

# Numerical Methods

## 8.1

2. The Euler formula for this problem is  $y_{n+1} = y_n + h(5t_n - 3\sqrt{y_n})$ , in which  $t_n = t_0 + nh$ . Since  $t_0 = 0$ , we can also write  $y_{n+1} = y_n + 5nh^2 - 3h\sqrt{y_n}$  with  $y_0 = 2$ .

(a) Euler method with  $h = 0.05$  :

	$n = 2$	$n = 4$	$n = 6$	$n = 8$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	1.59980	1.29288	1.07242	0.930175

(b) Euler method with  $h = 0.025$  :

	$n = 4$	$n = 8$	$n = 12$	$n = 16$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	1.61124	1.31361	1.10012	0.962552

The backward Euler formula is  $y_{n+1} = y_n + h(5t_{n+1} - 3\sqrt{y_{n+1}})$  in which  $t_n = t_0 + nh$ . Since  $t_0 = 0$ , we can also write  $y_{n+1} = y_n + 5(n+1)h^2 - 3h\sqrt{y_{n+1}}$ , with  $y_0 = 2$ . Solving for  $y_{n+1}$ , and choosing the positive root, we find that

$$y_{n+1} = \left[ -\frac{3}{2}h + \frac{1}{2}\sqrt{(20n+29)h^2 + 4y_n} \right]^2.$$

(c) Backward Euler method with  $h = 0.05$  :

	$n = 2$	$n = 4$	$n = 6$	$n = 8$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	1.64337	1.37164	1.17763	1.05334

(d) Backward Euler method with  $h = 0.025$  :

	$n = 4$	$n = 8$	$n = 12$	$n = 16$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	1.63301	1.35295	1.15267	1.02407

3. The Euler formula for this problem is  $y_{n+1} = y_n + h(2y_n - 3t_n)$ , in which  $t_n = t_0 + nh$ . Since  $t_0 = 0$ ,  $y_{n+1} = y_n + 2hy_n - 3nh^2$ , with  $y_0 = 1$ .

(a) Euler method with  $h = 0.05$  :

	$n = 2$	$n = 4$	$n = 6$	$n = 8$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	1.2025	1.41603	1.64289	1.88590

(b) Euler method with  $h = 0.025$  :

	$n = 4$	$n = 8$	$n = 12$	$n = 16$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	1.20388	1.41936	1.64896	1.89572

The backward Euler formula is  $y_{n+1} = y_n + h(2y_{n+1} - 3t_{n+1})$ , in which  $t_n = t_0 + nh$ . Since  $t_0 = 0$ , we can also write  $y_{n+1} = y_n + 2hy_{n+1} - 3(n+1)h^2$ , with  $y_0 = 1$ . Solving for  $y_{n+1}$ , we find that

$$y_{n+1} = \frac{y_n - 3(n+1)h^2}{1 - 2h}.$$

(c) Backward Euler method with  $h = 0.05$  :

	$n = 2$	$n = 4$	$n = 6$	$n = 8$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	1.20864	1.43104	1.67042	1.93076

(d) Backward Euler method with  $h = 0.025$  :

	$n = 4$	$n = 8$	$n = 12$	$n = 16$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	1.20693	1.42683	1.66265	1.91802

4. The Euler formula is  $y_{n+1} = y_n + h[2t_n + e^{-t_n}y_n]$ . Since  $t_n = t_0 + nh$  and  $t_0 = 0$ , we can also write  $y_{n+1} = y_n + 2nh^2 + he^{-nh}y_n$ , with  $y_0 = 1$ .

(a) Euler method with  $h = 0.05$  :

	$n = 2$	$n = 4$	$n = 6$	$n = 8$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	1.10244	1.21426	1.33484	1.46399

(b) Euler method with  $h = 0.025$  :

	$n = 4$	$n = 8$	$n = 12$	$n = 16$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	1.10365	1.21656	1.33817	1.46832

The backward Euler formula is  $y_{n+1} = y_n + h[2t_{n+1} + e^{-t_{n+1}}y_{n+1}]$ . Since  $t_0 = 0$  and  $t_{n+1} = (n+1)h$ , we can also write

$$y_{n+1} = y_n + 2h^2(n+1) + he^{-(n+1)h}y_{n+1},$$

with  $y_0 = 1$ . This equation cannot be solved explicitly for  $y_{n+1}$ . At each step, given the current value of  $y_n$ , the equation must be solved numerically for  $y_{n+1}$ .

(c) Backward Euler method with  $h = 0.05$  :

	$n = 2$	$n = 4$	$n = 6$	$n = 8$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	1.10720	1.22333	1.34797	1.48110

(d) Backward Euler method with  $h = 0.025$  :

	$n = 4$	$n = 8$	$n = 12$	$n = 16$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	1.10603	1.22110	1.34473	1.47688

6. The Euler formula for this problem is  $y_{n+1} = y_n + h(t_n^2 - y_n^2) \sin y_n$ . Here  $t_0 = 0$  and  $t_n = nh$ . So that  $y_{n+1} = y_n + h(n^2h^2 - y_n^2) \sin y_n$ , with  $y_0 = -1$ .

(a) Euler method with  $h = 0.05$  :

	$n = 2$	$n = 4$	$n = 6$	$n = 8$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	-0.920498	-0.857538	-0.808030	-0.770038

(b) Euler method with  $h = 0.025$  :

	$n = 4$	$n = 8$	$n = 12$	$n = 16$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	-0.922575	-0.860923	-0.82300	-0.774965

The backward Euler formula is  $y_{n+1} = y_n + h(t_{n+1}^2 - y_{n+1}^2) \sin y_{n+1}$ . Since  $t_0 = 0$  and  $t_{n+1} = (n+1)h$ , we can also write

$$y_{n+1} = y_n + h [(n+1)^2 h^2 - y_{n+1}^2] \sin y_{n+1},$$

with  $y_0 = -1$ . Note that this equation cannot be solved explicitly for  $y_{n+1}$ . Given  $y_n$ , the transcendental equation

$$y_{n+1} + h y_{n+1}^2 \sin y_{n+1} = y_n + h(n+1)^2 h^2$$

must be solved numerically for  $y_{n+1}$ .

(c) Backward Euler method with  $h = 0.05$  :

	$n = 2$	$n = 4$	$n = 6$	$n = 8$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	-0.928059	-0.870054	-0.824021	-0.788686

(d) Backward Euler method with  $h = 0.025$  :

	$n = 4$	$n = 8$	$n = 12$	$n = 16$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	-0.926341	-0.867163	-0.820279	-0.784275

8. The Euler formula  $y_{n+1} = y_n + h(5t_n - 3\sqrt{y_n})$ , in which  $t_n = t_0 + nh$ . Since  $t_0 = 0$ , we can also write  $y_{n+1} = y_n + 5nh^2 - 3h\sqrt{y_n}$  with  $y_0 = 2$ .

(a) Euler method with  $h = 0.025$  :

	$n = 20$	$n = 40$	$n = 60$	$n = 80$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	0.891830	1.25225	2.37818	4.07257

(b) Euler method with  $h = 0.0125$  :

	$n = 40$	$n = 80$	$n = 120$	$n = 160$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	0.908902	1.26872	2.39336	4.08799

The backward Euler formula is  $y_{n+1} = y_n + h(5t_{n+1} - 3\sqrt{y_{n+1}})$ , in which  $t_n = t_0 + nh$ . Since  $t_0 = 0$ , we can also write  $y_{n+1} = y_n + 5(n+1)h^2 - 3h\sqrt{y_{n+1}}$  with

$y_0 = 2$ . Solving for  $y_{n+1}$ , and choosing the positive root, we find that

$$y_{n+1} = \left[ -\frac{3}{2}h + \frac{1}{2}\sqrt{(20n+29)h^2 + 4y_n} \right]^2.$$

(c) Backward Euler method with  $h = 0.025$  :

	$n = 20$	$n = 40$	$n = 60$	$n = 80$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	0.958565	1.31786	2.43924	4.13474

(d) Backward Euler method with  $h = 0.0125$  :

	$n = 40$	$n = 80$	$n = 120$	$n = 160$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	0.942261	1.30153	2.24389	4.11908

10. The Euler formula is  $y_{n+1} = y_n + h[2t_n + e^{-t_n y_n}]$ . Since  $t_n = t_0 + nh$  and  $t_0 = 0$ , we can also write  $y_{n+1} = y_n + 2nh^2 + h e^{-nh y_n}$ , with  $y_0 = 1$ .

(a) Euler method with  $h = 0.025$  :

	$n = 20$	$n = 40$	$n = 60$	$n = 80$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	1.60729	2.46830	3.72167	5.45963

(b) Euler method with  $h = 0.0125$  :

	$n = 40$	$n = 80$	$n = 120$	$n = 160$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	1.60996	2.47460	3.73356	5.47774

The backward Euler formula is  $y_{n+1} = y_n + h[2t_{n+1} + e^{-t_{n+1} y_{n+1}}]$ . Since  $t_0 = 0$  and  $t_{n+1} = (n+1)h$ , we can also write

$$y_{n+1} = y_n + 2h^2(n+1) + h e^{-(n+1)h y_{n+1}},$$

with  $y_0 = 1$ . This equation cannot be solved explicitly for  $y_{n+1}$ . At each step, given the current value of  $y_n$ , the equation must be solved numerically for  $y_{n+1}$ .

(c) Backward Euler method with  $h = 0.025$  :

	$n = 20$	$n = 40$	$n = 60$	$n = 80$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	1.61792	2.49356	3.76940	5.53223

(d) Backward Euler method with  $h = 0.0125$  :

	$n = 40$	$n = 80$	$n = 120$	$n = 160$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	1.61528	2.48723	3.75742	5.51404

11. The Euler formula is  $y_{n+1} = y_n + h(4 - t_n y_n)/(1 + y_n^2)$ . Since  $t_n = t_0 + nh$  and  $t_0 = 0$ , we can also write  $y_{n+1} = y_n + (4h - nh^2 y_n)/(1 + y_n^2)$ , with  $y_0 = -2$ .

(a) Euler method with  $h = 0.025$  :

	$n = 20$	$n = 40$	$n = 60$	$n = 80$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	-1.45865	-0.217545	1.05715	1.41487

(b) Euler method with  $h = 0.0125$  :

	$n = 40$	$n = 80$	$n = 120$	$n = 160$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	-1.45322	-0.180813	1.05903	1.41244

The backward Euler formula is  $y_{n+1} = y_n + h(4 - t_{n+1} y_{n+1})/(1 + y_{n+1}^2)$ . Since  $t_0 = 0$  and  $t_{n+1} = (n+1)h$ , we can also write  $y_{n+1}(1 + y_{n+1}^2) = y_n(1 + y_{n+1}^2) + [4h - (n+1)h^2 y_{n+1}]$ , with  $y_0 = -2$ . This equation cannot be solved explicitly for  $y_{n+1}$ . At each step, given the current value of  $y_n$ , the equation must be solved numerically for  $y_{n+1}$ .

(c) Backward Euler method with  $h = 0.025$  :

	$n = 20$	$n = 40$	$n = 60$	$n = 80$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	-1.43600	-0.0681657	1.06489	1.40575

(d) Backward Euler method with  $h = 0.0125$  :

	$n = 40$	$n = 80$	$n = 120$	$n = 160$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	-1.44190	-0.105737	1.06290	1.40789

12. The Euler formula is  $y_{n+1} = y_n + h(y_n^2 + 2t_n y_n)/(3 + t_n^2)$ . Since  $t_n = t_0 + nh$  and  $t_0 = 0$ , we can also write  $y_{n+1} = y_n + (h y_n^2 + 2nh^2 y_n)/(3 + n^2 h^2)$ , with  $y_0 = 0.5$ .

