\ln t$.) Hence a second linearly independent solution is

$$\mathbf{x}^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} t\,\ln\,t + \begin{pmatrix} 1+2k \\ k \end{pmatrix} t = \begin{pmatrix} 2 \\ 1 \end{pmatrix} t\,\ln\,t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} t + k\begin{pmatrix} 2 \\ 1 \end{pmatrix} t.$$

Dropping the last term, the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} t + c_2\left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} t\,\ln\,t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} t\right].$$

16.(a) Using the result in Problem 15, the eigenvalues are

$$r_{1,2} = -\frac{1}{2RC} \pm \frac{\sqrt{L^2 - 4R^2CL}}{2RCL}\,.$$

The discriminant vanishes when $L = 4R^2C$.

(b) The system of differential equations is

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{4} \\ -1 & -1 \end{pmatrix}\begin{pmatrix} I \\ V \end{pmatrix}.$$

The associated eigenvalue problem is

$$\begin{pmatrix} -r & \frac{1}{4} \\ -1 & -1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 + r + 1/4 = 0$, with a single root of $r_{1,2} = -1/2$. Setting $r = -1/2$, the algebraic equations reduce to $2\xi_1 + \xi_2 = 0$. An eigenvector is given by $\boldsymbol{\xi} = (1\,, -2)^T$. Hence one solution is

$$\begin{pmatrix} I \\ V \end{pmatrix}^{(1)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t/2}.$$

A second solution is obtained from a generalized eigenvector whose components satisfy

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ -1 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

It follows that $\eta_1 = k$ and $\eta_2 = 4 - 2k$. A second linearly independent solution is

$$\begin{pmatrix} I \\ V \end{pmatrix}^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} t\, e^{-t/2} + \begin{pmatrix} k \\ 4-2k \end{pmatrix} e^{-t/2} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} t\, e^{-t/2} + \begin{pmatrix} 0 \\ 4 \end{pmatrix} e^{-t/2} + k\begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t/2}.$$

Dropping the last term, the general solution is

$$\begin{pmatrix} I \\ V \end{pmatrix} = c_1\begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t/2} + c_2\left[\begin{pmatrix} 1 \\ -2 \end{pmatrix} t\, e^{-t/2} + \begin{pmatrix} 0 \\ 4 \end{pmatrix} e^{-t/2}\right].$$

Imposing the initial conditions, we require that $c_1 = 1$, $-2c_1 + 4c_2 = 2$, which results in $c_1 = 1$ and $c_2 = 1$. Therefore the solution of the IVP is

$$\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t/2} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} te^{-t/2}.$$

19.(a) The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} 5-r & -3 & -2 \\ 8 & -5-r & -4 \\ -4 & 3 & 3-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $r^3 - 3r^2 + 3r - 1 = 0$, with a single root of multiplicity three, $r = 1$. Setting $r = 1$, we have

$$\begin{pmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The system of algebraic equations reduces to a single equation

$$4\xi_1 - 3\xi_2 - 2\xi_3 = 0.$$

An eigenvector vector is given by $\boldsymbol{\xi}^{(1)} = (1\,,0\,,2)^T$. Since the last equation has two free variables, a second linearly independent eigenvector (associated with $r = 1$) is $\boldsymbol{\xi}^{(2)} = (0\,,2\,,-3)^T$. Therefore two solutions are obtained as

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} e^t \text{ and } \mathbf{x}^{(2)} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} e^t.$$

(b) It follows directly that $\mathbf{x}' = \boldsymbol{\xi} te^t + \boldsymbol{\xi} e^t + \boldsymbol{\eta} e^t$. Hence the coefficient vectors must satisfy $\boldsymbol{\xi} te^t + \boldsymbol{\xi} e^t + \boldsymbol{\eta} e^t = \mathbf{A}\boldsymbol{\xi} te^t + \mathbf{A}\boldsymbol{\eta} e^t$. Rearranging the terms, we have

$$\boldsymbol{\xi} e^t = (\mathbf{A} - \mathbf{I})\boldsymbol{\xi} te^t + (\mathbf{A} - \mathbf{I})\boldsymbol{\eta} e^t.$$

Given an eigenvector $\boldsymbol{\xi}$, it follows that $(\mathbf{A} - \mathbf{I})\boldsymbol{\xi} = \mathbf{0}$ and $(\mathbf{A} - \mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}$.

