CHAPTER

9

Nonlinear Differential Equations and Stability

9.1

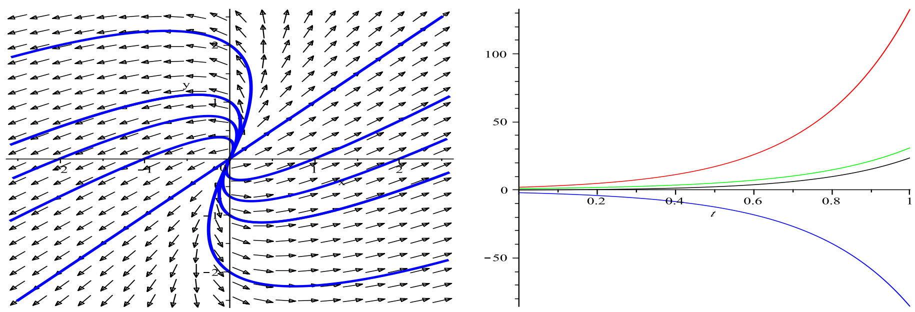
2.(a) Setting $\mathbf{x} = \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 5-r & -1 \\ 3 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\,\mathbf{I}) = r^2 - 6\,r + 8 = 0$. The roots of the characteristic equation are $r_1 = 2$ and $r_2 = 4$. For $r = 2$, the system of equations reduces to $3\xi_1 = \xi_2$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1\,,3)^T$. Substitution of $r = 4$ results in the single equation $\xi_1 = \xi_2$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (1\,,1)^T$.

(b) The eigenvalues are real and positive, hence the critical point is an unstable node.

(c,d)



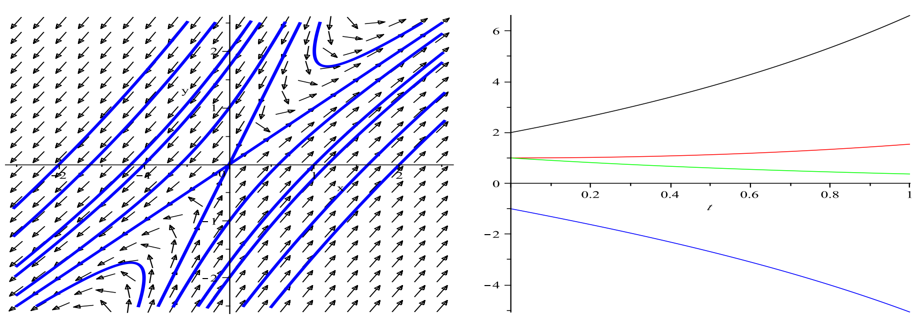
3.(a) Solution of the ODE requires analysis of the algebraic equations

$$\begin{pmatrix} 2-r & -1 \\ 3 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\,\mathbf{I}) = r^2 - 1 = 0$. The roots of the characteristic equation are $r_1 = 1$ and $r_2 = -1$. For $r = 1$, the system of equations reduces to $\xi_1 = \xi_2$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1\,,1)^T$. Substitution of $r = -1$ results in the single equation $3\,\xi_1 - \xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (1\,,3)^T$.

(b) The eigenvalues are real, with $r_1\,r_2 < 0$. Hence the critical point is an unstable saddle point.

(c,d)



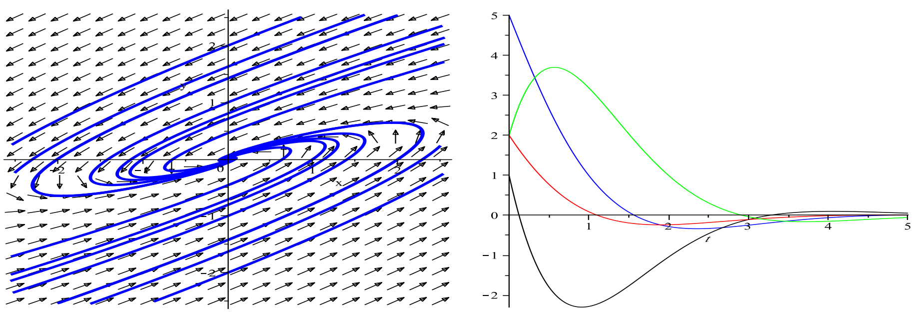
5.(a) The characteristic equation is given by

$$\begin{vmatrix} 1-r & -5 \\ 1 & -3-r \end{vmatrix} = r^2 + 2\,r + 2 = 0\,.$$

The equation has complex roots $r_1 = -1 + i$ and $r_2 = -1 - i$. For $r = -1 + i$, the components of the solution vector must satisfy $\xi_1 - (2+i)\xi_2 = 0$. Thus the corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (2+i\,,1)^T$. Substitution of $r = -1 - i$ results in the single equation $\xi_1 - (2-i)\xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (2-i\,,1)^T$.

(b) The eigenvalues are complex conjugates, with negative real part. Hence the origin is an asymptotically stable spiral.

(c,d)



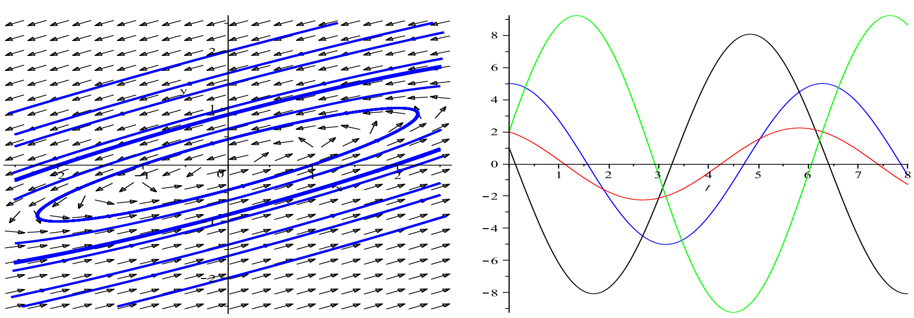
6.(a) Solution of the ODEs is based on the analysis of the algebraic equations

$$\begin{pmatrix} 2-r & -5 \\ 1 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we require that $\det(\mathbf{A} - r\,\mathbf{I}) = r^2 + 1 = 0$. The roots of the characteristic equation are $r = \pm i$. Setting $r = i$, the equations are equivalent to $\xi_1 - (2+i)\xi_2 = 0$. The eigenvectors are $\boldsymbol{\xi}^{(1)} = (2+i\,,1)^T$ and $\boldsymbol{\xi}^{(2)} = (2-i\,,1)^T$.

(b) The eigenvalues are purely imaginary. Hence the critical point is a stable center.

(c,d)



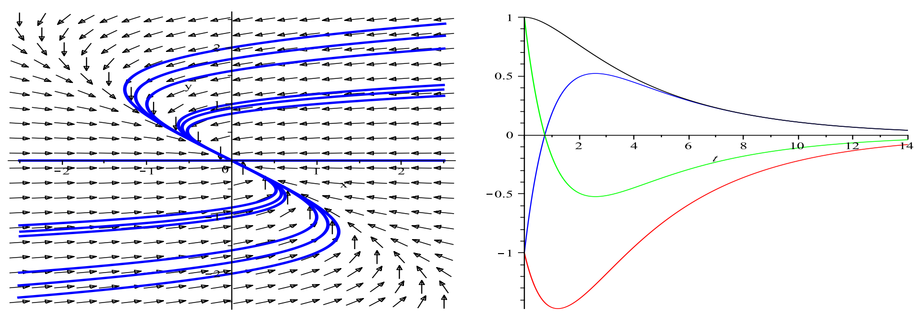
8.(a) The characteristic equation is given by

$$\begin{vmatrix} -1-r & -1 \\ 0 & -1/4-r \end{vmatrix} = (r+1)(r+1/4) = 0\,,$$

with roots $r_1 = -1$ and $r_2 = -1/4$. For $r = -1$, the components of the solution vector must satisfy $\xi_2 = 0$. Thus the corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1\,,0)^T$. Substitution of $r = -1/4$ results in the single equation $3\xi_1/4 + \xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (4\,,-3)^T$.

(b) The eigenvalues are real and both negative. Hence the critical point is an asymptotically stable node.

(c,d)



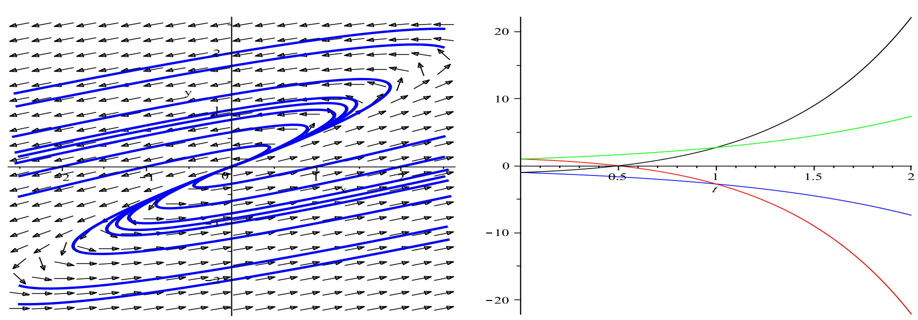
9.(a) Solution of the ODEs is based on the analysis of the algebraic equations

$$\begin{pmatrix} 3-r & -4 \\ 1 & -1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we require that $\det(\mathbf{A} - r\,\mathbf{I}) = r^2 - 2\,r + 1 = 0$. The single root of the characteristic equation is $r = 1$. Setting $r = 1$, the components of the solution vector must satisfy $\xi_1 - 2\,\xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi} = (2\,,1)^T$.

(b) Since there is only one linearly independent eigenvector, the critical point is an unstable, improper node.

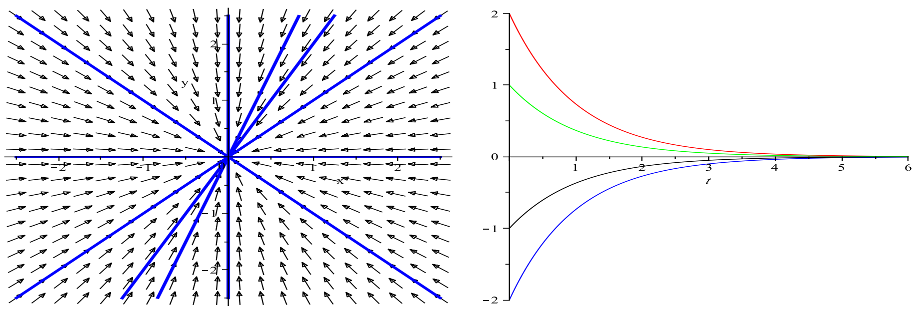
(c,d)



11.(a) The characteristic equation is $(r+1)^2 = 0$, with double root $r = -1$. It is easy to see that the two linearly independent eigenvectors are $\boldsymbol{\xi}^{(1)} = (1\,,0)^T$ and $\boldsymbol{\xi}^{(2)} = (0\,,1)^T$.

(b) Since there are two linearly independent eigenvectors, the critical point is an asymptotically stable proper node.

(c,d)



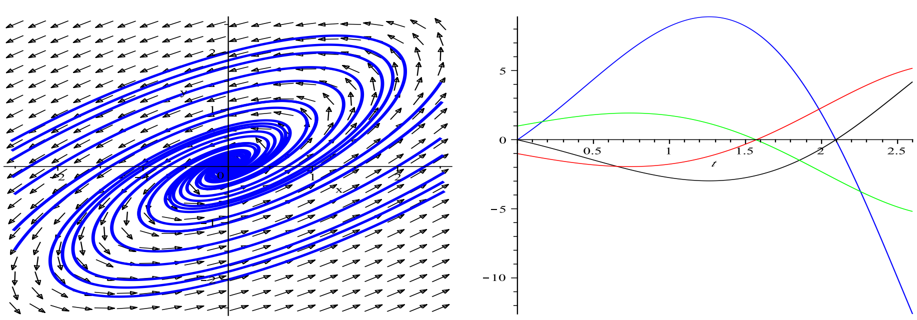
12.(a) Setting $\mathbf{x} = \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 2-r & -5/2 \\ 9/5 & -1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we require that $\det(\mathbf{A} - r\,\mathbf{I}) = r^2 - r + 5/2 = 0$. The roots of the characteristic equation are $r = 1/2 \pm 3i/2$. Substituting $r = 1/2 - 3i/2$, the equations reduce to $(3+3i)\xi_1 - 5\,\xi_2 = 0$. Therefore the two eigenvectors are $\boldsymbol{\xi}^{(1)} = (5\,, 3+3i)^T$ and $\boldsymbol{\xi}^{(2)} = (5\,, 3-3i)^T$.

(b) Since the eigenvalues are complex, with positive real part, the critical point is an unstable spiral.

(c,d)



14. Setting $\mathbf{x}' = \mathbf{0}$, that is,

$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 \\ -1 \end{pmatrix},$$

we find that the critical point is $\mathbf{x}^0 = (-1, 0)^T$. With the change of dependent variable, $\mathbf{x} = \mathbf{x}^0 + \mathbf{u}$, the differential equation can be written as

$$\frac{d\mathbf{u}}{dt} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{u}.$$

The critical point for the transformed equation is the origin. Setting $\mathbf{u}= \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} -2-r & 1 \\ 1 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we require that $\det(\mathbf{A} - r\,\mathbf{I}) = r^2 + 4r + 3 = 0$. The roots of the characteristic equation are $r = -3\,,\ -1\,.$ Hence the critical point is an asymptotically stable node.