(a) Euler method with  $h = 0.025$  :

	$n = 20$	$n = 40$	$n = 60$	$n = 80$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	0.587987	0.791589	1.14743	1.70973

(b) Euler method with  $h = 0.0125$  :

	$n = 40$	$n = 80$	$n = 120$	$n = 160$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	0.589440	0.795758	1.15693	1.72955

The backward Euler formula is  $y_{n+1} = y_n + h(y_{n+1}^2 + 2t_{n+1}y_{n+1})/(3 + t_{n+1}^2)$ . Since  $t_0 = 0$  and  $t_{n+1} = (n+1)h$ , we can also write

$$y_{n+1} [3 + (n+1)^2 h^2] - h y_{n+1}^2 = y_n [3 + (n+1)^2 h^2] + 2(n+1)h^2 y_{n+1},$$

with  $y_0 = 0.5$ . Note that although this equation can be solved explicitly for  $y_{n+1}$ , it is also possible to use a numerical equation solver. At each time step, given the current value of  $y_n$ , the equation may be solved numerically for  $y_{n+1}$ .

(c) Backward Euler method with  $h = 0.025$  :

	$n = 20$	$n = 40$	$n = 60$	$n = 80$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	0.593901	0.808716	1.18687	1.79291

(d) Backward Euler method with  $h = 0.0125$  :

	$n = 40$	$n = 80$	$n = 120$	$n = 160$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	0.592396	0.804319	1.17664	1.77111

13. The Euler formula for this problem is  $y_{n+1} = y_n + h(1 - t_n + 4y_n)$ , in which  $t_n = t_0 + nh$ . Since  $t_0 = 0$ , we can also write  $y_{n+1} = y_n + h - nh^2 + 4hy_n$ , with  $y_0 = 1$ . With  $h = 0.01$ , a total number of 200 iterations is needed to reach  $t = 2$ . With  $h = 0.001$ , a total of 2000 iterations are necessary.

14. The backward Euler formula is  $y_{n+1} = y_n + h(1 - t_{n+1} + 4y_{n+1})$ . Since the equation is linear, we can solve for  $y_{n+1}$  in terms of  $y_n$  :

$$y_{n+1} = \frac{y_n + h - h t_{n+1}}{1 - 4h}.$$

Here  $t_0 = 0$  and  $y_0 = 1$ . With  $h = 0.01$ , a total number of 200 iterations is needed to reach  $t = 2$ . With  $h = 0.001$ , a total of 2000 iterations are necessary.

18. Let  $\phi(t)$  be a solution of the initial value problem. The local truncation error for the Euler method, on the interval  $t_n \leq t \leq t_{n+1}$ , is given by

$$e_{n+1} = \frac{1}{2}\phi''(\bar{t}_n)h^2,$$

where  $t_n < \bar{t}_n < t_{n+1}$ . Since  $\phi'(t) = t^2 + [\phi(t)]^2$ , it follows that

$$\phi''(t) = 2t + 2\phi(t)\phi'(t) = 2t + 2t^2\phi(t) + 2[\phi(t)]^3$$

and so

$$e_{n+1} = [\bar{t}_n + \bar{t}_n^2\phi(\bar{t}_n) + \phi^3(\bar{t}_n)]h^2$$

where  $t_n < \bar{t}_n < t_{n+1}$ .

20. Given that  $\phi(t)$  is a solution of the initial value problem, the local truncation error for the Euler method, on the interval  $t_n \leq t \leq t_{n+1}$ , is

$$e_{n+1} = \frac{1}{2}\phi''(\bar{t}_n)h^2,$$

where  $t_n < \bar{t}_n < t_{n+1}$ . Based on the ODE,  $\phi'(t) = \sqrt{t + \phi(t)}$ , and hence

$$\phi''(t) = \frac{1 + \phi'(t)}{2\sqrt{t + \phi(t)}} = \frac{1}{2\sqrt{t + \phi(t)}} + \frac{1}{2}$$

and so

$$e_{n+1} = \frac{1}{4} \left[ 1 + \frac{1}{\sqrt{\bar{t}_n + \phi(\bar{t}_n)}} \right] h^2$$

where  $t_n < \bar{t}_n < t_{n+1}$ .

21. Let  $\phi(t)$  be a solution of the initial value problem. The local truncation error for the Euler method, on the interval  $t_n \leq t \leq t_{n+1}$ , is given by

$$e_{n+1} = \frac{1}{2}\phi''(\bar{t}_n)h^2,$$

where  $t_n < \bar{t}_n < t_{n+1}$ . Since  $\phi'(t) = 2t + e^{-t\phi(t)}$ , it follows that

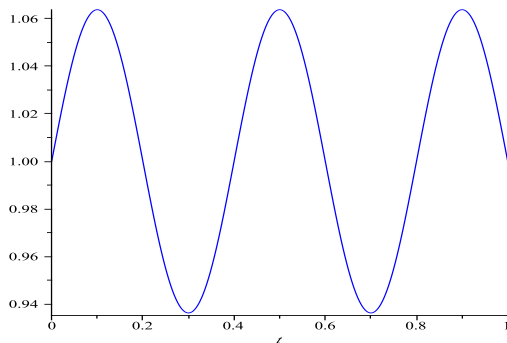
$$\phi''(t) = 2 - [\phi(t) + t\phi'(t)] \cdot e^{-t\phi(t)} = 2 - [\phi(t) + 2t^2 + te^{-t\phi(t)}] \cdot e^{-t\phi(t)}.$$

Hence

$$e_{n+1} = h^2 - \frac{h^2}{2} \left[ \phi(\bar{t}_n) + 2\bar{t}_n^2 + \bar{t}_n e^{-\bar{t}_n\phi(\bar{t}_n)} \right] \cdot e^{-\bar{t}_n\phi(\bar{t}_n)}.$$

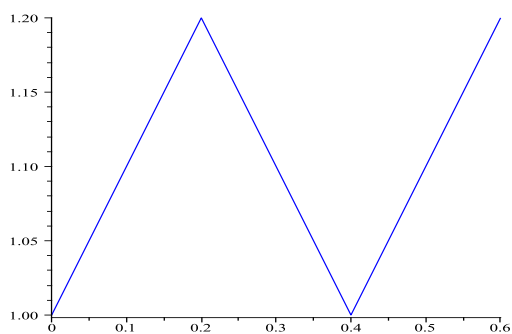
22.(a) Direct integration yields  $\phi(t) = (1/5\pi) \sin 5\pi t + 1$ .





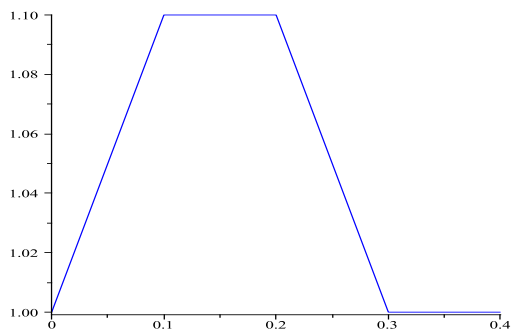
(b)

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
$t_n$	0.0	0.2	0.4	0.6
$y_n$	1.0	1.2	1.0	1.2



(c)

	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$
$t_n$	0.0	0.1	0.2	0.3	0.4
$y_n$	1.0	1.1	1.1	1.0	1.0



(d) Since  $\phi''(t) = -5\pi \sin 5\pi t$ , the local truncation error for the Euler method, on the interval  $t_n \leq t \leq t_{n+1}$ , is given by

$$e_{n+1} = -\frac{5\pi h^2}{2} \sin 5\pi \bar{t}_n.$$

In order to satisfy

$$|e_{n+1}| \leq \frac{5\pi h^2}{2} < 0.05,$$

it is necessary that

$$h < \frac{1}{\sqrt{50\pi}} \approx 0.08.$$

26.(a)

$$1000 \cdot \begin{vmatrix} 6.0 & 18 \\ 2.0 & 6.0 \end{vmatrix} = 1000 \cdot (0) = 0.$$

(b)

$$1000 \cdot \begin{vmatrix} 6.01 & 18.0 \\ 2.00 & 6.00 \end{vmatrix} = 1000(0.06) = 60.$$

(c)

$$1000 \cdot \begin{vmatrix} 6.010 & 18.04 \\ 2.004 & 6.000 \end{vmatrix} = 1000(-0.09216) = -92.16.$$

27. Rounding to three digits,  $a(b - c) \approx 0.224$ . Likewise, to three digits,  $ab \approx 0.702$  and  $ac \approx 0.477$ . It follows that  $ab - ac \approx 0.225$ .

## 8.2

2. The improved Euler formula is

$$y_{n+1} = y_n + \frac{h}{2}(5t_n - 3\sqrt{y_n}) + \frac{h}{2}(5t_{n+1} - 3\sqrt{K_n}),$$

in which  $K_n = y_n + h(5t_n - 3\sqrt{y_n})$ . Since  $t_n = t_0 + nh$  and  $t_0 = 0$ , we can also write

$$y_{n+1} = y_n + \frac{h}{2}(5nh - 3\sqrt{y_n}) + \frac{h}{2}[5(n+1)h - 3\sqrt{K_n}],$$

with  $y_0 = 2$ .

(a)  $h = 0.05$  :

	$n = 2$	$n = 4$	$n = 6$	$n = 8$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	1.62283	1.33460	1.12820	0.995445

(b)  $h = 0.025$  :

	$n = 4$	$n = 8$	$n = 12$	$n = 16$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	1.62243	1.33386	1.12718	0.994215

(c)  $h = 0.0125$  :

	$n = 8$	$n = 16$	$n = 24$	$n = 32$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	1.62234	1.33368	1.12693	0.993921

3. The improved Euler formula for this problem is

$$y_{n+1} = y_n + \frac{h}{2}(4y_n - 3t_n - 3t_{n+1}) + h^2(2y_n - 3t_n).$$

Since  $t_n = t_0 + nh$  and  $t_0 = 0$ , we can also write

$$y_{n+1} = y_n + 2hy_n + \frac{h^2}{2}(4y_n - 3 - 6n) - 3nh^3,$$

with  $y_0 = 1$ .

(a)  $h = 0.05$  :

	$n = 2$	$n = 4$	$n = 6$	$n = 8$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	1.20526	1.42273	1.65511	1.90570

(b)  $h = 0.025$  :

	$n = 4$	$n = 8$	$n = 12$	$n = 16$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	1.20533	1.42290	1.65542	1.90621

(c)  $h = 0.0125$  :

	$n = 8$	$n = 16$	$n = 24$	$n = 32$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	1.20534	1.42294	1.65550	1.90634

5. The improved Euler formula is

$$y_{n+1} = y_n + h \frac{y_n^2 + 2t_n y_n}{2(3 + t_n^2)} + h \frac{K_n^2 + 2t_{n+1}K_n}{2(3 + t_{n+1}^2)},$$

in which

$$K_n = y_n + h \frac{y_n^2 + 2t_n y_n}{3 + t_n^2}.$$

Since  $t_n = t_0 + nh$  and  $t_0 = 0$ , we can also write

$$y_{n+1} = y_n + h \frac{y_n^2 + 2nh y_n}{2(3 + n^2 h^2)} + h \frac{K_n^2 + 2(n+1)hK_n}{2[3 + (n+1)^2 h^2]},$$

with  $y_0 = 0.5$ .

(a)  $h = 0.05$  :

	$n = 2$	$n = 4$	$n = 6$	$n = 8$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	0.510164	0.524126	0.54083	0.564251

(b)  $h = 0.025$  :

	$n = 4$	$n = 8$	$n = 12$	$n = 16$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	0.510168	0.524135	0.542100	0.564277

(c)  $h = 0.0125$  :

	$n = 8$	$n = 16$	$n = 24$	$n = 32$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	0.510169	0.524137	0.542104	0.564284

6. The improved Euler formula for this problem is

$$y_{n+1} = y_n + \frac{h}{2}(t_n^2 - y_n^2) \sin y_n + \frac{h}{2}(t_{n+1}^2 - K_n^2) \sin K_n,$$

in which

$$K_n = y_n + h(t_n^2 - y_n^2) \sin y_n.$$

Since  $t_n = t_0 + nh$  and  $t_0 = 0$ , we can also write

$$y_{n+1} = y_n + \frac{h}{2}(n^2 h^2 - y_n^2) \sin y_n + \frac{h}{2}[(n+1)^2 h^2 - K_n^2] \sin K_n,$$

with  $y_0 = -1$ .