(c) Clearly, $(\mathbf{A} - \mathbf{I})^2\boldsymbol{\eta} = (\mathbf{A} - \mathbf{I})(\mathbf{A} - \mathbf{I})\boldsymbol{\eta} = (\mathbf{A} - \mathbf{I})\boldsymbol{\xi} = \mathbf{0}$. Also,

$$\begin{pmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{pmatrix} \begin{pmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(d) We get that

$$\boldsymbol{\xi} = (\mathbf{A} - \mathbf{I})\boldsymbol{\eta} = \begin{pmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ 2 \end{pmatrix}.$$

This is an eigenvector:

$$\begin{pmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{pmatrix} \begin{pmatrix} -2 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ 2 \end{pmatrix}.$$

(e) Given the three linearly independent solutions, a fundamental matrix is given by

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} e^t & 0 & -2t\,e^t \\ 0 & 2e^t & -4t\,e^t \\ 2e^t & -3e^t & 2t\,e^t + e^t \end{pmatrix}.$$

(f) We construct the transformation matrix

$$\mathbf{T} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & -4 & 0 \\ 2 & 2 & 1 \end{pmatrix},$$

with inverse

$$\mathbf{T}^{-1} = \begin{pmatrix} 1 & -1/2 & 0 \\ 0 & -1/4 & 0 \\ -2 & 3/2 & 1 \end{pmatrix}.$$

The Jordan form of the matrix $\mathbf{A}$ is

$$\mathbf{J} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

21.(a) Direct multiplication results in

$$\mathbf{J}^2 = \begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^2 & 2\lambda \\ 0 & 0 & \lambda^2 \end{pmatrix}, \ \mathbf{J}^3 = \begin{pmatrix} \lambda^3 & 0 & 0 \\ 0 & \lambda^3 & 3\lambda^2 \\ 0 & 0 & \lambda^3 \end{pmatrix}, \ \mathbf{J}^4 = \begin{pmatrix} \lambda^4 & 0 & 0 \\ 0 & \lambda^4 & 4\lambda^3 \\ 0 & 0 & \lambda^4 \end{pmatrix}.$$

(b) Suppose that

$$\mathbf{J}^n = \begin{pmatrix} \lambda^n & 0 & 0 \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}.$$

Then

$$\mathbf{J}^{n+1} = \begin{pmatrix} \lambda^n & 0 & 0 \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix} \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda \cdot \lambda^n & 0 & 0 \\ 0 & \lambda \cdot \lambda^n & \lambda^n + n\lambda \cdot \lambda^{n-1} \\ 0 & 0 & \lambda \cdot \lambda^n \end{pmatrix}.$$

Hence the result follows by mathematical induction.

(c) Note that $\mathbf{J}$ is block diagonal. Hence each block may be exponentiated. Using the result in Problem 20,

$$e^{\mathbf{J}t} = \begin{pmatrix} e^{\lambda t} & 0 & 0 \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{pmatrix}.$$

(d) Setting $\lambda = 1$, and using the transformation matrix $\mathbf{T}$ in Problem 19,

$$\mathbf{T}e^{\mathbf{J}t} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 2 & -2 & -1 \end{pmatrix} \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{pmatrix} = \begin{pmatrix} e^t & 2e^t & 2t\,e^t \\ 0 & 4e^t & 4t\,e^t \\ 2e^t & -2e^t & -2t\,e^t - e^t \end{pmatrix}.$$

Based on the form of $\mathbf{J}$, $e^{\mathbf{J}t}$ is the fundamental matrix associated with the solutions

$$\mathbf{y}^{(1)} = \boldsymbol{\xi}^{(1)}e^t, \mathbf{y}^{(2)} = (2\boldsymbol{\xi}^{(1)} + 2\boldsymbol{\xi}^{(2)})e^t \text{ and } \mathbf{y}^{(3)} = (2\boldsymbol{\xi}^{(1)} + 2\boldsymbol{\xi}^{(2)})te^t + \boldsymbol{\eta}e^t.$$