15. Setting $\mathbf{x}' = \mathbf{0}$, that is,

$$\begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ -5 \end{pmatrix},$$

we find that the critical point is $\mathbf{x}^0 = (-2, 1)^T$. With the change of dependent variable, $\mathbf{x}{=}\mathbf{x}^0{+}\mathbf{u}$, the differential equation can be written as

$$\frac{d\mathbf{u}}{dt} = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix} \mathbf{u}.$$

The characteristic equation is $\det(\mathbf{A} - r\,\mathbf{I}) = r^2 + 2r + 3 = 0$, with complex conjugate roots $r = -1 \pm i\sqrt{2}$. Since the real parts of the eigenvalues are negative, the critical point is an asymptotically stable spiral.

16. The critical point $\mathbf{x}^0$ satisfies the system of equations

$$\begin{pmatrix} 0 & -\beta \\ \delta & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -\alpha \\ \gamma \end{pmatrix}.$$

It follows that $x^0 = \gamma/\delta$ and $y^0 = \alpha/\beta$. Using the transformation, $\mathbf{x}{=}\mathbf{x}^0{+}\mathbf{u}$, the differential equation can be written as

$$\frac{d\mathbf{u}}{dt} = \begin{pmatrix} 0 & -\beta \\ \delta & 0 \end{pmatrix} \mathbf{u}.$$

The characteristic equation is $\det(\mathbf{A} - r\,\mathbf{I}) = r^2 + \beta\,\delta = 0$. Since $\beta\,\delta > 0$, the roots are purely imaginary, with $r = \pm\, i\sqrt{\beta\delta}$. Hence the critical point is a stable center.

21.(a) If $q > 0$ and $p < 0$, then the roots are either complex conjugates with negative real parts, or both real and negative.

(b) If $q > 0$ and $p = 0$, then the roots are purely imaginary.

(c) If $q < 0$, then the roots are real, with $r_1 \cdot r_2 < 0$. If $p > 0$, then either the roots are real, with $r_1 > 0$ or the roots are complex conjugates with positive real parts.

9.2

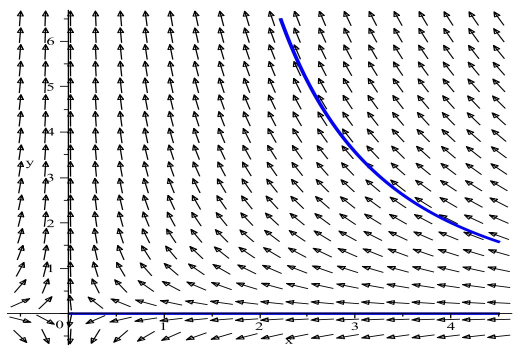
2. The differential equations can be combined to obtain a related ODE

$$\frac{dy}{dx} = -\frac{2y}{x}.$$

The equation is separable, with

$$\frac{dy}{y} = -\frac{2\,dx}{x}.$$

The solution is given by $y = C\,x^{-2}$. Note that the system is uncoupled, and hence we also have $x = x_0 e^{-t}$ and $y = y_0 e^{2t}$. Matching the initial conditions, for the first case we obtain $x(t) = 4e^{-t}$ and $y(t) = 2e^{2t}$, for the second case we obtain $x(t) = 4e^{-t}$ and $y(t) = 0$.



In order to determine the direction of motion along the trajectories, observe that for positive initial conditions, x will decrease, whereas y will increase.

4. The trajectories of the system satisfy the ODE

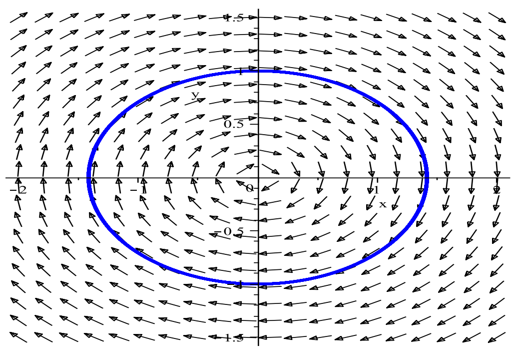
$$\frac{dy}{dx} = -\frac{bx}{ay}.$$

The equation is separable, with $ay\,dy = -bx\,dx$. Hence the trajectories are given by $b\,x^2 + a\,y^2 = C^2$, in which C is arbitrary. Evidently, the trajectories are ellipses. Invoking the initial condition, we find that $C^2 = ab$. The system of ODEs can also be written as

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix}\mathbf{x}.$$

Using the methods in Chapter 7, it is easy to show that

$$x = \sqrt{a}\,\cos\sqrt{ab}\,t, \qquad y = -\sqrt{b}\,\sin\sqrt{ab}\,t.$$



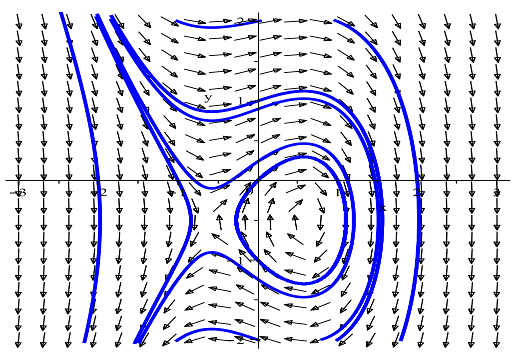
Note that for positive initial conditions, x will increase, whereas y will decrease.

6.(a) The critical points are solutions of the equations

$$1 + 2y = 0$$
$$1 - 3x^2 = 0\,.$$

There are two critical points, $(-1/\sqrt{3}\,, -1/2)$ and $(1/\sqrt{3}\,, -1/2)$.

(b)



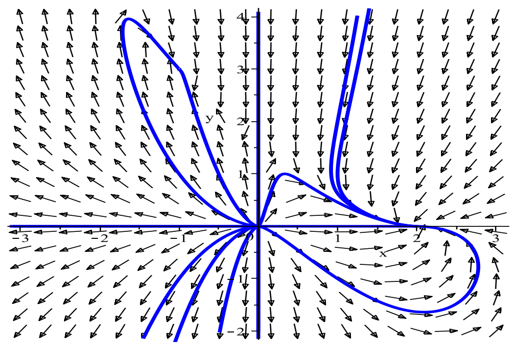
(c) Locally, the trajectories near the point $(-1/\sqrt{3}\,, -1/2)$ resemble the behavior near a saddle. Hence the critical point is unstable. Near the point $(1/\sqrt{3}\,, -1/2)$, the solutions are periodic. Therefore the second critical point is stable.

7.(a) The critical points are solutions of the equations

$$2x - x^2 - xy = 0$$
$$3y - 2y^2 - 3xy = 0\,.$$

There are four critical points, $(0, 0)$, $(0, 3/2)$, $(2, 0)$, and $(-1, 3)$.

(b)



(c) Examining the phase plot we can conclude that $(0,0)$ is an unstable node, $(0,3/2)$ is a saddle point (hence unstable), $(2,0)$ is an asymptotically stable node, and $(-1,3)$ is an asymptotically stable node.

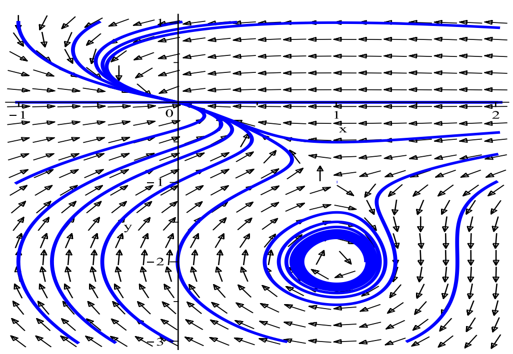
(d) Again, the phase plot shows us that the basin of $(2,0)$ is the (open) first and fourth quadrants and the basin of $(-1,3)$ is the (open) second quadrant.

8.(a) The critical points are solutions of the equations

$$-(2+y)(x+y) = 0$$
$$-y(1-x) = 0\,.$$

There are three critical points, $(0,0)$, $(1,-1)$, and $(1,-2)$.

(b)



(c) Examining the phase plot we can conclude that $(0,0)$ is an asymptotically stable node, $(1,-1)$ is a saddle point (hence unstable), and $(1,-2)$ is an asymptotically stable spiral.

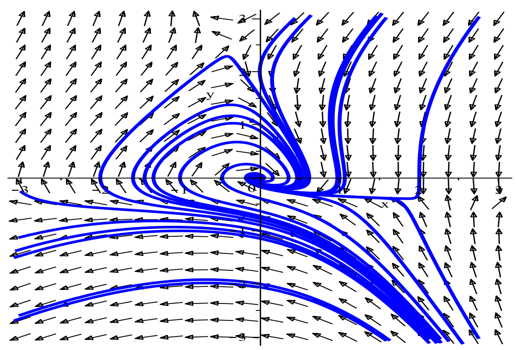
(d) The phase plot suggests that the basin of $(0,0)$ is the whole plane except for a subset of the fourth quadrant that is the basin of $(1,-2)$.

9.(a) The critical points are given by the solution set of the equations

$$y(2-x-y)=0$$
$$-x-y-2xy=0\,.$$

Clearly, $(0\,,0)$ is a critical point. If $x=2-y$, then it follows that $y(y-2)=1$. The additional critical points are $(1-\sqrt{2}\,,1+\sqrt{2})$ and $(1+\sqrt{2}\,,1-\sqrt{2})$.

(b)



(c) The behavior near the origin is that of a stable spiral. Hence the point $(0\,,0)$ is asymptotically stable. At the critical point $(1-\sqrt{2}\,,1+\sqrt{2})$, the trajectories resemble those near a saddle. Hence the critical point is unstable. Near the point $(1+\sqrt{2}\,,1-\sqrt{2})$, the trajectories resemble those near a saddle. Hence the critical point is also unstable.

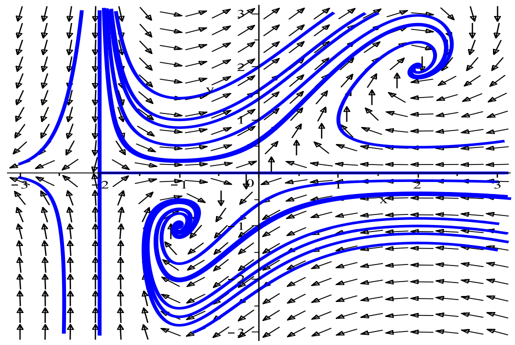
(d) Observing the direction field and the trajectories in (b), we can see that the basin of attraction of the origin is a complicated region including portions of all four quadrants.

10.(a) The critical points are solutions of the equations

$$(2+x)(y-x)=0$$
$$y(2+x-x^2)=0\,.$$

The origin is evidently a critical point. If $x=-2$, then $y=0$. If $x=y$, then either $y=0$ or $x=y=-1$ or $x=y=2$. Hence the other critical points are $(-2\,,0)$, $(-1\,,-1)$ and $(2\,,2)$.

(b)



(c) Based on the global phase portrait, the critical points $(0\,,0)$ and $(-2\,,0)$ have the characteristics of a saddle. Hence these points are unstable. The behavior near the remaining two critical points resembles those near a stable spiral. Hence the critical points $(-1\,,-1)$ and $(2\,,2)$ are asymptotically stable.

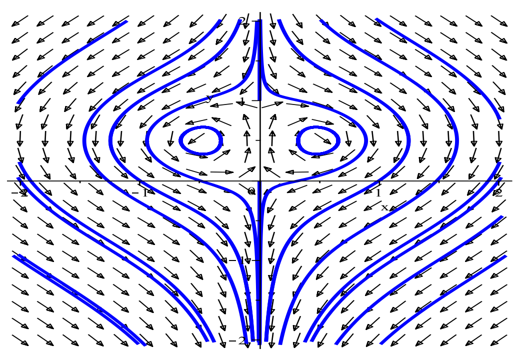
(d) The basin of $(2,2)$ is the part of the upper half plane where $x > -2$, the basin of $(-1,-1)$ is the part of the lower half plane where $x > -2$.

11.(a) The critical points are given by the solution set of the equations

$$x(1-2y) = 0$$
$$y - x^2 - y^2 = 0\,.$$

If $x = 0$, then either $y = 0$ or $y = 1$. If $y = 1/2$, then $x = \pm 1/2$. Hence the critical points are at $(0\,,0)$, $(0\,,1)$, $(-1/2\,,1/2)$ and $(1/2\,,1/2)$.

(b)



(c) The trajectories near the critical points $(-1/2\,,1/2)$ and $(1/2\,,1/2)$ are closed curves. Hence the critical points have the characteristics of a center, which is stable. The trajectories near the critical points $(0\,,0)$ and $(0,1)$ resemble those near a saddle. Hence these critical points are unstable.

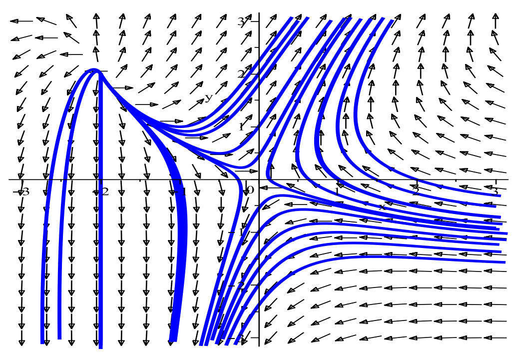
(d) As the two stable critical points are centers, they have no basins of attraction. Trajectories near the critical points are ovals around those points.

13.(a) The critical points are solutions of the equations

$$(2+x)(y-x)=0$$
$$(4-x)(y+x)=0\,.$$

If $y=x$, then either $x=y=0$ or $x=y=4$. If $x=-2$, then $y=2$. If $x=-y$, then $y=2$ or $y=0$. Hence the critical points are at $(0\,,0)$, $(4\,,4)$ and $(-2\,,2)$.

(b)



(c) The critical point at $(4\,,4)$ is evidently a stable spiral, which is asymptotically stable. Closer examination of the critical point at $(0\,,0)$ reveals that it is a saddle, which is unstable. The trajectories near the critical point $(-2\,,2)$ resemble those near an unstable node.