(a)  $h = 0.05$  :

	$n = 2$	$n = 4$	$n = 6$	$n = 8$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	-0.924650	-0.864338	-0.816642	-0.780008

(b)  $h = 0.025$  :

	$n = 4$	$n = 8$	$n = 12$	$n = 16$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	-0.924550	-0.864177	-0.816442	-0.779781

(c)  $h = 0.0125$  :

	$n = 8$	$n = 16$	$n = 24$	$n = 32$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	-0.924525	-0.864138	-0.816393	-0.779725

7. The improved Euler formula for this problem is

$$y_{n+1} = y_n + \frac{h}{2}(4y_n - t_n - t_{n+1} + 1) + h^2(2y_n - t_n + 0.5).$$

Since  $t_n = t_0 + nh$  and  $t_0 = 0$ , we can also write

$$y_{n+1} = y_n + h(2y_n + 0.5) + h^2(2y_n - n) - nh^3,$$

with  $y_0 = 1$ .

(a)  $h = 0.025$  :

	$n = 20$	$n = 40$	$n = 60$	$n = 80$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	2.96719	7.88313	20.8114	55.5106

(b)  $h = 0.0125$  :

	$n = 40$	$n = 80$	$n = 120$	$n = 160$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	2.96800	7.88755	20.8294	55.5758

8. The improved Euler formula is

$$y_{n+1} = y_n + \frac{h}{2}(5t_n - 3\sqrt{y_n}) + \frac{h}{2}(5t_{n+1} - 3\sqrt{K_n}),$$

in which  $K_n = y_n + h(5t_n - 3\sqrt{y_n})$ . Since  $t_n = t_0 + nh$  and  $t_0 = 0$ , we can also write

$$y_{n+1} = y_n + \frac{h}{2}(5nh - 3\sqrt{y_n}) + \frac{h}{2}[5(n+1)h - 3\sqrt{K_n}],$$

with  $y_0 = 2$ .

(a)  $h = 0.025$  :

	$n = 20$	$n = 40$	$n = 60$	$n = 80$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	0.926139	1.28558	2.40898	4.10386

(b)  $h = 0.0125$  :

	$n = 40$	$n = 80$	$n = 120$	$n = 160$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	0.925815	1.28525	2.40869	4.10359

9. The improved Euler formula for this problem is

$$y_{n+1} = y_n + \frac{h}{2} \sqrt{t_n + y_n} + \frac{h}{2} \sqrt{t_{n+1} + K_n},$$

in which  $K_n = y_n + h\sqrt{t_n + y_n}$ . Since  $t_n = t_0 + nh$  and  $t_0 = 0$ , we can also write

$$y_{n+1} = y_n + \frac{h}{2} \sqrt{nh + y_n} + \frac{h}{2} \sqrt{(n+1)h + K_n},$$

with  $y_0 = 3$ .

(a)  $h = 0.025$  :

	$n = 20$	$n = 40$	$n = 60$	$n = 80$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	3.96217	5.10887	6.43134	7.92332

(b)  $h = 0.0125$  :

	$n = 40$	$n = 80$	$n = 120$	$n = 160$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	3.96218	5.10889	6.43138	7.92337

10. The improved Euler formula is

$$y_{n+1} = y_n + \frac{h}{2} [2t_n + e^{-t_n} y_n] + \frac{h}{2} [2t_{n+1} + e^{-t_{n+1}} K_n],$$

in which  $K_n = y_n + h[2t_n + e^{-t_n} y_n]$ . Since  $t_n = t_0 + nh$  and  $t_0 = 0$ , we can also write

$$y_{n+1} = y_n + \frac{h}{2} [2nh + e^{-nh} y_n] + \frac{h}{2} [2(n+1)h + e^{-(n+1)h} K_n],$$

with  $y_0 = 1$ .

(a)  $h = 0.025$  :

	$n = 20$	$n = 40$	$n = 60$	$n = 80$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	1.61263	2.48097	3.74556	5.49595

(b)  $h = 0.0125$  :

	$n = 40$	$n = 80$	$n = 120$	$n = 160$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	1.61263	2.48092	3.74550	5.49589

12. The improved Euler formula is

$$y_{n+1} = y_n + h \frac{y_n^2 + 2t_n y_n}{2(3 + t_n^2)} + h \frac{K_n^2 + 2t_{n+1}K_n}{2(3 + t_{n+1}^2)},$$

in which

$$K_n = y_n + h \frac{y_n^2 + 2t_n y_n}{3 + t_n^2}.$$

Since  $t_n = t_0 + nh$  and  $t_0 = 0$ , we can also write

$$y_{n+1} = y_n + h \frac{y_n^2 + 2nh y_n}{2(3 + n^2h^2)} + h \frac{K_n^2 + 2(n+1)hK_n}{2[3 + (n+1)^2h^2]},$$

with  $y_0 = 0.5$ .

(a)  $h = 0.025$  :

	$n = 20$	$n = 40$	$n = 60$	$n = 80$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	0.590897	0.799950	1.16653	1.74969

(b)  $h = 0.0125$  :

	$n = 40$	$n = 80$	$n = 120$	$n = 160$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	0.590906	0.799988	1.16663	1.74992

16. The exact solution of the initial value problem is  $\phi(t) = \frac{1}{2} + \frac{1}{2}e^{2t}$ . Based on the result in Problem 14(c), the local truncation error for a linear differential equation is

$$e_{n+1} = \frac{1}{6}\phi'''(\bar{t}_n)h^3,$$

where  $t_n < \bar{t}_n < t_{n+1}$ . Since  $\phi'''(t) = 4e^{2t}$ , the local truncation error is

$$e_{n+1} = \frac{2}{3}e^{2\bar{t}_n}h^3.$$

Furthermore, with  $0 \leq \bar{t}_n \leq 1$ ,

$$|e_{n+1}| \leq \frac{2}{3}e^2h^3.$$

It also follows that for  $h = 0.1$ ,

$$|e_1| \leq \frac{2}{3}e^{0.2}(0.1)^3 = \frac{1}{1500}e^{0.2}.$$

Using the improved Euler method, with  $h = 0.1$ , we have  $y_1 \approx 1.11000$ . The exact value is given by  $\phi(0.1) = 1.1107014$ .

17. The exact solution of the initial value problem is given by  $\phi(t) = \frac{1}{2}t + e^{2t}$ . Using the modified Euler method, the local truncation error for a linear differential equation is

$$e_{n+1} = \frac{1}{6}\phi'''(\bar{t}_n)h^3,$$

where  $t_n < \bar{t}_n < t_{n+1}$ . Since  $\phi'''(t) = 8e^{2t}$ , the local truncation error is

$$e_{n+1} = \frac{4}{3}e^{2\bar{t}_n}h^3.$$

Furthermore, with  $0 \leq \bar{t}_n \leq 1$ , the local error is bounded by

$$|e_{n+1}| \leq \frac{4}{3}e^2h^3.$$

It also follows that for  $h = 0.1$ ,

$$|e_1| \leq \frac{4}{3}e^{0.2}(0.1)^3 = \frac{1}{750}e^{0.2}.$$

Using the improved Euler method, with  $h = 0.1$ , we have  $y_1 \approx 1.27000$ . The exact value is given by  $\phi(0.1) = 1.271403$ .

18. Using the Euler method,  $y_1 = 1 + 0.1(0.5 - 0 + 2 \cdot 1) = 1.25$ . Using the improved Euler method,  $y_1 = 1 + 0.05(0.5 - 0 + 2) + 0.05(0.5 - 0.1 + 2.5) = 1.27$ . The estimated error is  $e_1 \approx 1.27 - 1.25 = 0.02$ . The step size should be adjusted by a factor of  $\sqrt{0.0025/0.02} \approx 0.354$ . Hence the required step size is estimated as  $h \approx (0.1)(0.36) = 0.036$ .

20. Using the Euler method,  $y_1 = 3 + 0.1\sqrt{0+3} = 3.173205$ . Using the improved Euler method,

$$y_1 = 3 + 0.05\sqrt{0+3} + 0.05\sqrt{0.1+3.173205} = 3.177063.$$

The estimated error is  $e_1 \approx 3.177063 - 3.173205 = 0.003858$ . The step size should be adjusted by a factor of  $\sqrt{0.0025/0.003858} \approx 0.805$ . Hence the required step size is estimated as  $h \approx (0.1)(0.805) = 0.0805$ .

21. Using the Euler method,

$$y_1 = 0.5 + 0.1 \frac{(0.5)^2 + 0}{3 + 0} = 0.508334$$

Using the improved Euler method,

$$y_1 = 0.5 + 0.05 \frac{(0.5)^2 + 0}{3 + 0} + 0.05 \frac{(0.508334)^2 + 2(0.1)(0.508334)}{3 + (0.1)^2} = 0.510148.$$

The estimated error is  $e_1 \approx 0.510148 - 0.508334 = 0.0018$ . The local truncation error is less than the given tolerance. The step size can be adjusted by a factor of  $\sqrt{0.0025/0.0018} \approx 1.1785$ . Hence it is possible to use a step size of

$$h \approx (0.1)(1.1785) \approx 0.117.$$



22. Assuming that the solution has continuous derivatives at least to the third order,

$$\phi(t_{n+1}) = \phi(t_n) + \phi'(t_n)h + \frac{\phi''(t_n)}{2!}h^2 + \frac{\phi'''(\bar{t}_n)}{3!}h^3,$$

where  $t_n < \bar{t}_n < t_{n+1}$ . Suppose that  $y_n = \phi(t_n)$ .

(a) The local truncation error is given by

$$e_{n+1} = \phi(t_{n+1}) - y_{n+1}.$$

The modified Euler formula is defined as

$$y_{n+1} = y_n + h f \left[ t_n + \frac{1}{2}h, y_n + \frac{1}{2}h f(t_n, y_n) \right].$$

Observe that  $\phi'(t_n) = f(t_n, \phi(t_n)) = f(t_n, y_n)$ . It follows that

$$\begin{aligned} e_{n+1} &= \phi(t_{n+1}) - y_{n+1} = \\ &= h f(t_n, y_n) + \frac{\phi''(t_n)}{2!}h^2 + \frac{\phi'''(\bar{t}_n)}{3!}h^3 - h f \left[ t_n + \frac{1}{2}h, y_n + \frac{1}{2}h f(t_n, y_n) \right]. \end{aligned}$$

(b) As shown in Problem 14(b),

$$\phi''(t_n) = f_t(t_n, y_n) + f_y(t_n, y_n)f(t_n, y_n).$$

Furthermore,

$$\begin{aligned} f \left[ t_n + \frac{1}{2}h, y_n + \frac{1}{2}h f(t_n, y_n) \right] &= f(t_n, y_n) + f_t(t_n, y_n)\frac{h}{2} + f_y(t_n, y_n)k + \\ &+ \frac{1}{2!} \left[ \frac{h^2}{4}f_{tt} + hk f_{ty} + k^2 f_{yy} \right]_{t=\xi, y=\eta}, \end{aligned}$$

in which  $k = \frac{1}{2}h f(t_n, y_n)$  and  $t_n < \xi < t_n + h/2$ ,  $y_n < \eta < y_n + k$ . Therefore

$$e_{n+1} = \frac{\phi'''(\bar{t}_n)}{3!}h^3 - \frac{h}{2!} \left[ \frac{h^2}{4}f_{tt} + hk f_{ty} + k^2 f_{yy} \right]_{t=\xi, y=\eta}.$$

Note that each term in the brackets has a factor of  $h^2$ . Hence the local truncation error is proportional to  $h^3$ .