Hence the resulting matrix is the fundamental matrix associated with the solution set

$$\left\{\boldsymbol{\xi}^{(1)}e^t\,, (2\boldsymbol{\xi}^{(1)} + 2\boldsymbol{\xi}^{(2)})e^t, (2\boldsymbol{\xi}^{(1)} + 2\boldsymbol{\xi}^{(2)})te^t + \boldsymbol{\eta}e^t\right\},$$

as opposed to the solution set in Problem 19, given by

$$\left\{\boldsymbol{\xi}^{(1)}e^t\,, \boldsymbol{\xi}^{(2)}e^t, (2\boldsymbol{\xi}^{(1)} + 2\boldsymbol{\xi}^{(2)})te^t + \boldsymbol{\eta}e^t\right\}.$$

22.(a) Direct multiplication results in

$$\mathbf{J}^2 = \begin{pmatrix} \lambda^2 & 2\lambda & 1 \\ 0 & \lambda^2 & 2\lambda \\ 0 & 0 & \lambda^2 \end{pmatrix}, \ \mathbf{J}^3 = \begin{pmatrix} \lambda^3 & 3\lambda^2 & 3\lambda \\ 0 & \lambda^3 & 3\lambda^2 \\ 0 & 0 & \lambda^3 \end{pmatrix}, \ \mathbf{J}^4 = \begin{pmatrix} \lambda^4 & 4\lambda^3 & 6\lambda^2 \\ 0 & \lambda^4 & 4\lambda^3 \\ 0 & 0 & \lambda^4 \end{pmatrix}.$$

(b) Suppose that

$$\mathbf{J}^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & \frac{n(n-1)}{2}\lambda^{n-2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}.$$

Then

$$\begin{aligned} \mathbf{J}^{n+1} &= \begin{pmatrix} \lambda^n & n\lambda^{n-1} & \frac{n(n-1)}{2}\lambda^{n-2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix} \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \\ &= \begin{pmatrix} \lambda\cdot\lambda^n & \lambda^n + n\lambda\cdot\lambda^{n-1} & n\lambda^{n-1} + \frac{n(n-1)}{2}\lambda\cdot\lambda^{n-2} \\ 0 & \lambda\cdot\lambda^n & \lambda^n + n\lambda\cdot\lambda^{n-1} \\ 0 & 0 & \lambda\cdot\lambda^n \end{pmatrix}. \end{aligned}$$

The result follows by noting that

$$n\lambda^{n-1} + \frac{n(n-1)}{2}\lambda\cdot\lambda^{n-2} = \left[n + \frac{n(n-1)}{2}\right]\lambda^{n-1} = \frac{n^2+n}{2}\lambda^{n-1}.$$

(c) We first observe that

$$\begin{aligned} \sum_{n=0}^{\infty}\lambda^n\frac{t^n}{n!} &= e^{\lambda t} \\ \sum_{n=0}^{\infty} n\lambda^{n-1}\frac{t^n}{n!} &= t\sum_{n=1}^{\infty}\lambda^{n-1}\frac{t^{n-1}}{(n-1)!} = t\,e^{\lambda t} \\ \sum_{n=0}^{\infty}\frac{n(n-1)}{2}\lambda^{n-2}\frac{t^n}{n!} &= \frac{t^2}{2}\sum_{n=2}^{\infty}\lambda^{n-2}\frac{t^{n-2}}{(n-2)!} = \frac{t^2}{2}e^{\lambda t}. \end{aligned}$$

Therefore

$$e^{\mathbf{J}t} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{pmatrix}.$$