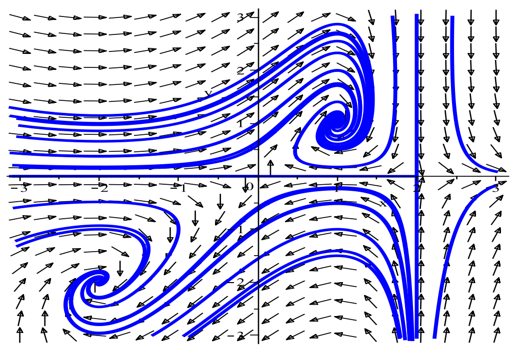
(d) The basin of attraction for $(4,4)$ consists of the points for which $x>-2$ and which lie over a curve (inferred from part (b)) passing through the origin and $(-2,2)$.

14.(a) The critical points are given by the solution set of the equations

$$(2-x)(y-x)=0$$
$$y(2-x-x^2)=0\,.$$

If $x=2$, then $y=0$. If $y=0$, then $x=0$. Also, $x=1$ and $x=-2$ are roots of the second equation, and then $y=x$ from the first equation. Hence the critical points are at $(2\,,0)$, $(0\,,0)$, $(1\,,1)$ and $(-2\,,-2)$.

(b)



(c) The critical points $(0,0)$ and $(2,0)$ are saddles, hence unstable. The other two critical points are asymptotically stable spirals.

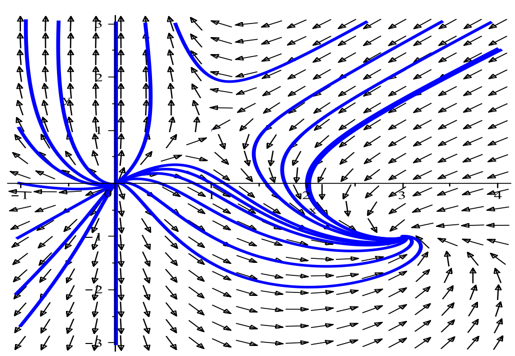
(d) The basin of $(1,1)$ is the part of the upper half plane where $x < 2$ (so all points (x,y) such that $x < 2$ and $y > 0$), the basin of $(-2,-2)$ is all points (x,y) such that $x < 2$ and $y < 0$.

15.(a) The critical points are given by the solution set of the equations

$$x(2-x-y) = 0$$
$$-x+3y-2xy = 0\,.$$

If $x = 0$, then $y = 0$. The other critical points can be found by setting $y = 2-x$ and substituting this into the second equation. This gives us $x = 3$ and $x = 1$. Hence the critical points are at $(0\,,0)$, $(1\,,1)$ and $(3\,,-1)$.

(b)



(c) The critical point $(1,1)$ is a saddle, hence unstable. $(0,0)$ is an unstable node and $(3,-1)$ is an asymptotically stable spiral.

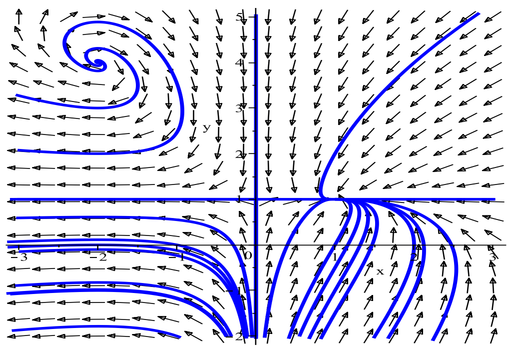
(d) The basin of $(3,-1)$ consists of all points (x,y) for which $x > 0$ and $x > y$.

16.(a) The critical points are given by the solution set of the equations

$$x(2-x-y)=0$$
$$(1-y)(2+x)=0\,.$$

If $x=0$, then $y=1$. Also, when $x=-2$, then $y=4$. The last critical point is given by $y=1$, $x=1$. Hence the critical points are at $(0\,,1)$, $(1\,,1)$ and $(-2\,,4)$.

(b)



(c) The critical point $(0,1)$ is a saddle, hence unstable. $(-2,4)$ is an unstable spiral and $(1,1)$ is an asymptotically stable node.

(d) The basin of $(1,1)$ is the right half plane, i.e. all the points (x,y) for which $x>0$.

18.(a) The trajectories are solutions of the differential equation

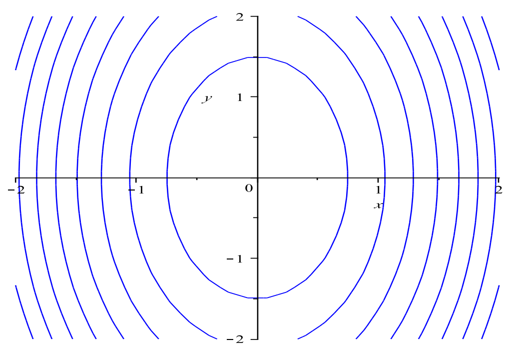
$$\frac{dy}{dx}=-\frac{4x}{y},$$

which can also be written as $4x\,dx+y\,dy=0$. Integrating, we obtain

$$4x^2+y^2=C^2.$$

Hence the trajectories are ellipses.

(b)



Based on the differential equations, the direction of motion on each trajectory is clockwise.

19.(a) The trajectories of the system satisfy the ODE

$$\frac{dy}{dx} = \frac{2x+y}{y}\,,$$

which can also be written as $(2x+y)dx - ydy = 0$. This differential equation is homogeneous. Setting $y = x\,v(x)$, we obtain

$$v + x\frac{dv}{dx} = \frac{2}{v} + 1\,,$$

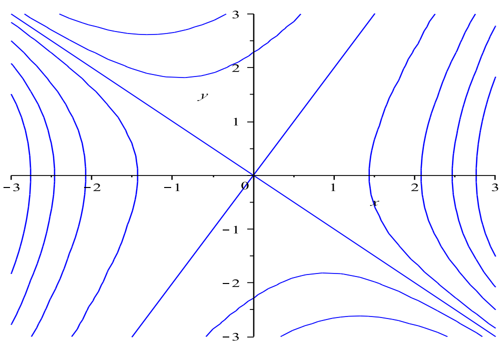
that is,

$$x\frac{dv}{dx} = \frac{2+v-v^2}{v}\,.$$

The resulting ODE is separable, with solution $x^3(v+1)(v-2)^2 = C$. Reverting back to the original variables, the trajectories are level curves of

$$H(x\,,y) = (x+y)(y-2x)^2.$$

(b)



The origin is a saddle. Along the line $y = 2x$, solutions increase without bound. Along the line $y = -x$, solutions converge toward the origin.

20.(a) The trajectories are solutions of the differential equation

$$\frac{dy}{dx} = \frac{x+y}{x-y},$$

which is homogeneous. Setting $y = x\,v(x)$, we obtain

$$v + x\frac{dv}{dx} = \frac{x+xv}{x-xv}\,,$$

that is,

$$x\frac{dv}{dx} = \frac{1+v^2}{1-v}\,.$$

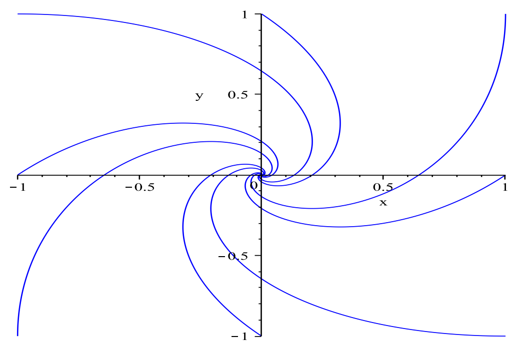
The resulting ODE is separable, with solution

$$\arctan(v) = \ln|x|\sqrt{1+v^2}\,.$$

Reverting back to the original variables, the trajectories are level curves of

$$H(x\,,y) = \arctan(y/x) - \ln\sqrt{x^2+y^2}\,.$$

(b)



The origin is a stable spiral.

22.(a) The trajectories are solutions of the differential equation

$$\frac{dy}{dx} = \frac{-2xy^2 + 6\,xy}{2x^2y - 3x^2 - 4y},$$

which can also be written as $(2xy^2 - 6\,xy)dx + (2x^2y - 3x^2 - 4y)dy = 0$. The resulting ODE is exact, with

$$\frac{\partial H}{\partial x} = 2xy^2 - 6\,xy \text{ and } \frac{\partial H}{\partial y} = 2x^2y - 3x^2 - 4y\,.$$

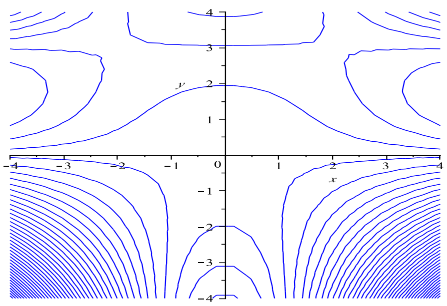
Integrating the first equation, we find that $H(x\,,y) = x^2y^2 - 3x^2y + f(y)$. It follows that

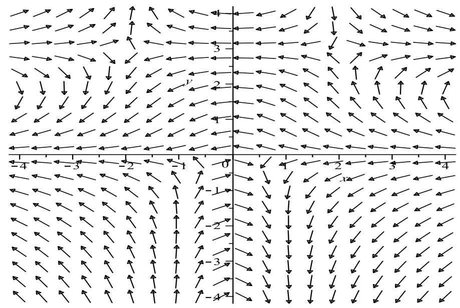
$$\frac{\partial H}{\partial y} = 2x^2y - 3x^2 + f'(y).$$

Comparing the two partial derivatives, we obtain $f(y) = -2y^2 + c$. Hence

$$H(x\,,y) = x^2y^2 - 3x^2y - 2y^2.$$

(b) The associated direction field shows the direction of motion along the trajectories.





24.(a) The trajectories are solutions of the differential equation

$$\frac{dy}{dx} = \frac{-6\,x + x^3}{6\,y},$$

which can also be written as $(6\,x - x^3)dx + 6\,y dy = 0$. The resulting ODE is exact, with

$$\frac{\partial H}{\partial x} = 6\,x - x^3 \;\text{ and }\; \frac{\partial H}{\partial y} = 6\,y\,.$$

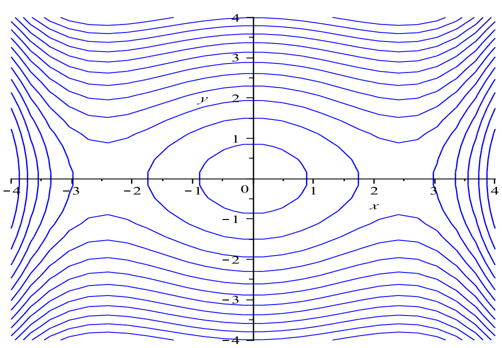
Integrating the first equation, we have $H(x\,,y) = 3x^2 - x^4/4 + f(y)$. It follows that

$$\frac{\partial H}{\partial y} = f\,'(y).$$

Comparing the two partial derivatives, we conclude that $f(y) = 3y^2 + c$. Hence

$$H(x\,,y) = 3x^2 - \frac{x^4}{4} + 3y^2.$$

(b)



9.3

1. Write the system in the form $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(\mathbf{x})$. In this case, it is evident that

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -y^2 \\ x^2 \end{pmatrix}.$$

That is, $\mathbf{g}(\mathbf{x}) = (-y^2, x^2)^T$. Using polar coordinates, $\|\mathbf{g}(\mathbf{x})\| = r^2\sqrt{\sin^4\theta + \cos^4\theta}$ and $\|\mathbf{x}\| = r$. Hence

$$\lim_{r\to 0}\frac{\|\mathbf{g}(\mathbf{x})\|}{\|\mathbf{x}\|} = \lim_{r\to 0} r\sqrt{\sin^4\theta + \cos^4\theta} = 0\,,$$

and the system is locally linear. The origin is an isolated critical point of the linear system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $r^2 + r - 2 = 0$, with roots $r_1 = 1$ and $r_2 = -2$. Hence the critical point is a saddle, which is unstable.

2. The system can be written as

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix}.$$

Following the discussion in Example 3, we note that $F(x\,,y) = -x + y + 2xy$ and $G(x\,,y) = -4x - y + x^2 - y^2$. Both of the functions F and G are twice differentiable, hence the system is locally linear. Furthermore,

$$F_x = -1 + 2y\,,\ F_y = 1 + 2x\,,\ G_x = -4 + 2x\,,\ G_y = -1 - 2y\,.$$

The origin is an isolated critical point, with

$$\begin{pmatrix} F_x(0\,,0) & F_y(0\,,0) \\ G_x(0\,,0) & G_y(0\,,0) \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -4 & -1 \end{pmatrix}.$$

The characteristic equation of the associated linear system is $r^2 + 2\,r + 5 = 0$, with complex conjugate roots $r_{1,2} = -1 \pm 2i$. The origin is a stable spiral, which is asymptotically stable.

5.(a) The critical points consist of the solution set of the equations

$$\begin{aligned}(2 + x)(y - x) &= 0 \\ (4 - x)(y + x) &= 0\,.\end{aligned}$$

As shown in Problem 13 of Section 9.2, the only critical points are at $(0\,,0)$, $(4\,,4)$ and $(-2\,,2)$.