(c) If  $f(t, y)$  is linear, then  $f_{tt} = f_{ty} = f_{yy} = 0$ , and

$$e_{n+1} = \frac{\phi'''(\bar{t}_n)}{3!}h^3.$$

23. The modified Euler formula for this problem is

$$\begin{aligned} y_{n+1} &= y_n + h \left\{ 3 + t_n + \frac{1}{2}h - \left[ y_n + \frac{1}{2}h(3 + t_n - y_n) \right] \right\} \\ &= y_n + h(3 + t_n - y_n) + \frac{h^2}{2}(y_n - t_n - 2). \end{aligned}$$

Since  $t_n = t_0 + nh$  and  $t_0 = 0$ , we can also write

$$y_{n+1} = y_n + h(3 + nh - y_n) + \frac{h^2}{2}(y_n - nh - 2),$$

with  $y_0 = 1$ . Setting  $h = 0.05$ , we obtain the following values:

	$n = 2$	$n = 4$	$n = 6$	$n = 8$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	1.19512	1.38120	1.55909	1.72956

25. The modified Euler formula is

$$\begin{aligned} y_{n+1} &= y_n + h \left[ 2y_n - 3t_n - \frac{3}{2}h + h(2y_n - 3t_n) \right] \\ &= y_n + h(2y_n - 3t_n) + \frac{h^2}{2}(4y_n - 6t_n - 3). \end{aligned}$$

Since  $t_n = t_0 + nh$  and  $t_0 = 0$ , we can also write

$$y_{n+1} = y_n + h(2y_n - 3nh) + \frac{h^2}{2}(4y_n - 6nh - 3),$$

with  $y_0 = 1$ . Setting  $h = 0.05$ , we obtain:

	$n = 2$	$n = 4$	$n = 6$	$n = 8$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	1.20526	1.42273	1.65511	1.90570

26. The modified Euler formula for this problem is

$$y_{n+1} = y_n + h \left[ 2t_n + h + e^{-(t_n + \frac{h}{2})K_n} \right],$$

in which  $K_n = y_n + (h/2)[2t_n + e^{-t_n}y_n]$ . Now  $t_n = t_0 + nh$ , with  $t_0 = 0$  and  $y_0 = 1$ . Setting  $h = 0.05$ , we obtain the following values :

	$n = 2$	$n = 4$	$n = 6$	$n = 8$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	1.10485	1.21886	1.34149	1.47264

27. Let  $f(t, y)$  be linear in both variables. The improved Euler formula is

$$\begin{aligned} y_{n+1} &= y_n + \frac{1}{2}h [f(t_n, y_n) + f(t_n + h, y_n + hf(t_n, y_n))] \\ &= y_n + \frac{1}{2}hf(t_n, y_n) + \frac{1}{2}hf(t_n, y_n) + \frac{1}{2}hf[h, hf(t_n, y_n)] \\ &= y_n + hf(h, y_n) + \frac{1}{2}hf[h, hf(t_n, y_n)]. \end{aligned}$$

The modified Euler formula is

$$y_{n+1} = y_n + hf \left[ t_n + \frac{1}{2}h, y_n + \frac{1}{2}hf(t_n, y_n) \right] = y_n + hf(t_n, y_n) + hf \left[ \frac{1}{2}h, \frac{1}{2}hf(t_n, y_n) \right].$$

Since  $f(t, y)$  is linear in both variables,

$$f\left[\frac{1}{2}h, \frac{1}{2}hf(t_n, y_n)\right] = \frac{1}{2}f[h, hf(t_n, y_n)].$$

## 8.3

1. The ODE is linear, with  $f(t, y) = 3 + t - y$ . The Runge-Kutta algorithm requires the evaluations

$$\begin{aligned} k_{n1} &= f(t_n, y_n) \\ k_{n2} &= f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_{n1}\right) \\ k_{n3} &= f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_{n2}\right) \\ k_{n4} &= f(t_n + h, y_n + hk_{n3}). \end{aligned}$$

The next estimate is given as the weighted average

$$y_{n+1} = y_n + \frac{h}{6}(k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4}).$$

(a) For  $h = 0.1$ :

	$n = 1$	$n = 2$	$n = 3$	$n = 4$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	1.19516	1.38127	1.55918	1.72968

(b) For  $h = 0.05$ :

	$n = 2$	$n = 4$	$n = 6$	$n = 8$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	1.19516	1.38127	1.55918	1.72968

The exact solution of the IVP is  $y(t) = 2 + t - e^{-t}$ .

2. In this problem,  $f(t, y) = 5t - 3\sqrt{y}$ . At each time step, the Runge-Kutta algorithm requires the evaluations

$$\begin{aligned} k_{n1} &= f(t_n, y_n) \\ k_{n2} &= f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_{n1}\right) \\ k_{n3} &= f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_{n2}\right) \\ k_{n4} &= f(t_n + h, y_n + hk_{n3}). \end{aligned}$$

The next estimate is given as the weighted average

$$y_{n+1} = y_n + \frac{h}{6}(k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4}).$$

(a) For  $h = 0.1$  :

	$n = 1$	$n = 2$	$n = 3$	$n = 4$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	1.62231	1.33362	1.12686	0.993839

(b) For  $h = 0.05$  :

	$n = 2$	$n = 4$	$n = 6$	$n = 8$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	1.62230	1.33362	1.12685	0.993826

The exact solution of the IVP is given implicitly as

$$\frac{1}{(2\sqrt{y} + 5t)^5(t - \sqrt{y})^2} = \frac{\sqrt{2}}{512}.$$

3. The ODE is linear, with  $f(t, y) = 2y - 3t$ . The Runge-Kutta algorithm requires the evaluations

$$\begin{aligned} k_{n1} &= f(t_n, y_n) \\ k_{n2} &= f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_{n1}\right) \\ k_{n3} &= f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_{n2}\right) \\ k_{n4} &= f(t_n + h, y_n + hk_{n3}). \end{aligned}$$

The next estimate is given as the weighted average

$$y_{n+1} = y_n + \frac{h}{6}(k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4}).$$

(a) For  $h = 0.1$  :

	$n = 1$	$n = 2$	$n = 3$	$n = 4$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	1.20535	1.42295	1.65553	1.90638

(b) For  $h = 0.05$  :

	$n = 2$	$n = 4$	$n = 6$	$n = 8$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	1.20535	1.42296	1.65553	1.90638

The exact solution of the IVP is  $y(t) = e^{2t}/4 + 3t/2 + 3/4$ .

5. In this problem,  $f(t, y) = (y^2 + 2ty)/(3 + t^2)$ . The Runge-Kutta algorithm requires the evaluations

$$\begin{aligned}k_{n1} &= f(t_n, y_n) \\k_{n2} &= f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_{n1}\right) \\k_{n3} &= f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_{n2}\right) \\k_{n4} &= f(t_n + h, y_n + hk_{n3}).\end{aligned}$$

The next estimate is given as the weighted average

$$y_{n+1} = y_n + \frac{h}{6}(k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4}).$$

(a) For  $h = 0.1$  :

	$n = 1$	$n = 2$	$n = 3$	$n = 4$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	0.510170	0.524138	0.542105	0.564286

(b) For  $h = 0.05$  :

	$n = 2$	$n = 4$	$n = 6$	$n = 8$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	0.520169	0.524138	0.542105	0.564286

The exact solution of the IVP is  $y(t) = (3 + t^2)/(6 - t)$ .

6. In this problem,  $f(t, y) = (t^2 - y^2) \sin y$ . At each time step, the Runge-Kutta algorithm requires the evaluations

$$\begin{aligned}k_{n1} &= f(t_n, y_n) \\k_{n2} &= f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_{n1}\right) \\k_{n3} &= f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_{n2}\right) \\k_{n4} &= f(t_n + h, y_n + hk_{n3}).\end{aligned}$$

The next estimate is given as the weighted average

$$y_{n+1} = y_n + \frac{h}{6}(k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4}).$$

(a) For  $h = 0.1$  :

	$n = 1$	$n = 2$	$n = 3$	$n = 4$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	-0.924517	-0.864125	-0.816377	-0.779706

(b) For  $h = 0.05$  :

	$n = 2$	$n = 4$	$n = 6$	$n = 8$
$t_n$	0.1	0.2	0.3	0.4
$y_n$	-0.924517	-0.864125	-0.816377	-0.779706

7.(a) For  $h = 0.1$  :

	$n = 5$	$n = 10$	$n = 15$	$n = 20$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	2.96825	7.88889	20.8349	55.5957

(b) For  $h = 0.05$  :

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	2.96828	7.88904	20.8355	55.5980

The exact solution of the IVP is  $y(t) = e^{2t} + t/2$ .

8. See Problem 2 for the exact solution.

(a) For  $h = 0.1$  :

	$n = 5$	$n = 10$	$n = 15$	$n = 20$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	0.925725	1.28516	2.40860	4.10350

(b) For  $h = 0.05$  :

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	0.925711	1.28515	2.40860	4.10350

9.(a) For  $h = 0.1$  :

	$n = 5$	$n = 10$	$n = 15$	$n = 20$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	3.96219	5.10890	6.43139	7.92338

(b) For  $h = 0.05$  :

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	3.96219	5.10890	6.43139	7.92338

The exact solution is given implicitly as

$$\ln \left[ \frac{2}{y+t-1} \right] + 2\sqrt{t+y} - 2 \operatorname{arctanh} \sqrt{t+y} = t + 2\sqrt{3} - 2 \operatorname{arctanh} \sqrt{3}.$$

10. See Problem 4.

(a) For  $h = 0.1$  :

	$n = 5$	$n = 10$	$n = 15$	$n = 20$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	1.61262	2.48091	3.74548	5.49587

(b) For  $h = 0.05$  :

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	1.61262	2.48091	3.74548	5.49587

12. See Problem 5 for the exact solution.

(a) For  $h = 0.1$  :

	$n = 5$	$n = 10$	$n = 15$	$n = 20$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	0.590909	0.800000	1.166667	1.75000

(b) For  $h = 0.05$  :

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	0.590909	0.800000	1.166667	1.75000

13. The ODE is linear, with  $f(t, y) = 1 - t + 4y$ . The Runge-Kutta algorithm requires the evaluations

$$\begin{aligned} k_{n1} &= f(t_n, y_n) \\ k_{n2} &= f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_{n1}\right) \\ k_{n3} &= f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_{n2}\right) \\ k_{n4} &= f(t_n + h, y_n + hk_{n3}). \end{aligned}$$

The next estimate is given as the weighted average

$$y_{n+1} = y_n + \frac{h}{6}(k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4}).$$

The exact solution of the IVP is  $y(t) = 19e^{4t}/16 + t/4 - 3/16$ .