(d) Setting $\lambda = 2$, and using the transformation matrix $\mathbf{T}$ in Problem 18,

$$\mathbf{T}e^{\mathbf{J}t} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ -1 & 0 & 3 \end{pmatrix} \begin{pmatrix} e^{2t} & te^{2t} & \frac{t^2}{2}e^{2t} \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{pmatrix} = \begin{pmatrix} 0 & e^{2t} & te^{2t} + 2e^{2t} \\ e^{2t} & te^{2t} + e^{2t} & \frac{t^2}{2}e^{2t} + te^{2t} \\ -e^{2t} & -te^{2t} & -\frac{t^2}{2}e^{2t} + 3e^{2t} \end{pmatrix}.$$

7.9

5. As shown in Problem 2, Section 7.8, the general solution of the homogeneous equation is

$$\mathbf{x}_c = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} t \\ 2t - \frac{1}{2} \end{pmatrix}.$$

An associated fundamental matrix is

$$\mathbf{\Psi}(t) = \begin{pmatrix} 1 & t \\ 2 & 2t - \frac{1}{2} \end{pmatrix}.$$

The inverse of the fundamental matrix is easily determined as

$$\mathbf{\Psi}^{-1}(t) = \begin{pmatrix} 4t-3 & -2t+2 \\ 8t-8 & -4t+5 \end{pmatrix}.$$

We can now compute

$$\mathbf{\Psi}^{-1}(t)\mathbf{g}(t) = -\frac{1}{t^3}\begin{pmatrix} 2t^2+4t-1 \\ -2t-4 \end{pmatrix},$$

and

$$\int \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)\,dt = \begin{pmatrix} -\frac{1}{2}t^{-2} + 4t^{-1} - 2\ln t \\ -2t^{-2} - 2t^{-1} \end{pmatrix}.$$

Finally,

$$\mathbf{v}(t) = \mathbf{\Psi}(t)\int \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)\,dt,$$

where

$$v_1(t) = -\frac{1}{2}t^{-2} + 2t^{-1} - 2\ln t - 2, \qquad v_2(t) = 5t^{-1} - 4\ln t - 4\,.$$

Note that the vector $(2\,,4)^T$ is a multiple of one of the fundamental solutions. Hence we can write the general solution as

$$\mathbf{x} = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} t \\ 2t-\frac{1}{2} \end{pmatrix} - \frac{1}{t^2}\begin{pmatrix} 1/2 \\ 0 \end{pmatrix} + \frac{1}{t}\begin{pmatrix} 2 \\ 5 \end{pmatrix} - 2\ln t\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

6. The eigenvalues of the coefficient matrix are $r_1 = 0$ and $r_2 = -5$. It follows that the solution of the homogeneous equation is

$$\mathbf{x}_c = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} -2e^{-5t} \\ e^{-5t} \end{pmatrix}.$$

The coefficient matrix is symmetric. Hence the system is diagonalizable. Using the normalized eigenvectors as columns, the transformation matrix, and its inverse, are

$$\mathbf{T} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \qquad \mathbf{T}^{-1} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}.$$

Setting $\mathbf{x}$=$\mathbf{Ty}$, and $\mathbf{h}(t)$ =$\mathbf{T}^{-1}\mathbf{g}(t)$, the transformed system is given, in scalar form, as

$$y_1' = \frac{5+8t}{\sqrt{5}\,t}$$

$$y_2' = -5y_2 + \frac{4}{\sqrt{5}}\,.$$

The solutions are readily obtained as

$$y_1(t) = \sqrt{5}\,\ln t + \frac{8}{\sqrt{5}}\,t + c_1 \text{ and } y_2(t) = c_2\,e^{-5t} + \frac{4}{5\sqrt{5}}\,.$$

Transforming back to the original variables, we have $\mathbf{x}$=$\mathbf{Ty}$, with

$$\mathbf{x} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix} y_1(t) + \frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix} y_2(t).$$

Hence the general solution is

$$\mathbf{x} = k_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + k_2\begin{pmatrix} -2e^{-5t} \\ e^{-5t} \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}\ln t + \frac{8}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \frac{4}{25}\begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

7. The solution of the homogeneous equation is

$$\mathbf{x}_c = c_1\begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix} + c_2\begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}.$$