(b,c) First note that $F(x\,,y) = (2 + x)(y - x)$ and $G(x\,,y) = (4 - x)(y + x)$. The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} F_x(x\,,y) & F_y(x\,,y) \\ G_x(x\,,y) & G_y(x\,,y) \end{pmatrix} = \begin{pmatrix} -2 - 2x + y & 2 + x \\ 4 - y - 2x & 4 - x \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0) = \begin{pmatrix} -2 & 2 \\ 4 & 4 \end{pmatrix},$$

with eigenvalues $r_1 = 1 - \sqrt{17}$ and $r_2 = 1 + \sqrt{17}$. The eigenvalues are real, with opposite sign. Hence the critical point is a saddle, which is unstable. At the point $(-2\,,2)$, the coefficient matrix of the linearized system is

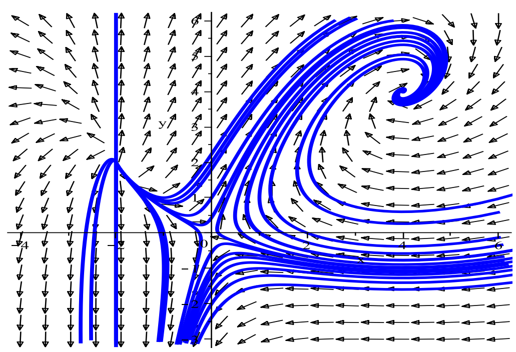
$$\mathbf{J}(-2\,,2) = \begin{pmatrix} 4 & 0 \\ 6 & 6 \end{pmatrix},$$

with eigenvalues $r_1 = 4$ and $r_2 = 6$. The eigenvalues are real, unequal and positive, hence the critical point is an unstable node. At the point $(4\,,4)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(4\,,4) = \begin{pmatrix} -6 & 6 \\ -8 & 0 \end{pmatrix},$$

with complex conjugate eigenvalues $r_{1,2} = -3 \pm i\sqrt{39}$. The critical point is a stable spiral, which is asymptotically stable. Based on Table 9.3.1, the nonlinear terms do not affect the stability and type of each critical point.

(d)



7.(a) The critical points are solutions of the equations

$$\begin{aligned} 1 - y &= 0 \\ (x - y)(x + y) &= 0\,. \end{aligned}$$

The first equation requires that $y = 1$. Based on the second equation, $x = \pm 1$. Hence the critical points are $(-1\,,1)$ and $(1\,,1)$.

(b,c) $F(x\,,y) = 1 - y$ and $G(x\,,y) = x^2 - y^2$. The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} F_x(x\,,y) & F_y(x\,,y) \\ G_x(x\,,y) & G_y(x\,,y) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 2x & -2y \end{pmatrix}.$$

At the critical point $(-1\,,1)$, the coefficient matrix of the linearized system is

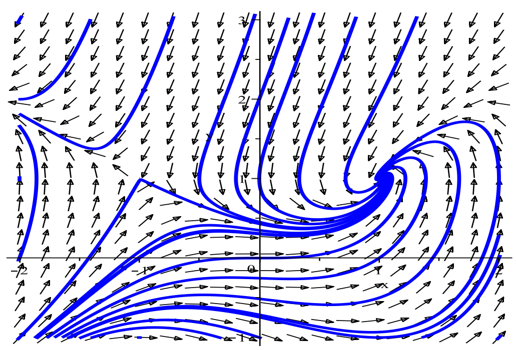
$$\mathbf{J}(-1\,,1) = \begin{pmatrix} 0 & -1 \\ -2 & -2 \end{pmatrix},$$

with eigenvalues $r_1 = -1 - \sqrt{3}$ and $r_2 = -1 + \sqrt{3}$. The eigenvalues are real, with opposite sign. Hence the critical point is a saddle, which is unstable. At the point $(1\,,1)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(1\,,1) = \begin{pmatrix} 0 & -1 \\ 2 & -2 \end{pmatrix},$$

with complex conjugate eigenvalues $r_{1,2} = -1 \pm i$. The critical point is a stable spiral, which is asymptotically stable.

(d)



Based on Table 9.3.1, the nonlinear terms do not affect the stability and type of each critical point.

8.(a) The critical points are given by the solution set of the equations

$$x(1 - x - y) = 0$$
$$y(2 - y - 3x) = 0\,.$$

If $x = 0$, then either $y = 0$ or $y = 2$. If $y = 0$, then $x = 0$ or $x = 1$. If $y = 1 - x$, then either $x = 1/2$ or $x = 1$. If $y = 2 - 3x$, then $x = 0$ or $x = 1/2$. Hence the critical points are at $(0\,,0)$, $(0\,,2)$, $(1\,,0)$ and $(1/2\,,1/2)$.

(b,c) Note that $F(x\,,y) = x - x^2 - xy$ and $G(x\,,y) = (2y - y^2 - 3xy)/4$. The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} F_x(x\,,y) & F_y(x\,,y) \\ G_x(x\,,y) & G_y(x\,,y) \end{pmatrix} = \begin{pmatrix} 1 - 2x - y & -x \\ -3y/4 & 1/2 - y/2 - 3x/4 \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix},$$

with eigenvalues $r_1 = 1$ and $r_2 = 1/2$. The eigenvalues are real and both positive. Hence the critical point is an unstable node. At the point $(0\,,2)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,2) = \begin{pmatrix} -1 & 0 \\ -\frac{3}{2} & -\frac{1}{2} \end{pmatrix},$$

with eigenvalues $r_1 = -1$ and $r_2 = -1/2$. The eigenvalues are both negative, hence the critical point is a stable node. At the point $(1\,,0)$, the coefficient matrix of the linearized system is

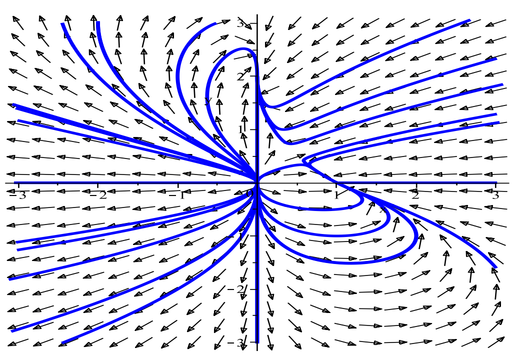
$$\mathbf{J}(1\,,0) = \begin{pmatrix} -1 & -1 \\ 0 & -\frac{1}{4} \end{pmatrix},$$

with eigenvalues $r_1 = -1$ and $r_2 = -1/4$. Both of the eigenvalues are negative, and hence the critical point is a stable node. At the critical point $(1/2\,,1/2)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(1/2\,,1/2) = \begin{pmatrix} -1/2 & -1/2 \\ -3/8 & -1/8 \end{pmatrix},$$

with eigenvalues $r_1 = -5/16 - \sqrt{57}/16$ and $r_2 = -5/16 + \sqrt{57}/16$. The eigenvalues are real, with opposite sign. Hence the critical point is a saddle, which is unstable.

(d)



Based on Table 9.3.1, the nonlinear terms do not affect the stability and type of each critical point.

9.(a) The critical points are given by the solution set of the equations

$$(2 + y)(y - x/2) = 0$$
$$(2 - x)(y + x/2) = 0\,.$$

If $y = -2$, then either $x = 2$ or $x = 4$. If $x = 2$, then $y = -2$ or $y = 1$. Also, $x = y = 0$ is a solution. Hence the critical points are at $(0\,,0)$, $(2\,,-2)$, $(4\,,-2)$ and $(2\,,1)$.

(b,c) Note that $F(x\,,y) = 2y + y^2 - x - xy/2$ and $G(x\,,y) = 2y - xy + x - x^2/2$. The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} F_x(x\,,y) & F_y(x\,,y) \\ G_x(x\,,y) & G_y(x\,,y) \end{pmatrix} = \begin{pmatrix} -1 - y/2 & 2 + 2y - x/2 \\ -y + 1 - x & 2 - x \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0) = \begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix},$$

with eigenvalues $r_1 = (1 + \sqrt{17})/2$ and $r_2 = (1 - \sqrt{17})/2$. The eigenvalues are real, with opposite sign. Hence the critical point is a saddle, which is unstable. At the point $(2\,,-2)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(2\,,-2) = \begin{pmatrix} 0 & -3 \\ 1 & 0 \end{pmatrix},$$

with eigenvalues $r_1 = \sqrt{3}i$ and $r_2 = -\sqrt{3}i$. The eigenvalues are purely imaginary, hence the critical point is either a center or a spiral point. At the point $(4\,,-2)$, the coefficient matrix of the linearized system is

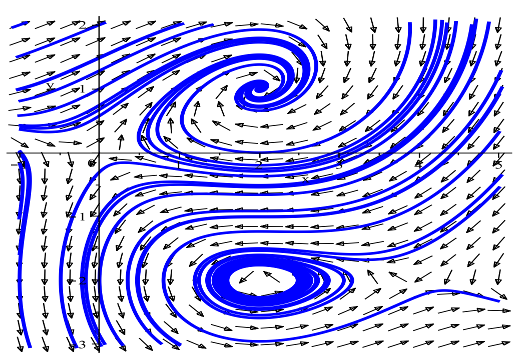
$$\mathbf{J}(4\,,-2) = \begin{pmatrix} 0 & -4 \\ -1 & -2 \end{pmatrix},$$

with eigenvalues $r_1 = -1 + \sqrt{5}$ and $r_2 = -1 - \sqrt{5}$. The eigenvalues are real, with opposite sign. Hence the critical point is a saddle, which is unstable. At the critical point $(2\,,1)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(2\,,1) = \begin{pmatrix} -3/2 & 3 \\ -2 & 0 \end{pmatrix},$$

with eigenvalues $r_1 = -3/4 + \sqrt{87}\,i/4$ and $r_2 = -3/4 - \sqrt{87}\,i/4$. The critical point is a stable spiral, which is asymptotically stable.

(d)



We observe that the point $(2, -2)$ is a spiral point.

11.(a) The critical points are solutions of the equations

$$2x + y + xy^3 = 0$$
$$x - 2y - xy = 0\,.$$

Substitution of $y = x/(x+2)$ into the first equation results in

$$3x^4 + 13x^3 + 28x^2 + 20x = 0\,.$$

One root of the resulting equation is $x = 0$. The only other real root of the equation is

$$x = \frac{1}{9}\left[(287 + 18\sqrt{2019}\,)^{1/3} - 83(287 + 18\sqrt{2019}\,)^{-1/3} - 13\right].$$

Hence the critical points are $(0\,,0)$ and $(-1.19345...\,,-1.4797...)$.

(b,c) $F(x\,,y) = 2x + y + xy^3$ and $G(x\,,y) = x - 2y - xy$. The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} F_x(x\,,y) & F_y(x\,,y) \\ G_x(x\,,y) & G_y(x\,,y) \end{pmatrix} = \begin{pmatrix} 2 + y^3 & 1 + 3xy^2 \\ 1 - y & -2 - x \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

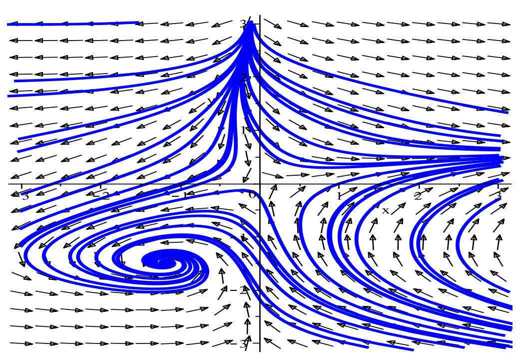
$$\mathbf{J}(0\,,0) = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix},$$

with eigenvalues $r_1 = \sqrt{5}$ and $r_2 = -\sqrt{5}$. The eigenvalues are real and of opposite sign. Hence the critical point is a saddle, which is unstable. At the point $(-1.19345...\,,-1.4797...)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(-1.19345\,,-1.4797) = \begin{pmatrix} -1.2399 & -6.8393 \\ 2.4797 & -0.8065 \end{pmatrix},$$

with complex conjugate eigenvalues $r_{1,2} = -1.0232 \pm 4.1125\,i$. The critical point is a stable spiral, which is asymptotically stable.

(d)



In both cases, the nonlinear terms do not affect the stability and type of the critical point.

12.(a) The critical points are given by the solution set of the equations

$$(1+x)\sin y = 0$$
$$1 - x - \cos y = 0\,.$$

If $x = -1$, then we must have $\cos y = 2$, which is impossible. Therefore $\sin y = 0$, which implies that $y = n\pi$, $n = 0, \pm 1, 2, \ldots$. Based on the second equation,

$$x = 1 - \cos n\pi\,.$$

It follows that the critical points are located at $(0, 2k\pi)$ and $(2, (2k+1)\pi)$, where $k = 0, \pm 1, \pm 2, \ldots$.

(b,c) Given that $F(x, y) = (1+x)\sin y$ and $G(x, y) = 1 - x - \cos y$, the Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} \sin y & (1+x)\cos y \\ -1 & \sin y \end{pmatrix}.$$

At the critical points $(0, 2k\pi)$, the coefficient matrix of the linearized system is

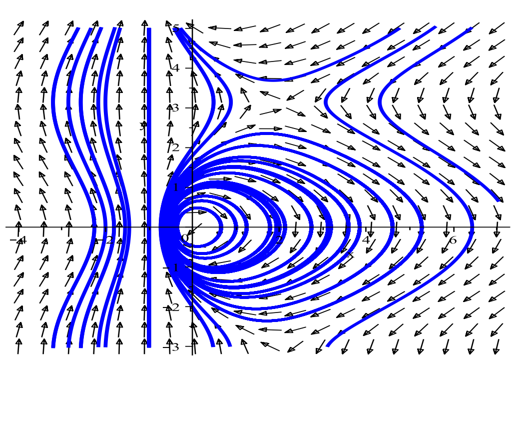
$$\mathbf{J}(0, 2k\pi) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

with purely complex eigenvalues $r_{1,2} = \pm\, i$. The critical points of the associated linear systems are centers, which are stable. Note that Theorem 9.3.2 does not provide a definite conclusion regarding the relation between the nature of the critical points of the nonlinear systems and their corresponding linearizations. At the points $(2, (2k+1)\pi)$, the coefficient matrix of the linearized system is

$$\mathbf{J}\,[2, (2k+1)\pi] = \begin{pmatrix} 0 & -3 \\ -1 & 0 \end{pmatrix},$$

with eigenvalues $r_1 = \sqrt{3}$ and $r_2 = -\sqrt{3}$. The eigenvalues are real, with opposite sign. Hence the critical points of the associated linear systems are saddles, which are unstable.