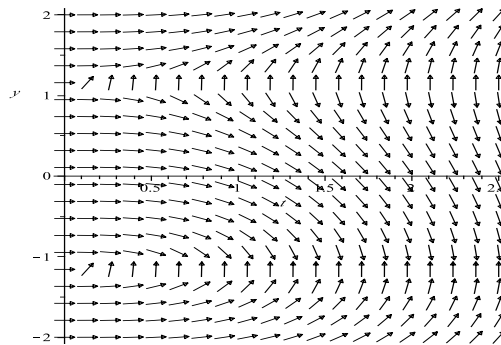
(a) For  $h = 0.1$  :

	$n = 5$	$n = 10$	$n = 15$	$n = 20$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	8.7093175	64.858107	478.81928	3535.8667

(b) For  $h = 0.05$  :

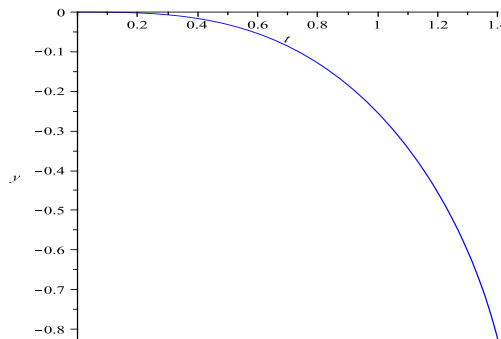
	$n = 10$	$n = 20$	$n = 30$	$n = 40$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	8.7118060	64.894875	479.22674	3539.8804

15.(a)



(b) For the integral curve starting at  $(0, 0)$ , the slope turns infinite near  $t_M \approx 1.5$ . Note that the exact solution of the IVP is defined implicitly as

$$y^3 - 4y = t^3.$$



(c) Using the classic Runge-Kutta algorithm with the given values of  $h$ , we obtain



the values

$$h = 0.1$$

	$n = 12$	$n = 13$	$n = 14$	$n = 15$	$n = 16$
$t_n$	1.2	1.3	1.4	1.5	1.6
$y_n$	-0.45566	-0.60448	-0.82806	-1.73807	-1.56618

$$h = 0.05$$

	$n = 26$	$n = 28$	$n = 30$	$n = 32$	$n = 34$
$t_n$	1.3	1.4	1.5	1.6	1.7
$y_n$	-0.60447	-0.82786	-1.06266	-1.42862	-1.17608

$$h = 0.025$$

	$n = 54$	$n = 56$	$n = 58$	$n = 60$	$n = 62$
$t_n$	1.35	1.4	1.45	1.5	1.55
$y_n$	-0.70134	-0.82783	-1.05986	-1.49336	-1.30986

$$h = 0.01$$

	$n = 142$	$n = 143$	$n = 144$	$n = 145$	$n = 146$
$t_n$	1.42	1.43	1.44	1.45	1.46
$y_n$	-0.89513	-0.93617	-0.98653	-1.05951	-0.76554

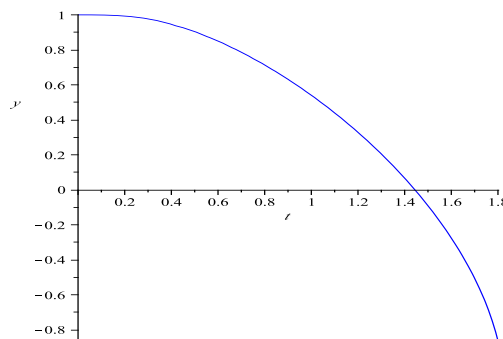
Based on the direction field, the solution should decrease monotonically to the limiting value  $y = -2/\sqrt{3}$ . In the following table, the value of  $t_M$  corresponds to the approximate time in the iteration process that the calculated values begin to increase.

$h$	$t_M$
0.1	1.55
0.05	1.65
0.025	1.525
0.01	1.455

(d) Numerical values will continue to be generated, although they will not be associated with the integral curve starting at  $(0, 0)$ . These values are approximations to nearby integral curves.

(e) We consider the solution associated with the initial condition  $y(0) = 1$ . The exact solution is given by

$$y^3 - 4y = t^3 - 3.$$



For the integral curve starting at  $(0, 1)$ , the slope becomes infinite near  $t_M \approx 2.0$ . In the following table, the values of  $t_M$  corresponds to the approximate time in the iteration process that the calculated values begin to increase.

$h$	$t_M$
0.1	1.85
0.05	1.85
0.025	1.86
0.01	1.835

## 8.4

1.(a) Using the notation  $f_n = f(t_n, y_n)$ , the predictor formula is

$$y_{n+1} = y_n + \frac{h}{24}(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}).$$

With  $f_{n+1} = f(t_{n+1}, y_{n+1})$ , the corrector formula is

$$y_{n+1} = y_n + \frac{h}{24}(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}).$$

We use the starting values generated by the Runge-Kutta method:

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
$t_n$	0.0	0.1	0.2	0.3
$y_n$	1.0	1.19516	1.38127	1.55918

	$n = 4$ (pre)	$n = 4$ (cor)	$n = 5$ (pre)	$n = 5$ (cor)
$t_n$	0.4	0.4	0.5	0.5
$y_n$	1.72967690	1.72986801	1.89346436	1.89346973

(b) With  $f_{n+1} = f(t_{n+1}, y_{n+1})$ , the fourth order Adams-Moulton formula is

$$y_{n+1} = y_n + \frac{h}{24}(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}).$$

In this problem,  $f_{n+1} = 3 + t_{n+1} - y_{n+1}$ . Since the ODE is linear, we can solve for

$$y_{n+1} = \frac{1}{24 + 9h} [24y_n + 27h + 9ht_{n+1} + h(19f_n - 5f_{n-1} + f_{n-2})].$$

	$n = 4$	$n = 5$
$t_n$	0.4	0.5
$y_n$	1.7296800	1.8934695

(c) The fourth order backward differentiation formula is

$$y_{n+1} = \frac{1}{25} [48y_n - 36y_{n-1} + 16y_{n-2} - 3y_{n-3} + 12hf_{n+1}].$$

In this problem,  $f_{n+1} = 3 + t_{n+1} - y_{n+1}$ . Since the ODE is linear, we can solve for

$$y_{n+1} = \frac{1}{25 + 12h} [36h + 12ht_{n+1} + 48y_n - 36y_{n-1} + 16y_{n-2} - 3y_{n-3}].$$

	$n = 4$	$n = 5$
$t_n$	0.4	0.5
$y_n$	1.7296805	1.8934711

The exact solution of the IVP is  $y(t) = 2 + t - e^{-t}$ .

2.(a) Using the notation  $f_n = f(t_n, y_n)$ , the predictor formula is

$$y_{n+1} = y_n + \frac{h}{24}(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}).$$

With  $f_{n+1} = f(t_{n+1}, y_{n+1})$ , the corrector formula is

$$y_{n+1} = y_n + \frac{h}{24}(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}).$$

We use the starting values generated by the Runge-Kutta method:

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
$t_n$	0.0	0.1	0.2	0.3
$y_n$	2.0	1.62231	1.33362	1.12686

	$n = 4$ (pre)	$n = 4$ (cor)	$n = 5$ (pre)	$n = 5$ (cor)
$t_n$	0.4	0.4	0.5	0.5
$y_n$	0.993751	0.993852	0.925469	0.925764

(b) With  $f_{n+1} = f(t_{n+1}, y_{n+1})$ , the fourth order Adams-Moulton formula is

$$y_{n+1} = y_n + \frac{h}{24}(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}).$$

In this problem,  $f_{n+1} = 5t_{n+1} - 3\sqrt{y_{n+1}}$ . Since the ODE is nonlinear, an equation solver is needed to approximate the solution of

$$y_{n+1} = y_n + \frac{h}{24} [45t_{n+1} - 27\sqrt{y_{n+1}} + 19f_n - 5f_{n-1} + f_{n-2}]$$

at each time step. We obtain the approximate values:

	$n = 4$	$n = 5$
$t_n$	0.4	0.5
$y_n$	0.993847	0.925746

(c) The fourth order backward differentiation formula is

$$y_{n+1} = \frac{1}{25} [48y_n - 36y_{n-1} + 16y_{n-2} - 3y_{n-3} + 12hf_{n+1}].$$

Since the ODE is nonlinear, an equation solver is used to approximate the solution of

$$y_{n+1} = \frac{1}{25} [48y_n - 36y_{n-1} + 16y_{n-2} - 3y_{n-3} + 12h(5t_{n+1} - 3\sqrt{y_{n+1}})]$$

at each time step.

	$n = 4$	$n = 5$
$t_n$	0.4	0.5
$y_n$	0.993869	0.925837

The exact solution of the IVP is given implicitly by

$$\frac{1}{(2\sqrt{y} + 5t)^5(t - \sqrt{y})^2} = \frac{\sqrt{2}}{512}.$$

3.(a) The predictor formula is

$$y_{n+1} = y_n + \frac{h}{24}(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}).$$

With  $f_{n+1} = f(t_{n+1}, y_{n+1})$ , the corrector formula is

$$y_{n+1} = y_n + \frac{h}{24}(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}).$$

Using the starting values generated by the Runge-Kutta method:

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
$t_n$	0.0	0.1	0.2	0.3
$y_n$	1.0	1.205350	1.422954	1.655527

	$n = 4$ (pre)	$n = 4$ (cor)	$n = 5$ (pre)	$n = 5$ (cor)
$t_n$	0.4	0.4	0.5	0.5
$y_n$	1.906340	1.906382	2.179455	2.179567

(b) With  $f_{n+1} = f(t_{n+1}, y_{n+1})$ , the fourth order Adams-Moulton formula is

$$y_{n+1} = y_n + \frac{h}{24}(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}).$$

In this problem,  $f_{n+1} = 2y_{n+1} - 3t_{n+1}$ . Since the ODE is linear, we can solve for

$$y_{n+1} = \frac{1}{24 - 18h} [24y_n - 27ht_{n+1} + h(19f_n - 5f_{n-1} + f_{n-2})].$$

	$n = 4$	$n = 5$
$t_n$	0.4	0.5
$y_n$	1.906385	2.179576

(c) The fourth order backward differentiation formula is

$$y_{n+1} = \frac{1}{25} [48y_n - 36y_{n-1} + 16y_{n-2} - 3y_{n-3} + 12hf_{n+1}].$$

In this problem,  $f_{n+1} = 2y_{n+1} - 3t_{n+1}$ . Since the ODE is linear, we can solve for

$$y_{n+1} = \frac{1}{25 - 24h} [48y_n - 36y_{n-1} + 16y_{n-2} - 3y_{n-3} - 36ht_{n+1}].$$

	$n = 4$	$n = 5$
$t_n$	0.4	0.5
$y_n$	1.906395	2.179611

The exact solution of the IVP is  $y(t) = e^{2t}/4 + 3t/2 + 3/4$ .

5.(a) The predictor formula is

$$y_{n+1} = y_n + \frac{h}{24}(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}).$$

With  $f_{n+1} = f(t_{n+1}, y_{n+1})$ , the corrector formula is

$$y_{n+1} = y_n + \frac{h}{24}(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}).$$

Using the starting values generated by the Runge-Kutta method:

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
$t_n$	0.0	0.1	0.2	0.3
$y_n$	0.5	0.51016950	0.52413795	0.54210529

	$n = 4$ (pre)	$n = 4$ (cor)	$n = 5$ (pre)	$n = 5$ (cor)
$t_n$	0.4	0.4	0.5	0.5
$y_n$	0.56428532	0.56428577	0.59090816	0.59090918

(b) With  $f_{n+1} = f(t_{n+1}, y_{n+1})$ , the fourth order Adams-Moulton formula is

$$y_{n+1} = y_n + \frac{h}{24}(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}).$$

In this problem,

$$f_{n+1} = \frac{y_{n+1}^2 + 2t_{n+1}y_{n+1}}{3 + t_{n+1}^2}.$$

Since the ODE is nonlinear, an equation solver is needed to approximate the solution of

$$y_{n+1} = y_n + \frac{h}{24} \left[ 9 \frac{y_{n+1}^2 + 2t_{n+1}y_{n+1}}{3 + t_{n+1}^2} + 19f_n - 5f_{n-1} + f_{n-2} \right]$$

at each time step.

	$n = 4$	$n = 5$
$t_n$	0.4	0.5
$y_n$	0.56428578	0.59090920

(c) The fourth order backward differentiation formula is

$$y_{n+1} = \frac{1}{25} [48y_n - 36y_{n-1} + 16y_{n-2} - 3y_{n-3} + 12hf_{n+1}].$$

Since the ODE is nonlinear, an equation solver is needed to approximate the solution of

$$y_{n+1} = \frac{1}{25} \left[ 48 y_n - 36 y_{n-1} + 16 y_{n-2} - 3 y_{n-3} + 12h \frac{y_{n+1}^2 + 2 t_{n+1} y_{n+1}}{3 + t_{n+1}^2} \right]$$

at each time step. We obtain the approximate values:

	$n = 4$	$n = 5$
$t_n$	0.4	0.5
$y_n$	0.56428588	0.59090952

The exact solution of the IVP is  $y(t) = (3 + t^2)/(6 - t)$ .