Based on the simple form of the right hand side, we use the method of undetermined coefficients. Set $\mathbf{v}$=$\mathbf{a}\,e^t$. Substitution into the ODE yields

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^t = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^t + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t.$$

In scalar form, after canceling the exponential, we have

$$\begin{aligned} a_1 &= a_1 + a_2 + 2 \\ a_2 &= 4a_1 + a_2 - 1\,, \end{aligned}$$

with $a_1 = 1/4$ and $a_2 = -2$. Hence the particular solution is

$$\mathbf{v} = \begin{pmatrix} 1/4 \\ -2 \end{pmatrix} e^t,$$

so that the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix} + c_2\begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix} + \frac{1}{4}\begin{pmatrix} e^t \\ -8e^t \end{pmatrix}.$$

9. Note that the coefficient matrix is symmetric. Hence the system is diagonalizable. The eigenvalues and eigenvectors are given by

$$r_1 = -\frac{1}{2}\ ,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } r_2 = -2\ ,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Using the normalized eigenvectors as columns, the transformation matrix, and its inverse, are

$$\mathbf{T} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \mathbf{T}^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Setting $\mathbf{x}$=$\mathbf{Ty}$, and $\mathbf{h}(t)$ =$\mathbf{T}^{-1}\mathbf{g}(t)$, the transformed system is given, in scalar form, as

$$\begin{aligned} y_1' &= -\frac{1}{2}y_1 + \sqrt{2}\,t + \frac{1}{\sqrt{2}}e^t \\ y_2' &= -2y_2 + \sqrt{2}\,t - \frac{1}{\sqrt{2}}e^t\,. \end{aligned}$$

Using any elementary method for first order linear equations, the solutions are

$$y_1(t) = k_1 e^{-t/2} + \frac{\sqrt{2}}{3} e^t - 4\sqrt{2} + 2\sqrt{2}\, t$$
$$y_2(t) = k_2 e^{-2t} - \frac{1}{3\sqrt{2}} e^t - \frac{1}{2\sqrt{2}} + \frac{1}{\sqrt{2}}\, t\,.$$

Transforming back to the original variables, $\mathbf{x}$=$\mathbf{Ty}$, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/2} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} - \frac{1}{4}\begin{pmatrix} 17 \\ 15 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 5 \\ 3 \end{pmatrix} t + \frac{1}{6}\begin{pmatrix} 1 \\ 3 \end{pmatrix} e^t.$$

10. Since the coefficient matrix is symmetric, the differential equations can be decoupled. The eigenvalues and eigenvectors are given by

$$r_1 = -4\ ,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} \text{ and } r_2 = -1\ ,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}.$$

Using the normalized eigenvectors as columns, the transformation matrix, and its inverse, are

$$\mathbf{T} = \frac{1}{\sqrt{3}}\begin{pmatrix} \sqrt{2} & 1 \\ -1 & \sqrt{2} \end{pmatrix} \qquad \mathbf{T}^{-1} = \frac{1}{\sqrt{3}}\begin{pmatrix} \sqrt{2} & -1 \\ 1 & \sqrt{2} \end{pmatrix}.$$

Setting $\mathbf{x}$=$\mathbf{Ty}$, and $\mathbf{h}(t)$ =$\mathbf{T}^{-1}\mathbf{g}(t)$, the transformed system is given, in scalar form, as

$$y_1' = -4y_1 + \frac{1}{\sqrt{3}}(1+\sqrt{2})e^{-t}$$
$$y_2' = -y_2 + \frac{1}{\sqrt{3}}(1-\sqrt{2})\, e^{-t}\,.$$

The solutions are easily obtained as

$$y_1(t) = k_1 e^{-4t} + \frac{1}{3\sqrt{3}}(1+\sqrt{2})e^{-t}, \qquad y_2(t) = k_2 e^{-t} + \frac{1}{\sqrt{3}}(1-\sqrt{2}\,)te^{-t}.$$