(d)



As asserted in Theorem 9.3.2, the trajectories near the critical points $(2\,,(2k+1)\pi)$ resemble those near a saddle. Upon closer examination, the critical points $(0\,,2k\pi)$ are indeed centers.

13.(a) The critical points are solutions of the equations

$$x - y^2 = 0$$
$$y - x^2 = 0\,.$$

Substitution of $y = x^2$ into the first equation results in

$$x - x^4 = 0\,,$$

with real roots $x = 0\,,\,1$. Hence the critical points are at $(0\,,0)$ and $(1\,,1)$.

(b,c) In this problem, $F(x\,,y) = x - y^2$ and $G(x\,,y) = y - x^2$. The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 1 & -2y \\ -2x & 1 \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

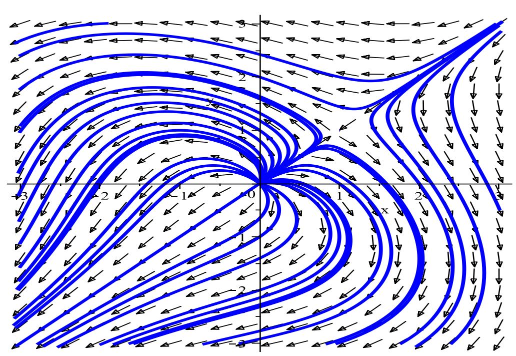
$$\mathbf{J}(0\,,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

with repeated eigenvalues $r_1 = 1$ and $r_2 = 1$. It is easy to see that the corresponding eigenvectors are linearly independent. Hence the critical point is an unstable proper node. Theorem 9.3.2 does not provide a definite conclusion regarding the relation between the nature of the critical point of the nonlinear system and the corresponding linearization. At the critical point $(1\,,1)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(1\,,1) = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix},$$

with eigenvalues $r_1 = 3$ and $r_2 = -1$. The eigenvalues are real, with opposite sign. Hence the critical point is a saddle, which is unstable.

(d)



Closer examination reveals that the critical point at the origin is indeed a proper node.

14.(a) The critical points are given by the solution set of the equations

$$1 - xy = 0$$
$$x - y^3 = 0\,.$$

After multiplying the second equation by y, it follows that $y = \pm 1$. Hence the critical points of the system are at $(1\,,1)$ and $(-1\,,-1)$.

(b,c) Note that $F(x\,,y) = 1 - xy$ and $G(x\,,y) = x - y^3$. The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} -y & -x \\ 1 & -3y^2 \end{pmatrix}.$$

At the critical point $(1\,,1)$, the coefficient matrix of the linearized system is

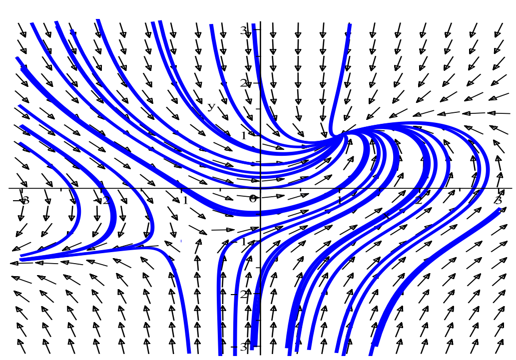
$$\mathbf{J}(1\,,1) = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix},$$

with eigenvalues $r_1 = -2$ and $r_2 = -2$. The eigenvalues are real and equal. It is easy to show that there is only one linearly independent eigenvector. Hence the critical point is a stable improper node. Theorem 9.3.2 does not provide a definite conclusion regarding the relation between the nature of the critical point of the nonlinear system and the corresponding linearization. At the point $(-1\,,-1)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(-1\,,-1) = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix},$$

with eigenvalues $r_1 = -1 + \sqrt{5}$ and $r_2 = -1 - \sqrt{5}$. The eigenvalues are real, with opposite sign. Hence the critical point of the associated linear system is a saddle, which is unstable.

(d)



Closer examination reveals that the critical point at $(1\,,1)$ is indeed a stable improper node, which is asymptotically stable.

15.(a) The critical points are given by the solution set of the equations

$$-2x - y - x(x^2 + y^2) = 0$$
$$x - y + y(x^2 + y^2) = 0\,.$$

It is clear that the origin is a critical point. Solving the first equation for y, we find that

$$y = \frac{-1 \pm \sqrt{1 - 8x^2 - 4x^4}}{2x}\,.$$

Substitution of these relations into the second equation results in two equations of the form $f_1(x) = 0$ and $f_2(x) = 0$. Plotting these functions, we note that only $f_1(x) = 0$ has real roots given by $x \approx \pm 0.33076$. It follows that the additional critical points are at $(-0.33076\,, 1.0924)$ and $(0.33076\,, -1.0924)$.

(b,c) Given that

$$F(x\,, y) = -2x - y - x(x^2 + y^2)$$
$$G(x\,, y) = x - y + y(x^2 + y^2),$$

the Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} -2 - 3x^2 - y^2 & -1 - 2xy \\ 1 + 2xy & -1 + x^2 + 3y^2 \end{pmatrix}.$$

At the critical point $(0\,,0)$, the coefficient matrix of the linearized system is

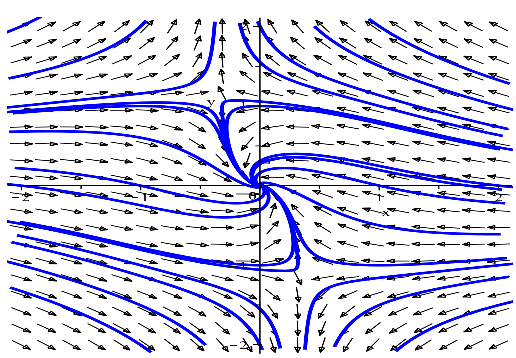
$$\mathbf{J}(0\,,0) = \begin{pmatrix} -2 & -1 \\ 1 & -1 \end{pmatrix},$$

with complex conjugate eigenvalues $r_{1,2} = (-3 \pm i\sqrt{3})/2$. Hence the critical point is a stable spiral, which is asymptotically stable. At the point $(-0.33076\,, 1.0924)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(-0.33076\,, 1.0924) = \begin{pmatrix} -3.5216 & -0.27735 \\ 0.27735 & 2.6895 \end{pmatrix},$$

with eigenvalues $r_1 = -3.5092$ and $r_2 = 2.6771$. The eigenvalues are real, with opposite sign. Hence the critical point of the associated linear system is a saddle, which is unstable. Identical results hold for the point at $(0.33076\,, -1.0924)$.

(d)



16.(a) The critical points are solutions of the equations

$$y + x(1 - x^2 - y^2) = 0$$
$$-x + y(1 - x^2 - y^2) = 0\,.$$

Multiply the first equation by y and the second equation by x. The difference of the two equations gives $x^2 + y^2 = 0$. Hence the only critical point is at the origin.

(b,c) With $F(x\,,y) = y + x(1 - x^2 - y^2)$ and $G(x\,,y) = -x + y(1 - x^2 - y^2)$, the Jacobian matrix of the vector field is

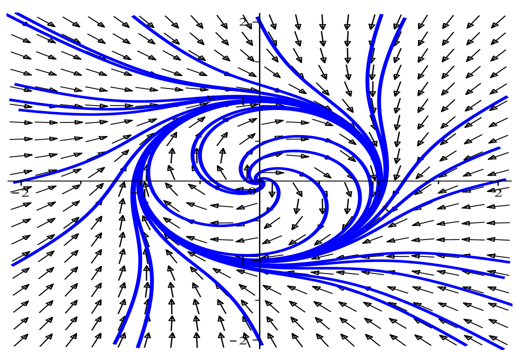
$$\mathbf{J} = \begin{pmatrix} 1 - 3x^2 - y^2 & 1 - 2xy \\ -1 - 2xy & 1 - x^2 - 3y^2 \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

with complex conjugate eigenvalues $r_{1,2} = 1 \pm i$. Hence the origin is an unstable spiral.

(d)



17.(a) The critical points are given by the solution set of the equations

$$4 - y^2 = 0$$
$$(x + 3/2)(y - x) = 0\,.$$

Clearly, $y = \pm 2$. The second equation tells us that $x = -3/2$ or $x = y$. Hence the critical points are at $(-3/2\,,2)$, $(-3/2\,,-2)$, $(2\,,2)$ and $(-2\,,-2)$.

(b,c) Note that $F(x\,,y) = 4 - y^2$ and $G(x\,,y) = xy + 3y/2 - x^2 - 3x/2$. The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} F_x(x\,,y) & F_y(x\,,y) \\ G_x(x\,,y) & G_y(x\,,y) \end{pmatrix} = \begin{pmatrix} 0 & -2y \\ y - 2x - 3/2 & x + 3/2 \end{pmatrix}.$$

At $(-3/2, 2)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(-3/2\,,2) = \begin{pmatrix} 0 & -4 \\ 7/2 & 0 \end{pmatrix},$$

with eigenvalues $r_1 = \sqrt{14}i$ and $r_2 = -\sqrt{14}i$. The eigenvalues are purely imaginary, hence the critical point is either a center or a spiral point. At the point $(-3/2\,,-2)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(-3/2\,,-2) = \begin{pmatrix} 0 & 4 \\ -1/2 & 0 \end{pmatrix},$$

with eigenvalues $r_1 = \sqrt{2}i$ and $r_2 = -\sqrt{2}i$. The eigenvalues are purely imaginary, hence the critical point is either a center or a spiral point. At the point $(2\,,2)$, the coefficient matrix of the linearized system is

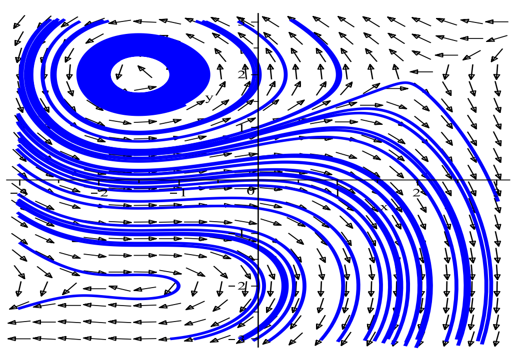
$$\mathbf{J}(2\,,2) = \begin{pmatrix} 0 & -4 \\ -7/2 & 7/2 \end{pmatrix},$$

with eigenvalues $r_1 = (7+\sqrt{273})/4$ and $r_2 = (7-\sqrt{273})/4$. The eigenvalues are real, with opposite sign. Hence the critical point is a saddle, which is unstable. At the critical point $(-2\,,-2)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(-2\,,-2) = \begin{pmatrix} 0 & 4 \\ 1/2 & -1/2 \end{pmatrix},$$

with eigenvalues $r_1 = (-1+\sqrt{33}\,)/4$ and $r_2 = -1-\sqrt{33}\,)/4$. Hence the critical point is a saddle, which is unstable.

(d)



Further observation indicates that $(-3/2, 2)$ and $(-3/2, -2)$ are unstable spiral points.

19.(a) The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ 1+6x^2 & 0 \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

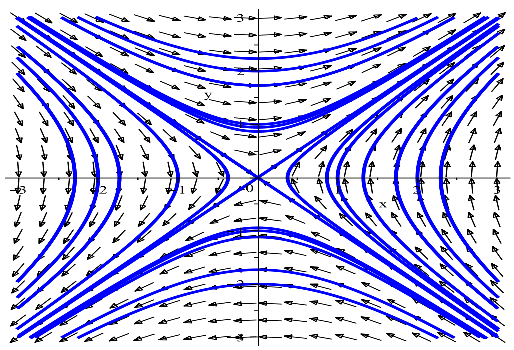
$$\mathbf{J}(0\,,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

with eigenvalues $r_1 = 1$ and $r_2 = -1$. The eigenvalues are real, with opposite sign. Hence the critical point is a saddle point.

(b) The trajectories of the linearized system are solutions of the differential equation

$$\frac{dy}{dx} = \frac{x}{y},$$

which is separable. Integrating both sides of the equation $x\,dx - y\,dy = 0$, the solution is $x^2 - y^2 = C$. The trajectories consist of a family of hyperbolas.



It is easy to show that the general solution is given by $x(t) = c_1 e^t + c_2 e^{-t}$ and $y(t) = c_1 e^t - c_2 e^{-t}$. The only bounded solutions consist of those for which $c_1 = 0$. In that case, $x(t) = c_2 e^{-t} = -y(t)$.

(c) The trajectories of the given system are solutions of the differential equation

$$\frac{dy}{dx} = \frac{x + 2x^3}{y},$$

which can also be written as $(x + 2x^3)dx - y\,dy = 0$. The resulting ODE is exact, with

$$\frac{\partial H}{\partial x} = x + 2x^3 \text{ and } \frac{\partial H}{\partial y} = -y\,.$$

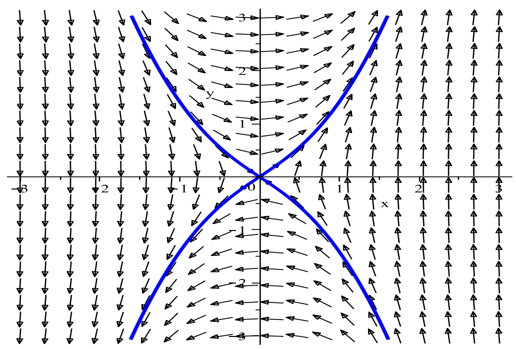
Integrating the first equation, we find that $H(x\,,y) = x^2/2 + x^4/2 + f(y)$. It follows that

$$\frac{\partial H}{\partial y} = f\,'(y).$$

Comparing the partial derivatives, we obtain $f(y) = -y^2/2 + c$. Hence the solutions are level curves of the function

$$H(x\,,y) = x^2/2 + x^4/2 - y^2/2\,.$$

(The equation is also separable. Separation of variables yields the same $H(x, y)$.) The trajectories approaching to, or diverging from, the origin are no longer straight lines.