6.(a) The predictor formula is

$$y_{n+1} = y_n + \frac{h}{24} (55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3}).$$

With  $f_{n+1} = f(t_{n+1}, y_{n+1})$ , the corrector formula is

$$y_{n+1} = y_n + \frac{h}{24} (9 f_{n+1} + 19 f_n - 5 f_{n-1} + f_{n-2}).$$

We use the starting values generated by the Runge-Kutta method:

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
$t_n$	0.0	0.1	0.2	0.3
$y_n$	-1.0	-0.924517	-0.864125	-0.816377

	$n = 4$ (pre)	$n = 4$ (cor)	$n = 5$ (pre)	$n = 5$ (cor)
$t_n$	0.4	0.4	0.5	0.5
$y_n$	-0.779832	-0.779693	-0.753311	-0.753135

(b) With  $f_{n+1} = f(t_{n+1}, y_{n+1})$ , the fourth order Adams-Moulton formula is

$$y_{n+1} = y_n + \frac{h}{24} (9 f_{n+1} + 19 f_n - 5 f_{n-1} + f_{n-2}).$$

In this problem,  $f_{n+1} = (t_{n+1}^2 - y_{n+1}^2) \sin y_{n+1}$ . Since the ODE is nonlinear, we obtain the implicit equation

$$y_{n+1} = y_n + \frac{h}{24} [9(t_{n+1}^2 - y_{n+1}^2) \sin y_{n+1} + 19 f_n - 5 f_{n-1} + f_{n-2}].$$

	$n = 4$	$n = 5$
$t_n$	0.4	0.5
$y_n$	-0.779700	-0.753144

(c) The fourth order backward differentiation formula is

$$y_{n+1} = \frac{1}{25} [48 y_n - 36 y_{n-1} + 16 y_{n-2} - 3 y_{n-3} + 12h f_{n+1}].$$

Since the ODE is nonlinear, we obtain the implicit equation

$$y_{n+1} = \frac{1}{25} [48 y_n - 36 y_{n-1} + 16 y_{n-2} - 3 y_{n-3} + 12h(t_{n+1}^2 - y_{n+1}^2) \sin y_{n+1}].$$

	$n = 4$	$n = 5$
$t_n$	0.4	0.5
$y_n$	-0.779680	-0.753089

8.(a) The predictor formula is

$$y_{n+1} = y_n + \frac{h}{24} (55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3}).$$

With  $f_{n+1} = f(t_{n+1}, y_{n+1})$ , the corrector formula is

$$y_{n+1} = y_n + \frac{h}{24} (9 f_{n+1} + 19 f_n - 5 f_{n-1} + f_{n-2}).$$

We use the starting values generated by the Runge-Kutta method:

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
$t_n$	0.0	0.05	0.1	0.15
$y_n$	2.0	1.7996296	1.6223042	1.4672503

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	0.9257133	1.285148	2.408595	4.103495

(b) Since the ODE is nonlinear, an equation solver is needed to approximate the solution of

$$y_{n+1} = y_n + \frac{h}{24} [45t_{n+1} - 27\sqrt{y_{n+1}} + 19 f_n - 5 f_{n-1} + f_{n-2}]$$

at each time step. We obtain the approximate values:

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	0.9257125	1.285148	2.408595	4.103495

(c) The fourth order backward differentiation formula is

$$y_{n+1} = \frac{1}{25} [48 y_n - 36 y_{n-1} + 16 y_{n-2} - 3 y_{n-3} + 12h f_{n+1}].$$

Since the ODE is nonlinear, an equation solver is needed to approximate the solution of

$$y_{n+1} = \frac{1}{25} [48 y_n - 36 y_{n-1} + 16 y_{n-2} - 3 y_{n-3} + 12h(5t_{n+1} - 3\sqrt{y_{n+1}})]$$

at each time step.

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	0.9257248	1.285158	2.408594	4.103493

The exact solution of the IVP is given implicitly by

$$\frac{1}{(2\sqrt{y} + 5t)^5(t - \sqrt{y})^2} = \frac{\sqrt{2}}{512}.$$

9.(a) The predictor formula is

$$y_{n+1} = y_n + \frac{h}{24}(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}).$$

With  $f_{n+1} = f(t_{n+1}, y_{n+1})$ , the corrector formula is

$$y_{n+1} = y_n + \frac{h}{24}(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}).$$

Using the starting values generated by the Runge-Kutta method:

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
$t_n$	0.0	0.05	0.1	0.15
$y_n$	3.0	3.087586	3.177127	3.268609

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	3.962186	5.108903	6.431390	7.923385

(b) With  $f_{n+1} = f(t_{n+1}, y_{n+1})$ , the fourth order Adams-Moulton formula is

$$y_{n+1} = y_n + \frac{h}{24}(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}).$$

In this problem,  $f_{n+1} = \sqrt{t_{n+1} + y_{n+1}}$ . Since the ODE is nonlinear, an equation solver must be implemented in order to approximate the solution of

$$y_{n+1} = y_n + \frac{h}{24} \left[ 9\sqrt{t_{n+1} + y_{n+1}} + 19f_n - 5f_{n-1} + f_{n-2} \right]$$

at each time step.

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	3.962186	5.108903	6.431390	7.923385

(c) The fourth order backward differentiation formula is

$$y_{n+1} = \frac{1}{25} [48y_n - 36y_{n-1} + 16y_{n-2} - 3y_{n-3} + 12hf_{n+1}].$$



Since the ODE is nonlinear, an equation solver is needed to approximate the solution of

$$y_{n+1} = \frac{1}{25} \left[ 48y_n - 36y_{n-1} + 16y_{n-2} - 3y_{n-3} + 12h\sqrt{t_{n+1} + y_{n+1}} \right]$$

at each time step.

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	3.962186	5.108903	6.431390	7.923385

The exact solution is given implicitly by

$$\ln \left[ \frac{2}{y+t-1} \right] + 2\sqrt{t+y} - 2 \operatorname{arctanh} \sqrt{t+y} = t + 2\sqrt{3} - 2 \operatorname{arctanh} \sqrt{3}.$$

10.(a) The predictor formula is

$$y_{n+1} = y_n + \frac{h}{24}(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}).$$

With  $f_{n+1} = f(t_{n+1}, y_{n+1})$ , the corrector formula is

$$y_{n+1} = y_n + \frac{h}{24}(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}).$$

We use the starting values generated by the Runge-Kutta method:

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
$t_n$	0.0	0.05	0.1	0.15
$y_n$	1.0	1.051230	1.104843	1.160740

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	1.612622	2.480909	3.7451479	5.495872

(b) With  $f_{n+1} = f(t_{n+1}, y_{n+1})$ , the fourth order Adams-Moulton formula is

$$y_{n+1} = y_n + \frac{h}{24}(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}).$$

In this problem,  $f_{n+1} = 2t_{n+1} + e^{-t_{n+1}}y_{n+1}$ . Since the ODE is nonlinear, an equation solver must be implemented in order to approximate the solution of

$$y_{n+1} = y_n + \frac{h}{24} \{ 9 [ 2t_{n+1} + e^{-t_{n+1}}y_{n+1} ] + 19f_n - 5f_{n-1} + f_{n-2} \}$$

at each time step.

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	1.612622	2.480909	3.7451479	5.495872

(c) The fourth order backward differentiation formula is

$$y_{n+1} = \frac{1}{25} [48 y_n - 36 y_{n-1} + 16 y_{n-2} - 3 y_{n-3} + 12h f_{n+1}].$$

Since the ODE is nonlinear, we obtain the implicit equation

$$y_{n+1} = \frac{1}{25} \{48 y_n - 36 y_{n-1} + 16 y_{n-2} - 3 y_{n-3} + 12h [2 t_{n+1} + e^{-t_{n+1} y_{n+1}}]\}.$$

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	1.612623	2.480905	3.7451473	5.495869

11.(a) The predictor formula is

$$y_{n+1} = y_n + \frac{h}{24} (55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3}).$$

With  $f_{n+1} = f(t_{n+1}, y_{n+1})$ , the corrector formula is

$$y_{n+1} = y_n + \frac{h}{24} (9 f_{n+1} + 19 f_n - 5 f_{n-1} + f_{n-2}).$$

Using the starting values generated by the Runge-Kutta method:

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
$t_n$	0.0	0.05	0.1	0.15
$y_n$	-2.0	-1.958833	-1.915221	-1.868975

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	-1.447639	-0.1436281	1.060946	1.410122

(b) With  $f_{n+1} = f(t_{n+1}, y_{n+1})$ , the fourth order Adams-Moulton formula is

$$y_{n+1} = y_n + \frac{h}{24} (9 f_{n+1} + 19 f_n - 5 f_{n-1} + f_{n-2}).$$

In this problem,

$$f_{n+1} = \frac{4 - t_{n+1} y_{n+1}}{1 + y_{n+1}^2}.$$

Since the differential equation is nonlinear, an equation solver is used to approximate the solution of

$$y_{n+1} = y_n + \frac{h}{24} \left[ 9 \frac{4 - t_{n+1} y_{n+1}}{1 + y_{n+1}^2} + 19 f_n - 5 f_{n-1} + f_{n-2} \right]$$

at each time step.

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	-1.447638	-0.1436767	1.060913	1.410103

(c) The fourth order backward differentiation formula is

$$y_{n+1} = \frac{1}{25} [48 y_n - 36 y_{n-1} + 16 y_{n-2} - 3 y_{n-3} + 12h f_{n+1}].$$

Since the ODE is nonlinear, an equation solver must be implemented in order to approximate the solution of

$$y_{n+1} = \frac{1}{25} \left[ 48 y_n - 36 y_{n-1} + 16 y_{n-2} - 3 y_{n-3} + 12h \frac{4 - t_{n+1} y_{n+1}}{1 + y_{n+1}^2} \right]$$

at each time step.

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	-1.447621	-0.1447619	1.060717	1.410027

12.(a) The predictor formula is

$$y_{n+1} = y_n + \frac{h}{24} (55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3}).$$

With  $f_{n+1} = f(t_{n+1}, y_{n+1})$ , the corrector formula is

$$y_{n+1} = y_n + \frac{h}{24} (9 f_{n+1} + 19 f_n - 5 f_{n-1} + f_{n-2}).$$

We use the starting values generated by the Runge-Kutta method:

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
$t_n$	0.0	0.05	0.1	0.15
$y_n$	0.5	0.5046218	0.5101695	0.5166666

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	0.5909091	0.8000000	1.166667	1.750000

(b) With  $f_{n+1} = f(t_{n+1}, y_{n+1})$ , the fourth order Adams-Moulton formula is

$$y_{n+1} = y_n + \frac{h}{24} (9 f_{n+1} + 19 f_n - 5 f_{n-1} + f_{n-2}).$$

In this problem,

$$f_{n+1} = \frac{y_{n+1}^2 + 2 t_{n+1} y_{n+1}}{3 + t_{n+1}^2}.$$

Since the ODE is nonlinear, an equation solver is needed to approximate the solution of

$$y_{n+1} = y_n + \frac{h}{24} \left[ 9 \frac{y_{n+1}^2 + 2 t_{n+1} y_{n+1}}{3 + t_{n+1}^2} + 19 f_n - 5 f_{n-1} + f_{n-2} \right]$$

at each time step.

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	0.5909091	0.8000000	1.166667	1.750000

(c) The fourth order backward differentiation formula is

$$y_{n+1} = \frac{1}{25} [48 y_n - 36 y_{n-1} + 16 y_{n-2} - 3 y_{n-3} + 12h f_{n+1}].$$

Since the ODE is nonlinear, we obtain the implicit equation

$$y_{n+1} = \frac{1}{25} \left[ 48 y_n - 36 y_{n-1} + 16 y_{n-2} - 3 y_{n-3} + 12h \frac{y_{n+1}^2 + 2 t_{n+1} y_{n+1}}{3 + t_{n+1}^2} \right].$$

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
$t_n$	0.5	1.0	1.5	2.0
$y_n$	0.5909092	0.8000002	1.166667	1.750001

The exact solution of the IVP is  $y(t) = (3 + t^2)/(6 - t)$ .