Transforming back to the original variables, the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} e^{-4t} + c_2 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t} + \frac{1}{9}\begin{pmatrix} 2+\sqrt{2}+3\sqrt{3} \\ 3\sqrt{6}-\sqrt{2}-1 \end{pmatrix} e^{-t} + \frac{1}{3}\begin{pmatrix} 1-\sqrt{2} \\ \sqrt{2}-2 \end{pmatrix} te^{-t}.$$

Note that

$$\begin{pmatrix} 2+\sqrt{2}+3\sqrt{3} \\ 3\sqrt{6}-\sqrt{2}-1 \end{pmatrix} = \begin{pmatrix} 2+\sqrt{2} \\ -\sqrt{2}-1 \end{pmatrix} + 3\sqrt{3}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}.$$

The second vector is an eigenvector, hence the solution may be written as

$$\mathbf{x} = c_1\begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} e^{-4t} + c_2 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t} + \frac{1}{9}\begin{pmatrix} 2+\sqrt{2} \\ -\sqrt{2}-1 \end{pmatrix} e^{-t} + \frac{1}{3}\begin{pmatrix} 1-\sqrt{2} \\ \sqrt{2}-2 \end{pmatrix} te^{-t}.$$

11. Based on the solution of Problem 3 of Section 7.6, a fundamental matrix is given by

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} 5\cos t & 5\sin t \\ 2\cos t + \sin t & -\cos t + 2\sin t \end{pmatrix}.$$

The inverse of the fundamental matrix is easily determined as

$$\mathbf{\Psi}^{-1}(t) = \frac{1}{5}\begin{pmatrix} \cos t - 2\sin t & 5\sin t \\ 2\cos t + \sin t & -5\cos t \end{pmatrix}.$$

It follows that

$$\mathbf{\Psi}^{-1}(t)\mathbf{g}(t) = \begin{pmatrix} \cos t \sin t \\ -\cos^2 t \end{pmatrix},$$

and

$$\int \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)\,dt = \begin{pmatrix} \frac{1}{2}\sin^2 t \\ -\frac{1}{2}\cos t \sin t - \frac{1}{2}t \end{pmatrix}.$$

A particular solution is constructed as

$$\mathbf{v}(t) = \mathbf{\Psi}(t)\int \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)\,dt,$$

where

$$v_1(t) = \frac{5}{2}\cos t \sin t - \cos^2 t + \frac{5}{2}t + 1, \qquad v_2(t) = \cos t \sin t - \frac{1}{2}\cos^2 t + t + \frac{1}{2}.$$

Hence the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2\begin{pmatrix} 5\sin t \\ -\cos t + 2\sin t \end{pmatrix} - \\ - t\sin t\begin{pmatrix} 5/2 \\ 1 \end{pmatrix} + t\cos t\begin{pmatrix} 0 \\ 1/2 \end{pmatrix} - \cos t\begin{pmatrix} 5/2 \\ 1 \end{pmatrix}.$$

13.(a) As shown in Problem 25 of Section 7.6, the solution of the homogeneous system is

$$\begin{pmatrix} x_1^{(c)} \\ x_2^{(c)} \end{pmatrix} = c_1 e^{-t/2}\begin{pmatrix} \cos(t/2) \\ 4\sin(t/2) \end{pmatrix} + c_2 e^{-t/2}\begin{pmatrix} \sin(t/2) \\ -4\cos(t/2) \end{pmatrix}.$$

Therefore the associated fundamental matrix is given by

$$\mathbf{\Psi}(t) = e^{-t/2}\begin{pmatrix} \cos(t/2) & \sin(t/2) \\ 4\sin(t/2) & -4\cos(t/2) \end{pmatrix}.$$

(b) The inverse of the fundamental matrix is

$$\mathbf{\Psi}^{-1}(t) = \frac{e^{t/2}}{4}\begin{pmatrix} 4\cos(t/2) & \sin(t/2) \\ 4\sin(t/2) & -\cos(t/2) \end{pmatrix}.$$