21.(a) The solutions of the system of equations

$$y = 0$$
$$-\omega^2 \sin x = 0$$

consist of the points $(\pm n\pi\,, 0)$, $n = 0\,, 1\,, 2\,, \ldots$. The functions $F(x\,, y) = y$ and $G(x\,, y) = -\omega^2 \sin x$ are analytic on the entire plane. It follows that the system is locally linear near each of the critical points.

(b) The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -\omega^2 \cos x & 0 \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,, 0) = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix},$$

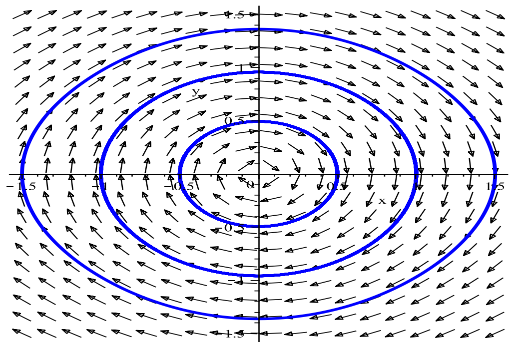
with purely complex eigenvalues $r_{1,2} = \pm i\omega$. Hence the origin is a center. Since the eigenvalues are purely complex, Theorem 9.3.2 gives no definite conclusion about the critical point of the nonlinear system. Physically, the critical point corresponds to the state $\theta = 0$, $\theta' = 0$. That is, the rest configuration of the pendulum.

(c) At the critical point $(\pi\,, 0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(\pi\,, 0) = \begin{pmatrix} 0 & 1 \\ \omega^2 & 0 \end{pmatrix},$$

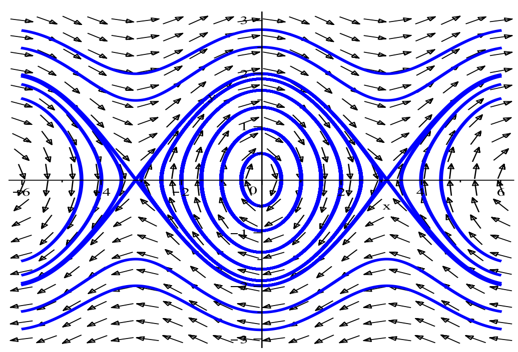
with eigenvalues $r_{1,2} = \pm\omega$. The eigenvalues are real and of opposite sign. Hence the critical point is a saddle. Theorem 9.3.2 asserts that the critical point for the nonlinear system is also a saddle, which is unstable. This critical point corresponds to the state $\theta = \pi$, $\theta' = 0$. That is, the upright rest configuration.

(d) Let $\omega^2 = 1$. The following is a plot of the phase curves near $(0\,, 0)$.



The local phase portrait shows that the origin is indeed a center.

(e)



It should be noted that the phase portrait has a periodic pattern, since $\theta = x$ mod 2π.

22.(a) The trajectories of the system in Problem 21 are solutions of the differential equation

$$\frac{dy}{dx} = \frac{-\omega^2 \sin x}{y},$$

which can also be written as $\omega^2 \sin x\, dx + y\, dy = 0$. The resulting ODE is exact, with

$$\frac{\partial H}{\partial x} = \omega^2 \sin x \;\text{ and }\; \frac{\partial H}{\partial y} = y\,.$$

Integrating the first equation, we find that $H(x\,,y) = -\omega^2 \cos x + f(y)$. It follows that

$$\frac{\partial H}{\partial y} = f\,'(y).$$

Comparing the partial derivatives, we obtain $f(y) = y^2/2 + C$. Hence the solutions are level curves of the function

$$H(x\,,y) = -\omega^2 \cos x + y^2/2\,.$$

Adding an arbitrary constant, say ω^2, to the function $H(x\,,y)$ does not change the nature of the level curves. Hence the trajectories are can be written as

$$\frac{1}{2}y^2 + \omega^2(1 - \cos\, x) = c\,,$$

in which c is an arbitrary constant.

(b) Multiplying by mL^2 and reverting to the original physical variables, we obtain

$$\frac{1}{2}mL^2(\frac{d\theta}{dt})^2 + mL^2\omega^2(1 - \cos\,\theta) = mL^2c\,.$$

Since $\omega^2 = g/L$, the equation can be written as

$$\frac{1}{2}mL^2(\frac{d\theta}{dt})^2 + mgL(1 - \cos\,\theta) = E\,,$$

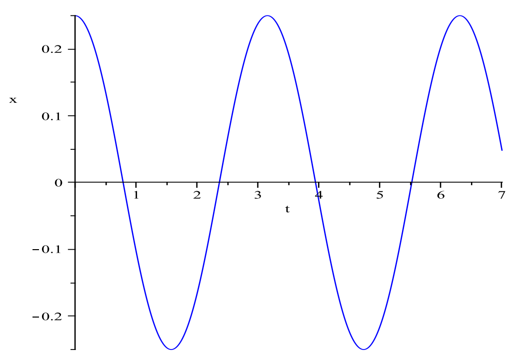
in which $E = mL^2c$.

(c) The absolute velocity of the point mass is given by $v = L\,d\theta/dt$. The kinetic energy of the mass is $T = mv^2/2$. Choosing the rest position as the datum, that is, the level of zero potential energy, the gravitational potential energy of the point mass is

$$V = mgL(1 - \cos\,\theta).$$

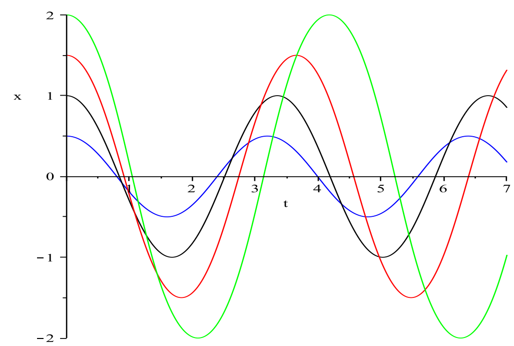
It follows that the total energy, $T + V$, is constant along the trajectories.

23.(a) $A = 0.25$



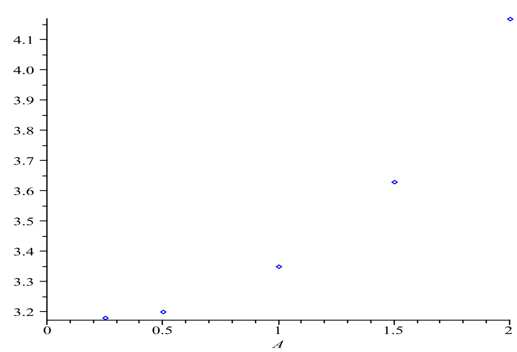
Since the system is undamped, and $y(0) = 0$, the amplitude is 0.25. The period is estimated at $\tau \approx 3.16$.

(b)



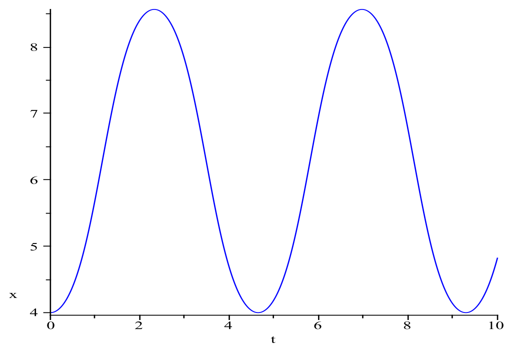
	R	τ
$A = 0.5$	0.5	3.20
$A = 1.0$	1.0	3.35
$A = 1.5$	1.5	3.63
$A = 2.0$	2.0	4.17

(c) Since the system is conservative, the amplitude is equal to the initial amplitude. On the other hand, the period of the pendulum is a monotone increasing function of the initial position A.



It appears that as $A \to 0$, the period approaches π, the period of the corresponding linear pendulum $(2\pi/\omega)$.

(d)



The pendulum is released from rest, at an inclination of $4-\pi$ radians from the vertical. Based on conservation of energy, the pendulum will swing past the lower equilibrium position ($\theta = 2\pi$) and come to rest, momentarily, at a maximum rotational displacement of $\theta_{max} = 3\pi - (4-\pi) = 4\pi - 4$. The transition between the two dynamics occurs at $A = \pi$, that is, once the pendulum is released beyond the upright configuration.

26.(a) It is evident that the origin is a critical point of each system. Furthermore, it is easy to see that the corresponding linear system, in each case, is given by

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -x\,.$$

The eigenvalues of the coefficient matrix are $r_{1,2} = \pm i$. Hence the critical point of the linearized system is a center.

(b) Using polar coordinates, it is also easy to show that

$$\lim_{r\to 0} \frac{\|\mathbf{g}(\mathbf{x})\|}{\|\mathbf{x}\|} = 0\,.$$

Alternatively, the nonlinear terms are analytic in the entire plane. Hence both systems are locally linear near the origin.

(c) For system (ii), note that

$$x\frac{dx}{dt} + y\frac{dy}{dt} = xy - x^2(x^2+y^2) - xy - y^2(x^2+y^2).$$

Converting to polar coordinates, and differentiating the equation $r^2 = x^2 + y^2$ with respect to t, we find that

$$r\frac{dr}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt} = -(x^2+y^2)^2 = -r^4.$$

That is, $r' = -r^3$. It follows that $r^2 = 1/(2t+c)$, where $c = 1/r_0^2$. Since $r \to 0$ as $t \to \infty$ and $dr/dt < 0$, regardless of the value of r_0, the origin is an asymptotically stable critical point.

On the other hand, for system (i),

$$r\frac{dr}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt} = (x^2+y^2)^2 = r^4.$$

That is, $r' = r^3$. Solving the differential equation results in

$$r^2 = \frac{c-2t}{(2t-c)^2}.$$

Imposing the initial condition $r(0) = r_0$, we obtain a specific solution

$$r^2 = -\frac{r_0^2}{2\,r_0^2\,t - 1}.$$

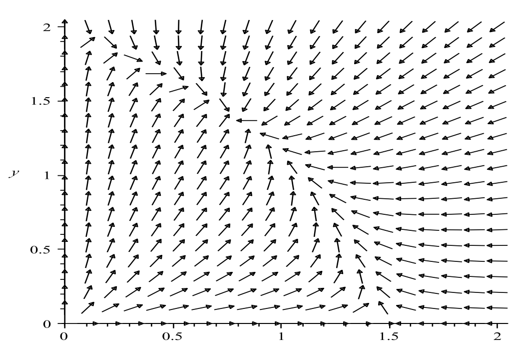
Since the solution becomes unbounded as $t \to 1/2r_0^2$, the critical point is unstable.

27. The characteristic equation of the coefficient matrix is $r^2+1=0$, with complex roots $r_{1,2} = \pm i$. Hence the critical point at the origin is a center. The characteristic equation of the perturbed matrix is $r^2 - 2\,\epsilon\, r + 1 + \epsilon^2 = 0$, with complex conjugate roots $r_{1,2} = \epsilon \pm i$. As long as $\epsilon \neq 0$, the critical point of the perturbed system is a spiral point. Its stability depends on the sign of ϵ.

28. The characteristic equation of the coefficient matrix is $(r+1)^2 = 0$, with roots $r_1 = r_2 = -1$. Hence the critical point is an asymptotically stable node. On the other hand, the characteristic equation of the perturbed system is $r^2+2r+1+\epsilon = 0$, with roots $r_{1,2} = -1 \pm \sqrt{-\epsilon}$. If $\epsilon > 0$, then $r_{1,2} = -1 \pm i\sqrt{\epsilon}$ are complex roots. The critical point is a stable spiral. If $\epsilon < 0$, then $r_{1,2} = -1 \pm \sqrt{|\epsilon|}$ are real and both negative ($|\epsilon| < 1$). The critical point remains a stable node.

9.4

1.(a)



(b) The critical points are solutions of the system of equations

$$\begin{aligned} x(1.5 - x - 0.5\,y) &= 0 \\ y(2 - y - 0.75\,x) &= 0\,. \end{aligned}$$

The four critical points are $(0\,,0)$, $(0\,,2)$, $(1.5\,,0)$ and $(0.8\,,1.4)$.