13. Both Adams methods entail the approximation of  $f(t, y)$ , on the interval  $[t_n, t_{n+1}]$ , by a polynomial. Approximating  $\phi'(t) = P_1(t) \equiv A$ , which is a constant polynomial, we have

$$\phi(t_{n+1}) - \phi(t_n) = \int_{t_n}^{t_{n+1}} A dt = A(t_{n+1} - t_n) = Ah.$$

Setting  $A = \lambda f_n + (1 - \lambda)f_{n-1}$ , where  $0 \leq \lambda \leq 1$ , we obtain the approximation

$$y_{n+1} = y_n + h [\lambda f_n + (1 - \lambda)f_{n-1}].$$

An appropriate choice of  $\lambda$  yields the familiar Euler formula. Similarly, setting

$$A = \lambda f_n + (1 - \lambda)f_{n+1},$$

where  $0 \leq \lambda \leq 1$ , we obtain the approximation

$$y_{n+1} = y_n + h [\lambda f_n + (1 - \lambda)f_{n+1}].$$

14. For a third order Adams-Bashforth formula, we approximate  $f(t, y)$ , on the interval  $[t_n, t_{n+1}]$ , by a quadratic polynomial using the points  $(t_{n-2}, y_{n-2})$ ,  $(t_{n-1}, y_{n-1})$  and  $(t_n, y_n)$ . Let  $P_3(t) = At^2 + Bt + C$ . We obtain the system of equations

$$\begin{aligned} At_{n-2}^2 + Bt_{n-2} + C &= f_{n-2} \\ At_{n-1}^2 + Bt_{n-1} + C &= f_{n-1} \\ At_n^2 + Bt_n + C &= f_n. \end{aligned}$$

For computational purposes, assume that  $t_0 = 0$ , and  $t_n = nh$ . It follows that

$$\begin{aligned} A &= \frac{f_n - 2f_{n-1} + f_{n-2}}{2h^2} \\ B &= \frac{(3-2n)f_n + (4n-4)f_{n-1} + (1-2n)f_{n-2}}{2h} \\ C &= \frac{n^2 - 3n + 2}{2}f_n + (2n - n^2)f_{n-1} + \frac{n^2 - n}{2}f_{n-2}. \end{aligned}$$

We then have

$$y_{n+1} - y_n = \int_{t_n}^{t_{n+1}} [At^2 + Bt + C] dt = Ah^3(n^2 + n + \frac{1}{3}) + Bh^2(n + \frac{1}{2}) + Ch,$$

which yields

$$y_{n+1} - y_n = \frac{h}{12}(23f_n - 16f_{n-1} + 5f_{n-2}).$$

15. For a third order Adams-Moulton formula, we approximate  $f(t, y)$ , on the interval  $[t_n, t_{n+1}]$ , by a quadratic polynomial using the points  $(t_{n-1}, y_{n-1})$ ,  $(t_n, y_n)$  and  $(t_{n+1}, y_{n+1})$ . Let  $P_3(t) = \alpha t^2 + \beta t + \gamma$ . This time we obtain the system of algebraic equations

$$\begin{aligned} \alpha t_{n-1}^2 + \beta t_{n-1} + \gamma &= f_{n-1} \\ \alpha t_n^2 + \beta t_n + \gamma &= f_n \\ \alpha t_{n+1}^2 + \beta t_{n+1} + \gamma &= f_{n+1}. \end{aligned}$$

For computational purposes, again assume that  $t_0 = 0$ , and  $t_n = nh$ . It follows that

$$\begin{aligned} \alpha &= \frac{f_{n-1} - 2f_n + f_{n+1}}{2h^2} \\ \beta &= \frac{-(2n+1)f_{n-1} + 4nf_n + (1-2n)f_{n+1}}{2h} \\ \gamma &= \frac{n^2 + n}{2}f_{n-1} + (1 - n^2)f_n + \frac{n^2 - n}{2}f_{n+1}. \end{aligned}$$

We then have

$$y_{n+1} - y_n = \int_{t_n}^{t_{n+1}} [\alpha t^2 + \beta t + \gamma] dt = \alpha h^3(n^2 + n + \frac{1}{3}) + \beta h^2(n + \frac{1}{2}) + \gamma h,$$

which results in

$$y_{n+1} - y_n = \frac{h}{12}(5f_{n+1} + 8f_n - f_{n-1}).$$

## 8.5

1. In vector notation, the initial value problem can be written as

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y + t \\ 4x - 2y \end{pmatrix}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(a) The Euler formula is

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} + h \begin{pmatrix} x_n + y_n + t_n \\ 4x_n - 2y_n \end{pmatrix}.$$

That is,

$$\begin{aligned} x_{n+1} &= x_n + h(x_n + y_n + t_n) \\ y_{n+1} &= y_n + h(4x_n - 2y_n). \end{aligned}$$

With  $h = 0.1$ , we obtain the values

	$n = 2$	$n = 4$	$n = 6$	$n = 8$	$n = 10$
$t_n$	0.2	0.4	0.6	0.8	1.0
$x_n$	1.26	1.7714	2.58991	3.82374	5.64246
$y_n$	0.76	1.4824	2.3703	3.60413	5.38885

(b) The Runge-Kutta method uses the following intermediate calculations:

$$\begin{aligned} \mathbf{k}_{n1} &= (x_n + y_n + t_n, 4x_n - 2y_n)^T \\ \mathbf{k}_{n2} &= \left[ x_n + \frac{h}{2}k_{n1}^1 + y_n + \frac{h}{2}k_{n1}^2 + t_n + \frac{h}{2}, 4\left(x_n + \frac{h}{2}k_{n1}^1\right) - 2\left(y_n + \frac{h}{2}k_{n1}^2\right) \right]^T \\ \mathbf{k}_{n3} &= \left[ x_n + \frac{h}{2}k_{n2}^1 + y_n + \frac{h}{2}k_{n2}^2 + t_n + \frac{h}{2}, 4\left(x_n + \frac{h}{2}k_{n2}^1\right) - 2\left(y_n + \frac{h}{2}k_{n2}^2\right) \right]^T \\ \mathbf{k}_{n4} &= \left[ x_n + hk_{n3}^1 + y_n + hk_{n3}^2 + t_n + h, 4\left(x_n + hk_{n3}^1\right) - 2\left(y_n + hk_{n3}^2\right) \right]^T. \end{aligned}$$

With  $h = 0.2$ , we obtain the values:

	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$t_n$	0.2	0.4	0.6	0.8	1.0
$x_n$	1.32493	1.93679	2.93414	4.48318	6.84236
$y_n$	0.758933	1.57919	2.66099	4.22639	6.56452

(c) With  $h = 0.1$ , we obtain

	$n = 2$	$n = 4$	$n = 6$	$n = 8$	$n = 10$
$t_n$	0.2	0.4	0.6	0.8	1.0
$x_n$	1.32489	1.9369	2.93459	4.48422	6.8444
$y_n$	0.759516	1.57999	2.66201	4.22784	6.56684

The exact solution of the IVP is

$$\begin{aligned} x(t) &= e^{2t} + \frac{2}{9}e^{-3t} - \frac{1}{3}t - \frac{2}{9} \\ y(t) &= e^{2t} - \frac{8}{9}e^{-3t} - \frac{2}{3}t - \frac{1}{9}. \end{aligned}$$

3.(a) The Euler formula is

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} + h \begin{pmatrix} -t_n x_n - y_n - 1 \\ x_n \end{pmatrix}.$$

That is,

$$\begin{aligned}x_{n+1} &= x_n + h(-t_n x_n - y_n - 1) \\ y_{n+1} &= y_n + h(x_n).\end{aligned}$$

With  $h = 0.1$ , we obtain the values

	$n = 2$	$n = 4$	$n = 6$	$n = 8$	$n = 10$
$t_n$	0.2	0.4	0.6	0.8	1.0
$x_n$	0.582	0.117969	-0.336912	-0.730007	-1.02134
$y_n$	1.18	1.27344	1.27382	1.18572	1.02371

(b) The Runge-Kutta method uses the following intermediate calculations:

$$\begin{aligned}\mathbf{k}_{n1} &= (-t_n x_n - y_n - 1, x_n)^T \\ \mathbf{k}_{n2} &= \left[ -(t_n + \frac{h}{2})(x_n + \frac{h}{2}k_{n1}^1) - (y_n + \frac{h}{2}k_{n1}^2) - 1, x_n + \frac{h}{2}k_{n1}^1 \right]^T \\ \mathbf{k}_{n3} &= \left[ -(t_n + \frac{h}{2})(x_n + \frac{h}{2}k_{n2}^1) - (y_n + \frac{h}{2}k_{n2}^2) - 1, x_n + \frac{h}{2}k_{n2}^1 \right]^T \\ \mathbf{k}_{n4} &= \left[ -(t_n + h)(x_n + hk_{n3}^1) - (y_n + hk_{n3}^2) - 1, x_n + hk_{n3}^1 \right]^T.\end{aligned}$$

With  $h = 0.2$ , we obtain the values:

	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$t_n$	0.2	0.4	0.6	0.8	1.0
$x_n$	0.568451	0.109776	-0.32208	-0.681296	-0.937852
$y_n$	1.15775	1.22556	1.20347	1.10162	0.937852

(c) With  $h = 0.1$ , we obtain

	$n = 2$	$n = 4$	$n = 6$	$n = 8$	$n = 10$
$t_n$	0.2	0.4	0.6	0.8	1.0
$x_n$	0.56845	0.109773	-0.322081	-0.681291	-0.937841
$y_n$	1.15775	1.22557	1.20347	1.10161	0.93784

4.(a) The Euler formula gives

$$\begin{aligned}x_{n+1} &= x_n + h(x_n - y_n + x_n y_n) \\ y_{n+1} &= y_n + h(3x_n - 2y_n - x_n y_n).\end{aligned}$$

With  $h = 0.1$ , we obtain the values

	$n = 2$	$n = 4$	$n = 6$	$n = 8$	$n = 10$
$t_n$	0.2	0.4	0.6	0.8	1.0
$x_n$	-0.198	-0.378796	-0.51932	-0.594324	-0.588278
$y_n$	0.618	0.28329	-0.0321025	-0.326801	-0.57545

(b) Given

$$\begin{aligned} f(t, x, y) &= x - y + xy \\ g(t, x, y) &= 3x - 2y - xy, \end{aligned}$$

the Runge-Kutta method uses the following intermediate calculations:

$$\begin{aligned} \mathbf{k}_{n1} &= [f(t_n, x_n, y_n), g(t_n, x_n, y_n)]^T \\ \mathbf{k}_{n2} &= \left[ f\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_{n1}^1, y_n + \frac{h}{2}k_{n1}^2\right), g\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_{n1}^1, y_n + \frac{h}{2}k_{n1}^2\right) \right]^T \\ \mathbf{k}_{n3} &= \left[ f\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_{n2}^1, y_n + \frac{h}{2}k_{n2}^2\right), g\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_{n2}^1, y_n + \frac{h}{2}k_{n2}^2\right) \right]^T \\ \mathbf{k}_{n4} &= [f(t_n + h, x_n + hk_{n3}^1, y_n + hk_{n3}^2), g(t_n + h, x_n + hk_{n3}^1, y_n + hk_{n3}^2)]^T. \end{aligned}$$

With  $h = 0.2$ , we obtain the values:

	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$t_n$	0.2	0.4	0.6	0.8	1.0
$x_n$	-0.196904	-0.372643	-0.501302	-0.561270	-0.547053
$y_n$	0.630936	0.298888	-0.0111429	-0.288943	-0.508303

(c) With  $h = 0.1$ , we obtain

	$n = 2$	$n = 4$	$n = 6$	$n = 8$	$n = 10$
$t_n$	0.2	0.4	0.6	0.8	1.0
$x_n$	-0.196935	-0.372687	-0.501345	-0.561292	-0.547031
$y_n$	0.630939	0.298866	-0.0112184	-0.28907	-0.508427

5.(a) The Euler formula gives

$$\begin{aligned} x_{n+1} &= x_n + h[x_n(1 - 0.5x_n - 0.5y_n)] \\ y_{n+1} &= y_n + h[y_n(-0.25 + 0.5x_n)]. \end{aligned}$$

With  $h = 0.1$ , we obtain the values

	$n = 2$	$n = 4$	$n = 6$	$n = 8$	$n = 10$
$t_n$	0.2	0.4	0.6	0.8	1.0
$x_n$	2.96225	2.34119	1.90236	1.56602	1.29768
$y_n$	1.34538	1.67121	1.97158	2.23895	2.46732