It follows that

$$\mathbf{\Psi}^{-1}(t)\mathbf{g}(t) = \frac{1}{2}\begin{pmatrix} \cos(t/2) \\ \sin(t/2) \end{pmatrix},$$

and

$$\int \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)\,dt = \begin{pmatrix} \sin(t/2) \\ -\cos(t/2) \end{pmatrix}.$$

A particular solution is constructed as

$$\mathbf{v}(t) = \mathbf{\Psi}(t)\int \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)\,dt,$$

where $v_1(t) = 0$, $v_2(t) = 4\,e^{-t/2}$. Hence the general solution is

$$\mathbf{x} = c_1 e^{-t/2}\begin{pmatrix}\cos(t/2)\\ 4\,\sin(t/2)\end{pmatrix} + c_2 e^{-t/2}\begin{pmatrix}\sin(t/2)\\ -4\,\cos(t/2)\end{pmatrix} + 4\,e^{-t/2}\begin{pmatrix}0\\1\end{pmatrix}.$$

Imposing the initial conditions, we require that $c_1 = 0$, $-4c_2 + 4 = 0$, which results in $c_1 = 0$ and $c_2 = 1$. Therefore the solution of the IVP is

$$\mathbf{x} = e^{-t/2}\begin{pmatrix}\sin(t/2)\\ 4 - 4\,\cos(t/2)\end{pmatrix}.$$

15. The general solution of the homogeneous problem is

$$\begin{pmatrix}x_1^{(c)}\\ x_2^{(c)}\end{pmatrix} = c_1\begin{pmatrix}1\\2\end{pmatrix}t^{-1} + c_2\begin{pmatrix}2\\1\end{pmatrix}t^2,$$

which can be verified by substitution into the system of ODEs. Since the vectors are linearly independent, a fundamental matrix is given by

$$\mathbf{\Psi}(t) = \begin{pmatrix}t^{-1} & 2t^2\\ 2t^{-1} & t^2\end{pmatrix}.$$

The inverse of the fundamental matrix is

$$\mathbf{\Psi}^{-1}(t) = \frac{1}{3}\begin{pmatrix}-t & 2t\\ 2t^{-2} & -t^{-2}\end{pmatrix}.$$

Dividing both equations by t, we obtain

$$\mathbf{g}(t) = \begin{pmatrix}-2\\ t^3 - t^{-1}\end{pmatrix}.$$

Proceeding with the method of variation of parameters,

$$\mathbf{\Psi}^{-1}(t)\mathbf{g}(t) = \begin{pmatrix}\frac{2}{3}t^4 + \frac{2}{3}t - \frac{2}{3}\\ -\frac{1}{3}t - \frac{4}{3}t^{-2} + \frac{1}{3}t^{-3}\end{pmatrix},$$

and

$$\int \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)\,dt = \begin{pmatrix}\frac{2}{15}t^5 + \frac{1}{3}t^2 - \frac{2}{3}t\\ -\frac{1}{6}t^2 + \frac{4}{3}t^{-1} - \frac{1}{6}t^{-2}\end{pmatrix}.$$

Hence a particular solution is obtained as

$$\mathbf{v} = \begin{pmatrix}-\frac{1}{5}t^4 + 3t - 1\\ \frac{1}{10}t^4 + 2t - \frac{3}{2}\end{pmatrix}.$$

The general solution is

$$\mathbf{x} = c_1\begin{pmatrix}1\\2\end{pmatrix}t^{-1} + c_2\begin{pmatrix}2\\1\end{pmatrix}t^2 + \frac{1}{10}\begin{pmatrix}-2\\1\end{pmatrix}t^4 + \begin{pmatrix}3\\2\end{pmatrix}t - \begin{pmatrix}1\\3/2\end{pmatrix}.$$

16. Based on the hypotheses,

$$\phi'(t) = \mathbf{P}(t)\phi(t) + \mathbf{g}(t) \text{ and } \mathbf{v}'(t) = \mathbf{P}(t)\mathbf{v}(t) + \mathbf{g}(t)\ .$$

Subtracting the two equations results in

$$\phi'(t) - \mathbf{v}'(t) = \mathbf{P}(t)\phi(t) - \mathbf{P}(t)\mathbf{v}(t)\,,$$

that is,

$$[\phi(t) - \mathbf{v}(t)]' = \mathbf{P}(t)\,[\phi(t) - \mathbf{v}(t)]\,.$$

It follows that $\phi(t) - \mathbf{v}(t)$ is a solution of the homogeneous equation. According to Theorem 7.4.2,

$$\phi(t) - \mathbf{v}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t).$$

Hence

$$\phi(t) = \mathbf{u}(t) + \mathbf{v}(t),$$

in which $\mathbf{u}(t)$ is the general solution of the homogeneous problem.