(c) The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 3/2 - 2x - y/2 & -x/2 \\ -3y/4 & 2 - 3x/4 - 2y \end{pmatrix}.$$

At the critical point $(0\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0) = \begin{pmatrix} 3/2 & 0 \\ 0 & 2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 3/2\ ,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\ ;\ r_2 = 2\ ,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are positive, hence the origin is an unstable node.
At the critical point $(0\,,2)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,2) = \begin{pmatrix} 1/2 & 0 \\ -3/2 & -2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 1/2\ ,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -0.6 \end{pmatrix}\ ;\ r_2 = -2\ ,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are of opposite sign. Hence the critical point is a saddle, which is unstable.
At the critical point $(1.5\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(1.5\,,0) = \begin{pmatrix} -1.5 & -0.75 \\ 0 & 0.875 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = -1.5\ ,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\ ;\ r_2 = 0.875\ ,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} -0.31579 \\ 1 \end{pmatrix}.$$

The eigenvalues are of opposite sign. Hence the critical point is also a saddle, which is unstable.
At the critical point $(0.8\,,1.4)$, the coefficient matrix of the linearized system is

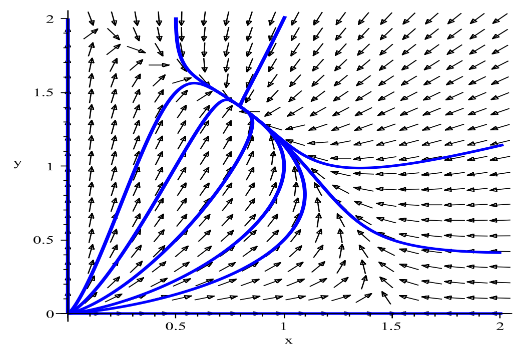
$$\mathbf{J}(0.8\,,1.4) = \begin{pmatrix} -0.8 & -0.4 \\ -1.05 & -1.4 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = -\frac{11}{10} + \frac{\sqrt{51}}{10}\ ,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ \frac{3-\sqrt{51}}{4} \end{pmatrix}\ ;\ r_2 = -\frac{11}{10} - \frac{\sqrt{51}}{10}\ ,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ \frac{3+\sqrt{51}}{4} \end{pmatrix}.$$

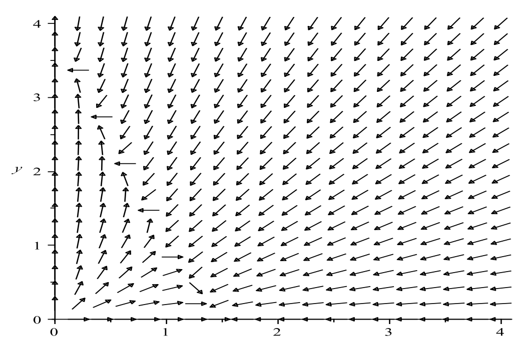
The eigenvalues are both negative. Hence the critical point is a stable node, which is asymptotically stable.

(d,e)



(f) Except for initial conditions lying on the coordinate axes, almost all trajectories converge to the stable node at $(0.8\,,1.4)$. Thus the species can coexist.

2.(a)



(b) The critical points are the solution set of the system of equations

$$x(1.5 - x - 0.5\,y) = 0$$
$$y(2 - 0.5\,y - 1.5\,x) = 0\,.$$

The four critical points are $(0\,,0)$, $(0\,,4)$, $(1.5\,,0)$ and $(1\,,1)$.

(c) The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 3/2 - 2x - y/2 & -x/2 \\ -3y/2 & 2 - 3x/2 - y \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0) = \begin{pmatrix} 3/2 & 0 \\ 0 & 2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 3/2\ ,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix}1\\0\end{pmatrix}\ ;\ r_2 = 2\ ,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix}0\\1\end{pmatrix}.$$

The eigenvalues are positive, hence the origin is an unstable node.
At the critical point $(0\,,4)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,4) = \begin{pmatrix}-1/2 & 0\\ -6 & -2\end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = -1/2\ ,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix}1\\-4\end{pmatrix}\ ;\ r_2 = -2\ ,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix}0\\1\end{pmatrix}.$$

The eigenvalues are both negative, hence the critical point $(0\,,4)$ is a stable node, which is asymptotically stable.
At the critical point $(3/2\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(3/2\,,0) = \begin{pmatrix}-3/2 & -3/4\\ 0 & -1/4\end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = -3/2\ ,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix}1\\0\end{pmatrix}\ ;\ r_2 = -1/4\ ,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix}3\\-5\end{pmatrix}.$$

The eigenvalues are both negative, hence the critical point is a stable node, which is asymptotically stable.
At the critical point $(1,1)$, the coefficient matrix of the linearized system is

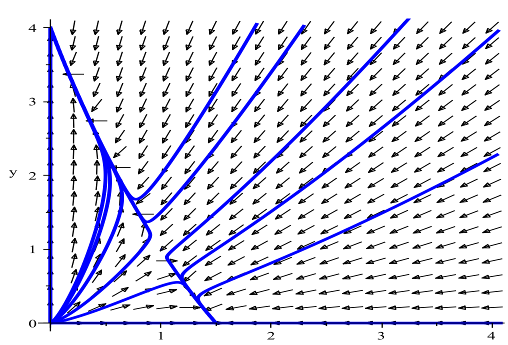
$$\mathbf{J}(1\,,1) = \begin{pmatrix}-1 & -1/2\\ -3/2 & -1/2\end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = \frac{-3+\sqrt{13}}{4}\ ,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix}1\\ -\frac{1+\sqrt{13}}{2}\end{pmatrix}\ ;\ r_2 = -\frac{3+\sqrt{13}}{4}\ ,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix}0\\ \frac{-1+\sqrt{13}}{2}\end{pmatrix}.$$

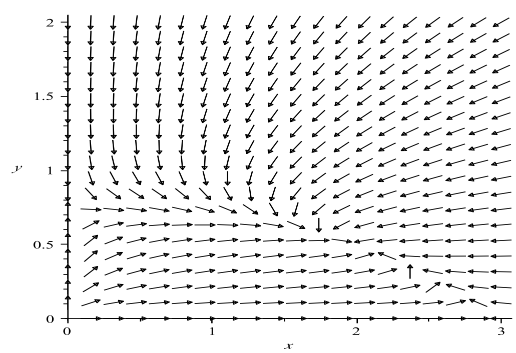
The eigenvalues are of opposite sign, hence $(1,1)$ is a saddle, which is unstable.

(d,e)



(f) Trajectories approaching the critical point $(1\,,1)$ form a separatrix. Solutions on either side of the separatrix approach either $(0\,,4)$ or $(1.5\,,0)$. Thus depending on the initial conditions, one species will drive the other to extinction.

4.(a)



(b) The critical points are solutions of the system of equations

$$x(1.5 - 0.5\,x - y) = 0$$
$$y(0.75 - y - 0.125\,x) = 0\,.$$

The four critical points are $(0\,,0)$, $(0\,,3/4)$, $(3\,,0)$ and $(2\,,1/2)$.

(c) The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 3/2 - x - y & -x \\ -y/8 & 3/4 - x/8 - 2y \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0) = \begin{pmatrix} 3/2 & 0 \\ 0 & 3/4 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 3/2\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\ ;\ r_2 = 3/4\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are positive, hence the origin is an unstable node.
At the critical point $(0\,,3/4)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,3/4) = \begin{pmatrix} 3/4 & 0 \\ -3/32 & -3/4 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 3/4\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} -16 \\ 1 \end{pmatrix}\ ;\ r_2 = -3/4\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the critical point $(0\,,3/4)$ is a saddle, which is unstable.

At the critical point $(3\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(3\,,0)=\begin{pmatrix}-3/2 & -3\\ 0 & 3/8\end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1=-3/2\ ,\ \boldsymbol{\xi}^{(1)}=\begin{pmatrix}1\\0\end{pmatrix}\ ;\ r_2=3/8\ ,\ \boldsymbol{\xi}^{(2)}=\begin{pmatrix}-8\\5\end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the critical point $(0\,,3/4)$ is a saddle, which is unstable.
At the critical point $(2\,,1/2)$, the coefficient matrix of the linearized system is

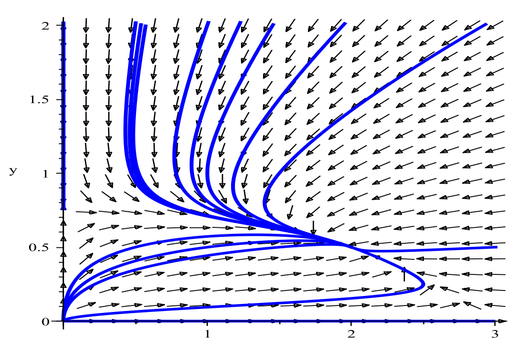
$$\mathbf{J}(2\,,1/2)=\begin{pmatrix}-1 & -2\\ -1/16 & -1/2\end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1=\frac{-3+\sqrt{3}}{4}\ ,\ \boldsymbol{\xi}^{(1)}=\begin{pmatrix}1\\ -\frac{1+\sqrt{3}}{8}\end{pmatrix}\ ;\ \ r_2=-\frac{3+\sqrt{3}}{4}\ ,\ \boldsymbol{\xi}^{(2)}=\begin{pmatrix}0\\ \frac{-1+\sqrt{3}}{8}\end{pmatrix}.$$

The eigenvalues are negative, hence the critical point $(2\,,1/2)$ is a stable node, which is asymptotically stable.

(d,e)



(f) Except for initial conditions along the coordinate axes, almost all solutions converge to the stable node $(2\,,1/2)$. Thus the species can coexist.

7. It follows immediately that

$$(\sigma_1X+\sigma_2Y)^2-4\sigma_1\sigma_2XY=\sigma_1^2X^2+2\sigma_1\sigma_2XY+\sigma_2^2Y^2-4\sigma_1\sigma_2XY=(\sigma_1X-\sigma_2Y)^2,$$

so

$$(\sigma_1X+\sigma_2Y)^2-4(\sigma_1\sigma_2-\alpha_1\alpha_2)XY=(\sigma_1X-\sigma_2Y)^2+4\alpha_1\alpha_2XY.$$

Since all parameters and variables are positive, it follows that

$$(\sigma_1X+\sigma_2Y)^2-4(\sigma_1\sigma_2-\alpha_1\alpha_2)XY\geq 0\,.$$

Hence the radicand in Eq.(39) is nonnegative.

10.(a) The critical points consist of the solution set of the equations

$$x(\epsilon_1 - \sigma_1 x - \alpha_1 y) = 0$$
$$y(\epsilon_2 - \sigma_2 y - \alpha_2 x) = 0\,.$$

If $x = 0$, then either $y = 0$ or $y = \epsilon_2/\sigma_2$. If $\epsilon_1 - \sigma_1 x - \alpha_1 y = 0$, then solving for x results in $x = (\epsilon_1 - \alpha_1 y)/\sigma_1$. Substitution into the second equation yields

$$(\sigma_1\sigma_2 - \alpha_1\alpha_2)y^2 - (\sigma_1\epsilon_2 - \epsilon_1\alpha_2)y = 0\,.$$

Based on the hypothesis, it follows that $(\sigma_1\epsilon_2 - \epsilon_1\alpha_2)y = 0$. So if $\sigma_1\epsilon_2 - \epsilon_1\alpha_2 \neq 0$, then $y = 0$, and the critical points are located at $(0\,,0)$, $(0\,,\epsilon_2/\sigma_2)$ and $(\epsilon_1/\sigma_1, 0)$. For the case $\sigma_1\epsilon_2 - \epsilon_1\alpha_2 = 0$, y can be arbitrary. From the relation

$$x = (\epsilon_1 - \alpha_1 y)/\sigma_1,$$

we conclude that all points on the line $\sigma_1 x + \alpha_1 y = \epsilon_1$ are critical points, in addition to the point $(0\,,0)$.

(b) The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} \epsilon_1 - 2\sigma_1 x - \alpha_1 y & -\alpha_1 x \\ -\alpha_2 y & \epsilon_2 - 2\sigma_2 y - \alpha_2 x \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0) = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix},$$

with eigenvalues $r_1 = \epsilon_1$ and $r_2 = \epsilon_2$. Since both eigenvalues are positive, the origin is an unstable node.
At the point $(0\,,\epsilon_2/\sigma_2)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,\epsilon_2/\sigma_2) = \begin{pmatrix} (\epsilon_1\sigma_2 - \alpha_1\epsilon_2)/\sigma_2 & 0 \\ \epsilon_2\alpha_2/\sigma_2 & -\epsilon_2 \end{pmatrix} = \begin{pmatrix} (\epsilon_1\alpha_2 - \sigma_1\epsilon_2)/\alpha_2 & 0 \\ \epsilon_2\alpha_2/\sigma_2 & -\epsilon_2 \end{pmatrix},$$

since $\sigma_1\sigma_2 - \alpha_1\alpha_2 = 0$ implies that $\alpha_1/\sigma_2 = \sigma_1/\alpha_2$. Thus the matrix has eigenvalues $r_1 = (\epsilon_1\alpha_2 - \sigma_1\epsilon_2)/\alpha_2$ and $r_2 = -\epsilon_2$. If $\sigma_1\epsilon_2 - \epsilon_1\alpha_2 > 0$, then both eigenvalues are negative. Hence the point $(0\,,\epsilon_2/\sigma_2)$ is a stable node, which is asymptotically stable. If $\sigma_1\epsilon_2 - \epsilon_1\alpha_2 < 0$, then the eigenvalues are of opposite sign. Hence the point $(0\,,\epsilon_2/\sigma_2)$ is a saddle, which is unstable.
At the point $(\epsilon_1/\sigma_1, 0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(\epsilon_1/\sigma_1, 0) = \begin{pmatrix} -\epsilon_1 & -\epsilon_1\alpha_1/\sigma_1 \\ 0 & (\sigma_1\epsilon_2 - \epsilon_1\alpha_2)/\sigma_1 \end{pmatrix},$$

with eigenvalues $r_1 = (\sigma_1\epsilon_2 - \epsilon_1\alpha_2)/\sigma_1$ and $r_2 = -\epsilon_1$. If $\sigma_1\epsilon_2 - \epsilon_1\alpha_2 > 0$, then the eigenvalues are of opposite sign. Hence the point $(\epsilon_1/\sigma_1, 0)$ is a saddle, which is unstable. If $\sigma_1\epsilon_2 - \epsilon_1\alpha_2 < 0$, then both eigenvalues are negative. In that case the point $(\epsilon_1/\sigma_1, 0)$ is a stable node, which is asymptotically stable.