(b) Given

$$\begin{aligned} f(t, x, y) &= x(1 - 0.5x - 0.5y) \\ g(t, x, y) &= y(-0.25 + 0.5x), \end{aligned}$$



the Runge-Kutta method uses the following intermediate calculations:

$$\begin{aligned} \mathbf{k}_{n1} &= [f(t_n, x_n, y_n), g(t_n, x_n, y_n)]^T \\ \mathbf{k}_{n2} &= \left[ f\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_{n1}^1, y_n + \frac{h}{2}k_{n1}^2\right), g\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_{n1}^1, y_n + \frac{h}{2}k_{n1}^2\right) \right]^T \\ \mathbf{k}_{n3} &= \left[ f\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_{n2}^1, y_n + \frac{h}{2}k_{n2}^2\right), g\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_{n2}^1, y_n + \frac{h}{2}k_{n2}^2\right) \right]^T \\ \mathbf{k}_{n4} &= [f(t_n + h, x_n + hk_{n3}^1, y_n + hk_{n3}^2), g(t_n + h, x_n + hk_{n3}^1, y_n + hk_{n3}^2)]^T. \end{aligned}$$

With  $h = 0.2$ , we obtain the values:

	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$t_n$	0.2	0.4	0.6	0.8	1.0
$x_n$	3.06339	2.44497	1.9911	1.63818	1.3555
$y_n$	1.34858	1.68638	2.00036	2.27981	2.5175

(c) With  $h = 0.1$ , we obtain

	$n = 2$	$n = 4$	$n = 6$	$n = 8$	$n = 10$
$t_n$	0.2	0.4	0.6	0.8	1.0
$x_n$	3.06314	2.44465	1.99075	1.63781	1.35514
$y_n$	1.34899	1.68699	2.00107	2.28057	2.51827

6.(a) The Euler formula gives

$$\begin{aligned} x_{n+1} &= x_n + h [e^{-x_n + y_n} - \cos x_n] \\ y_{n+1} &= y_n + h [\sin(x_n - 3y_n)]. \end{aligned}$$

With  $h = 0.1$ , we obtain the values

	$n = 2$	$n = 4$	$n = 6$	$n = 8$	$n = 10$
$t_n$	0.2	0.4	0.6	0.8	1.0
$x_n$	1.42386	1.82234	2.21728	2.61118	2.9955
$y_n$	2.18957	2.36791	2.53329	2.68763	2.83354

(b) The Runge-Kutta method uses the following intermediate calculations:

$$\begin{aligned} \mathbf{k}_{n1} &= [f(t_n, x_n, y_n), g(t_n, x_n, y_n)]^T \\ \mathbf{k}_{n2} &= \left[ f\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_{n1}^1, y_n + \frac{h}{2}k_{n1}^2\right), g\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_{n1}^1, y_n + \frac{h}{2}k_{n1}^2\right) \right]^T \\ \mathbf{k}_{n3} &= \left[ f\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_{n2}^1, y_n + \frac{h}{2}k_{n2}^2\right), g\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_{n2}^1, y_n + \frac{h}{2}k_{n2}^2\right) \right]^T \\ \mathbf{k}_{n4} &= [f(t_n + h, x_n + hk_{n3}^1, y_n + hk_{n3}^2), g(t_n + h, x_n + hk_{n3}^1, y_n + hk_{n3}^2)]^T. \end{aligned}$$

With  $h = 0.2$ , we obtain the values:

	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$t_n$	0.2	0.4	0.6	0.8	1.0
$x_n$	1.41513	1.81208	2.20635	2.59826	2.97806
$y_n$	2.18699	2.36233	2.5258	2.6794	2.82487

(c) With  $h = 0.1$ , we obtain

	$n = 2$	$n = 4$	$n = 6$	$n = 8$	$n = 10$
$t_n$	0.2	0.4	0.6	0.8	1.0
$x_n$	1.41513	1.81209	2.20635	2.59826	2.97806
$y_n$	2.18699	2.36233	2.52581	2.67941	2.82488

7. The Runge-Kutta method uses the following intermediate calculations:

$$\begin{aligned} \mathbf{k}_{n1} &= [x_n - 4y_n, -x_n + y_n]^T \\ \mathbf{k}_{n2} &= \left[ x_n + \frac{h}{2}k_{n1}^1 - 4(y_n + \frac{h}{2}k_{n1}^2), -(x_n + \frac{h}{2}k_{n1}^1) + y_n + \frac{h}{2}k_{n1}^2 \right]^T \\ \mathbf{k}_{n3} &= \left[ x_n + \frac{h}{2}k_{n2}^1 - 4(y_n + \frac{h}{2}k_{n2}^2), -(x_n + \frac{h}{2}k_{n2}^1) + y_n + \frac{h}{2}k_{n2}^2 \right]^T \\ \mathbf{k}_{n4} &= [x_n + hk_{n3}^1 - 4(y_n + hk_{n3}^2), -(x_n + hk_{n3}^1) + y_n + hk_{n3}^2]^T. \end{aligned}$$

Using  $h = 0.05$ , we obtain the following values:

	$n = 4$	$n = 8$	$n = 12$	$n = 16$	$n = 20$
$t_n$	0.2	0.4	0.6	0.8	1.0
$x_n$	1.3204	1.9952	3.2992	5.7362	10.227
$y_n$	-0.25085	-0.66245	-1.3752	-2.6434	-4.9294

Using  $h = 0.025$ , we obtain the following values:

	$n = 8$	$n = 16$	$n = 24$	$n = 32$	$n = 40$
$t_n$	0.2	0.4	0.6	0.8	1.0
$x_n$	1.3204	1.9952	3.2992	5.7362	10.227
$y_n$	-0.25085	-0.66245	-1.3752	-2.6435	-4.9294

The exact solution is given by

$$\phi(t) = \frac{e^{-t} + e^{3t}}{2}, \quad \psi(t) = \frac{e^{-t} - e^{3t}}{4},$$

and the associated tabulated values:

	$n = 5$	$n = 10$	$n = 15$	$n = 20$	$n = 25$
$t_n$	0.2	0.4	0.6	0.8	1.0
$\phi(t_n)$	1.3204	1.9952	3.2992	5.7362	10.227
$\psi(t_n)$	-0.25085	-0.66245	-1.3752	-2.6435	-4.9294

9. The predictor formulas are

$$x_{n+1} = x_n + \frac{h}{24}(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3})$$

$$y_{n+1} = y_n + \frac{h}{24}(55g_n - 59g_{n-1} + 37g_{n-2} - 9g_{n-3}).$$

With  $f_{n+1} = x_{n+1} - 4y_{n+1}$  and  $g_{n+1} = -x_{n+1} + y_{n+1}$ , the corrector formulas are

$$x_{n+1} = x_n + \frac{h}{24}(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2})$$

$$y_{n+1} = y_n + \frac{h}{24}(9g_{n+1} + 19g_n - 5g_{n-1} + g_{n-2}).$$

We use the starting values from the exact solution:

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
$t_n$	0	0.1	0.2	0.3
$x_n$	1.0	1.12883	1.32042	1.60021
$y_n$	0.0	-0.11057	-0.250847	-0.429696

One time step using the predictor-corrector method results in the approximate values:

	$n = 4$ (pre)	$n = 4$ (cor)
$t_n$	0.4	0.4
$x_n$	1.99445	1.99521
$y_n$	-0.662064	-0.662442

## 8.6

3. The solution of the initial value problem is  $\phi(t) = e^{-100t} + t$ .

(a,b) Based on the exact solution, the local truncation error for both of the Euler methods is

$$|e_{loc}| \leq \frac{10^4}{2} e^{-100\bar{t}_n} h^2.$$

Hence  $|e_n| \leq 5000h^2$ , for all  $0 < \bar{t}_n < 1$ . Furthermore, the local truncation error is greatest near  $t = 0$ . Therefore  $|e_1| \leq 5000h^2 < 0.0005$  for  $h < 0.0003$ . Now the truncation error accumulates at each time step. Therefore the actual time step should be much smaller than  $h \approx 0.0003$ . For example, with  $h = 0.00025$ , we obtain the data

	Euler	B.Euler	$\phi(t)$
$t = 0.05$	0.056323	0.057165	0.056738
$t = 0.1$	0.100040	0.100051	0.100045

Note that the total number of time steps needed to reach  $t = 0.1$  is  $N = 400$ .

(c) Using the Runge-Kutta method, comparisons are made for several values of  $h$ ;  $h = 0.005$  is sufficient.

$h = 0.1$  :

	$\phi(t)$	$y_n$	$y_n - \phi(t_n)$
$t = 0.05$	0.056738	0.057416	0.000678
$t = 0.1$	0.100045	0.100055	0.000010

$h = 0.005$  :

	$\phi(t)$	$y_n$	$y_n - \phi(t_n)$
$t = 0.05$	0.056738	0.056766	0.000027
$t = 0.1$	0.100045	0.100046	0.0000004

6.(a) Using the method of undetermined coefficients, it is easy to show that the general solution of the ODE is  $y(t) = ce^{\lambda t} + t^2$ . Imposing the initial condition, it follows that  $c = 0$  and hence the solution of the IVP is  $\phi(t) = t^2$ .

(b) Using the Runge-Kutta method, with  $h = 0.01$ , numerical solutions are generated for various values of  $\lambda$  :

$\lambda = 1$  :

	$\phi(t)$	$y_n$	$ y_n - \phi(t_n) $
$t = 0.25$	0.0625	0.0624999	$2 \times 10^{-11}$
$t = 0.5$	0.25	0.25	0
$t = 0.75$	0.5625	0.5625	0
$t = 1.0$	1.0	1.0	0

$\lambda = 10$  :

	$\phi(t)$	$y_n$	$ y_n - \phi(t_n) $
$t = 0.25$	0.0625	0.0624998	$2.215 \times 10^{-7}$
$t = 0.5$	0.25	0.249997	$2.920 \times 10^{-6}$
$t = 0.75$	0.5625	0.562464	$3.579 \times 10^{-5}$
$t = 1.0$	1.0	0.999564	$4.362 \times 10^{-4}$

$\lambda = 20$  :

	$\phi(t)$	$y_n$	$ y_n - \phi(t_n) $
$t = 0.25$	0.0625	0.062489	$1.10 \times 10^{-5}$
$t = 0.5$	0.25	0.248342	$1.658 \times 10^{-3}$
$t = 0.75$	0.5625	0.316455	0.246045
$t = 1.0$	1.0	-35.5143	36.5143

$\lambda = 50$  :

	$\phi(t)$	$y_n$	$ y_n - \phi(t_n) $
$t = 0.25$	0.0625	-0.044803	0.107303
$t = 0.5$	0.25	-28669.55	28669.804
$t = 0.75$	0.5625	$-7.66014 \times 10^9$	$7.66014 \times 10^9$
$t = 1.0$	1.0	$-2.04668 \times 10^{15}$	$2.04668 \times 10^{15}$

The following table shows the calculated value,  $y_1$ , at the first time step:

$\phi(t)$	$y_1(\lambda = 1)$	$y_1(\lambda = 10)$	$y_1(\lambda = 20)$	$y_1(\lambda = 50)$
$10^{-2}$	$9.99999 \times 10^{-5}$	$9.99979 \times 10^{-5}$	$9.99833 \times 10^{-5}$	$9.97396 \times 10^{-5}$

(c) Referring back to the exact solution given in part (a), if a nonzero initial condition, say  $y(0) = \varepsilon$ , is specified, the solution of the IVP becomes

$$\phi_\varepsilon(t) = \varepsilon e^{\lambda t} + t^2.$$

We then have  $|\phi(t) - \phi_\varepsilon(t)| = |\varepsilon| e^{\lambda t}$ . It is evident that for any  $t > 0$ ,

$$\lim_{\lambda \rightarrow \infty} |\phi(t) - \phi_\varepsilon(t)| = \infty.$$

This implies that virtually any error introduced early in the calculations will be magnified as  $\lambda \rightarrow \infty$ . The initial value problem is inherently unstable.