17.(a) Setting $t_0 = 0$ in Equation (34),

$$\mathbf{x} = \mathbf{\Phi}(t)\mathbf{x}^0 + \mathbf{\Phi}(t)\int_0^t \mathbf{\Phi}^{-1}(s)\mathbf{g}(s)ds = \mathbf{\Phi}(t)\mathbf{x}^0 + \int_0^t \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(s)\mathbf{g}(s)ds\,.$$

It was shown in Problem 15(c) in Section 7.7 that $\mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(s) = \mathbf{\Phi}(t-s)$. Therefore

$$\mathbf{x} = \mathbf{\Phi}(t)\mathbf{x}^0 + \int_0^t \mathbf{\Phi}(t-s)\mathbf{g}(s)ds\,.$$

(b) The principal fundamental matrix is identified as $\mathbf{\Phi}(t) = e^{\mathbf{A}t}$. Hence

$$\mathbf{x} = e^{\mathbf{A}t}\mathbf{x}^0 + \int_0^t e^{\mathbf{A}(t-s)}\mathbf{g}(s)ds\,.$$

In Problem 27 of Section 3.6, the particular solution is given as

$$y(t) = \int_{t_0}^t K(t-s)g(s)ds\,,$$

in which the kernel $K(t)$ depends on the nature of the fundamental solutions.

18. Similarly to Eq.(43), here

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{G}(s) + \begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix},$$

where

$$\mathbf{G}(s) = \begin{pmatrix} 2/(s+1) \\ 3/s^2 \end{pmatrix} \quad \text{and} \quad s\mathbf{I} - \mathbf{A} = \begin{pmatrix} s+2 & -1 \\ -1 & s+2 \end{pmatrix}.$$

The transfer matrix is given by Eq.(46):

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(s+1)(s+3)}\begin{pmatrix} s+2 & 1 \\ 1 & s+2 \end{pmatrix}.$$

From these equations we obtain that

$$\mathbf{X}(s) = \begin{pmatrix} \frac{2(s+2)}{(s+1)^2(s+3)} + \frac{3}{s^2(s+1)(s+3)} + \frac{\alpha_1(s+2)}{(s+1)(s+3)} + \frac{\alpha_2}{(s+1)(s+3)} \\ \frac{2}{(s+1)^2(s+3)} + \frac{3(s+2)}{s^2(s+1)(s+3)} + \frac{\alpha_1}{(s+1)(s+3)} + \frac{\alpha_2(s+2)}{(s+1)(s+3)} \end{pmatrix}.$$

The inverse Laplace transform gives us that

$$\mathbf{x}(t) = \begin{pmatrix} \frac{4+\alpha_1+\alpha_2}{2} e^{-t} + \frac{-4+3\alpha_1-3\alpha_2}{6} e^{-3t} + t + te^{-t} - \frac{4}{3} \\ \frac{2+\alpha_1+\alpha_2}{2} e^{-t} + \frac{4-3\alpha_1+3\alpha_2}{6} e^{-3t} + 2t + te^{-t} - \frac{5}{3} \end{pmatrix},$$

so α_1 and α_2 should be chosen so that

$$\frac{4+\alpha_1+\alpha_2}{2} = c_2 + \frac{1}{2} \text{ and } \frac{-4+3\alpha_1-3\alpha_2}{6} = c_1.$$

This gives us $\alpha_1 = (-5 + 6c_1 + 6c_2)/6$ and $\alpha_2 = -c_1 + c_2 - 13/6$.