(c) As shown in part (a), when $\sigma_1\epsilon_2 - \epsilon_1\alpha_2 = 0$, the set of critical points consists of $(0, 0)$ and all the points on the straight line $\sigma_1 x + \alpha_1 y = \epsilon_1$. Based on part (b), the origin is still an unstable node. Setting $y = (\epsilon_1 - \sigma_1 x)/\alpha_1$, the Jacobian matrix of the vector field, along the given straight line, is

$$\mathbf{J}(\epsilon_1/\sigma_1, 0) = \begin{pmatrix} -\sigma_1 x & -\alpha_1 x \\ -\alpha_2(\epsilon_1 - \sigma_1 x)/\alpha_1 & \epsilon_2 - 2\sigma_2(\epsilon_1 - \sigma_1 x)/\alpha_1 - \alpha_2 x \end{pmatrix}$$
$$= \begin{pmatrix} -\sigma_1 x & -\alpha_1 x \\ -\alpha_2(\epsilon_1 - \sigma_1 x)/\alpha_1 & \alpha_2 x - \epsilon_1\alpha_2/\sigma_1 \end{pmatrix},$$

since $\sigma_1\sigma_2 - \alpha_1\alpha_2 = 0$ implies that $\sigma_2/\alpha_1 = \alpha_2/\sigma_1$ and since $\sigma_1\epsilon_2 - \epsilon_1\alpha_2 = 0$ implies that $\epsilon_2 = \epsilon_1\alpha_2/\sigma_1$. The characteristic equation of the matrix is

$$r^2 + \left[\frac{\epsilon_1\alpha_2 - \alpha_2\sigma_1 x + \sigma_1^2 x}{\sigma_1}\right] r = 0\,.$$

Since $\epsilon_2 = \epsilon_1\alpha_2/\sigma_1$, we have that $(\epsilon_1\alpha_2 - \alpha_2\sigma_1 x + \sigma_1^2 x)/\sigma_1 = \epsilon_2 - \alpha_2 x + \sigma_1 x$. Hence the characteristic equation can be written as

$$r^2 + [\epsilon_2 - \alpha_2 x + \sigma_1 x]\, r = 0\,.$$

First note that $0 \le x \le \epsilon_1/\sigma_1$. Since the coefficient in the quadratic equation is linear, and

$$\epsilon_2 - \alpha_2 x + \sigma_1 x = \begin{cases} \epsilon_2 \text{ at } x = 0 \\ \epsilon_1 \text{ at } x = \epsilon_1/\sigma_1\,, \end{cases}$$

it follows that the coefficient is positive for $0 \le x \le \epsilon_1/\sigma_1$. Therefore, along the straight line $\sigma_1 x + \alpha_1 y = \epsilon_1$, one eigenvalue is zero and the other one is negative. Hence the continuum of critical points consists of stable nodes, which are asymptotically stable.

11.(a) The critical points are solutions of the system of equations

$$x(1 - x - y) + \delta a = 0$$
$$y(0.75 - y - 0.5\,x) + \delta b = 0\,.$$

Assume solutions of the form

$$x = x_0 + x_1\delta + x_2\delta^2 + \ldots$$
$$y = y_0 + y_1\delta + y_2\delta^2 + \ldots\,.$$

Substitution of the series expansions results in

$$x_0(1 - x_0 - y_0) + (x_1 - 2x_1x_0 - x_0y_1 - x_1y_0 + a)\delta + \ldots = 0$$
$$y_0(0.75 - y_0 - 0.5\,x_0\,) + (0.75\,y_1 - 2y_0y_1 - x_1y_0/2 - x_0y_1/2 + b)\delta + \ldots = 0\,.$$

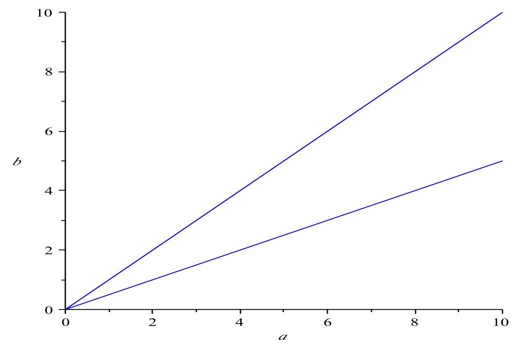
(b) Taking a limit as $\delta \to 0$, the equations reduce to the original system of equations. It follows that $x_0 = y_0 = 0.5$.

(c) Setting the coefficients of the linear terms equal to zero, we find that

$$-y_1/2 - x_1/2 + a = 0$$
$$-x_1/4 - y_1/2 + b = 0\,,$$

with solution $x_1 = 4a - 4b$ and $y_1 = -2a + 4b$.

(d) Consider the ab-parameter space. The collection of points for which $b < a$ represents an increase in the level of species 1. At points where $b > a$, $x_1\delta < 0$. Likewise, the collection of points for which $2b > a$ represents an increase in the level of species 2. At points where $2b < a$, $y_1\delta < 0$.



It follows that if $b < a < 2b$, the level of both species will increase. This condition is represented by the wedge-shaped region on the graph. Otherwise, the level of one species will increase, whereas the level of the other species will simultaneously decrease. Only for $a = b = 0$ will both populations remain the same.

12.(a) The critical points consist of the solution set of the equations

$$-y = 0$$
$$-\gamma\, y - x(x - 0.15)(x - 2) = 0\,.$$

Setting $y = 0$, the second equation becomes $x(x - 0.15)(x - 2) = 0$, with roots $x = 0$, 0.15 and 2. Hence the critical points are located at $(0\,,0)$, $(0.15\,,0)$ and $(2\,,0)$. The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 0 & -1 \\ -3x^2 + 4.3\,x - 0.3 & -\gamma \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0) = \begin{pmatrix} 0 & -1 \\ -0.3 & -\gamma \end{pmatrix},$$

with eigenvalues

$$r_{1,2} = -\frac{\gamma}{2} \pm \frac{1}{10}\sqrt{25\,\gamma^2 + 30}\,.$$

Regardless of the value of γ, the eigenvalues are real and of opposite sign. Hence $(0\,,0)$ is a saddle, which is unstable.
At the critical point $(0.15\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0.15\,,0) = \begin{pmatrix} 0 & -1 \\ 0.2775 & -\gamma \end{pmatrix},$$

with eigenvalues

$$r_{1,2} = -\frac{\gamma}{2} \pm \frac{1}{20}\sqrt{100\,\gamma^2 - 111}\,.$$

If $100\,\gamma^2 - 111 \geq 0$, then the eigenvalues are real. Furthermore, since $r_1 r_2 = 0.2775$, both eigenvalues will have the same sign. Therefore the critical point is a node, with its stability dependent on the sign of γ. If $100\,\gamma^2 - 111 < 0$, the eigenvalues are complex conjugates. In that case the critical point $(0.15\,,0)$ is a spiral, with its stability dependent on the sign of γ.
At the critical point $(2\,,0)$, the coefficient matrix of the linearized system is

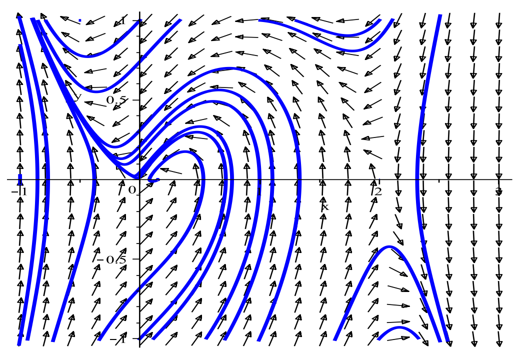
$$\mathbf{J}(2\,,0) = \begin{pmatrix} 0 & -1 \\ -3.7 & -\gamma \end{pmatrix},$$

with eigenvalues

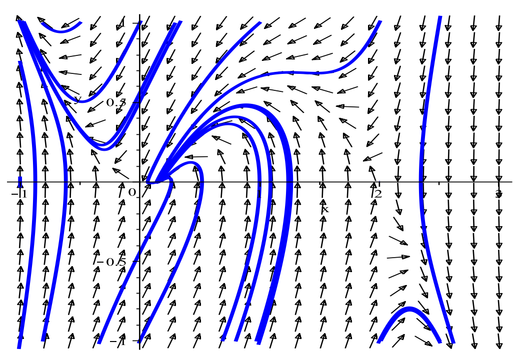
$$r_{1,2} = -\frac{\gamma}{2} \pm \frac{1}{10}\sqrt{25\,\gamma^2 + 370}\,.$$

Regardless of the value of γ, the eigenvalues are real and of opposite sign. Hence $(2\,,0)$ is a saddle, which is unstable.

(b)

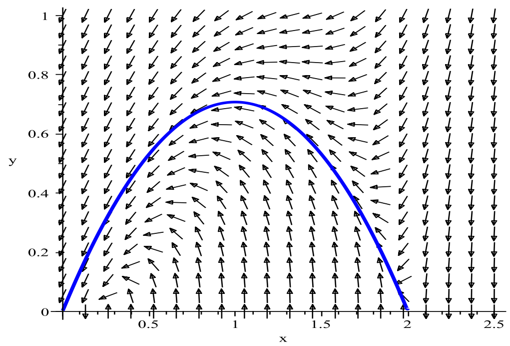


For $\gamma = 0.8$, the critical point $(0.15\,,0)$ is a stable spiral.

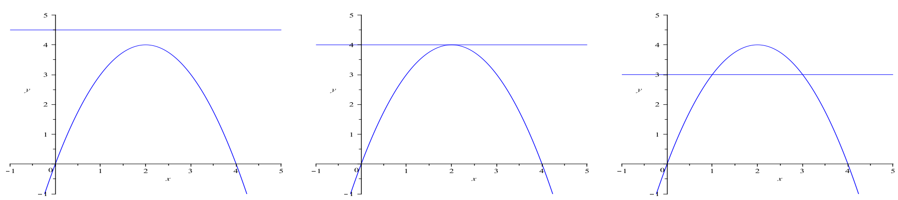


Closer examination shows that for $\gamma = 1.5$, the critical point $(0.15\,,0)$ is a stable node.

(c) Based on the phase portraits in part (b), it is apparent that the required value of γ satisfies $0.8 < \gamma < 1.5$. Using the initial condition $x(0) = 2$ and $y(0) = 0.01$, it is possible to solve the initial value problem for various values of γ. A reasonable first guess is $\gamma = \sqrt{1.11}$. This value marks the change in qualitative behavior of the critical point $(0.15\,,0)$. Numerical experiments show that the solution remains positive for $\gamma \approx 1.20$.



14.(a) Nullclines:



(a) $\alpha = 3$ (b) $\alpha = 8/3$ (c) $\alpha = 2$

(b) The critical points are solutions of the algebraic system

$$\frac{3}{2}\alpha - y = 0$$
$$-4x + y + x^2 = 0$$

which are

$$\left(2 \pm \sqrt{4 - \frac{3}{2}\alpha}\,,\, \frac{3}{2}\alpha\right),$$

and exist for $\alpha \le 8/3$.

(c) For $\alpha = 2$, the critical points are at $(1, 3)$ and $(3, 3)$. The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 0 & -1 \\ -4 + 2\,x & 1 \end{pmatrix}.$$

At the critical point $(1\,,3)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(1\,,3) = \begin{pmatrix} 0 & -1 \\ -2 & 1 \end{pmatrix}.$$

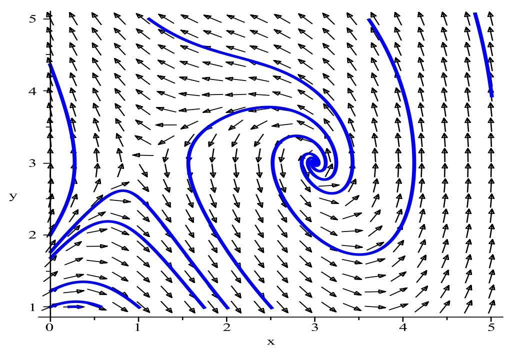
The eigenvalues of the Jacobian, $r = -1$ and 2, are real and opposite in sign; hence the critical point is a saddle point.
At the critical point $(3\,,3)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(3\,,3) = \begin{pmatrix} 0 & -1 \\ 2 & 1 \end{pmatrix}.$$

The eigenvalues of the Jacobian are

$$r = \frac{1 \pm i\sqrt{7}}{2}\,,$$

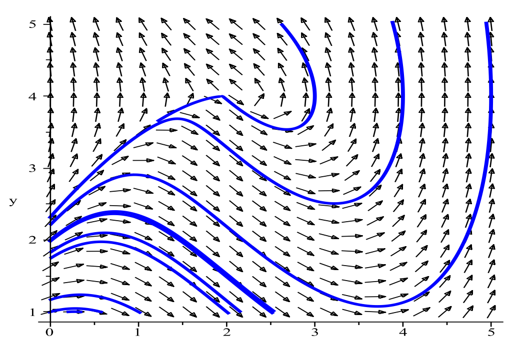
hence the critical point is an unstable spiral.



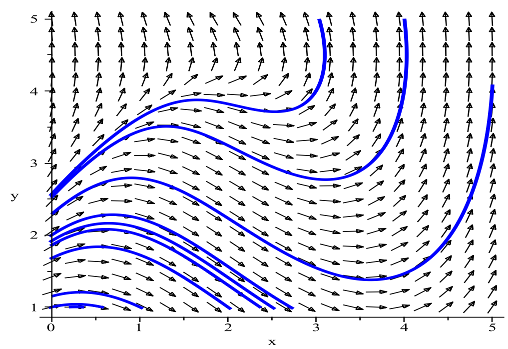
(d) The bifurcation value is $\alpha_0 = 8/3$. The coincident critical points are at $(2, 4)$. The coefficient matrix of the linearized system is

$$\mathbf{J}(2\,,4) = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}.$$

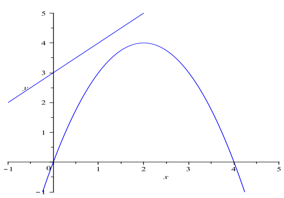
The eigenvalues of the Jacobian are $r = 0$ and 1.



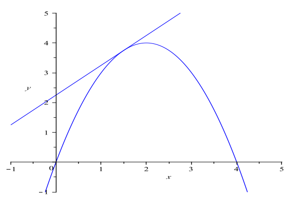
(e)



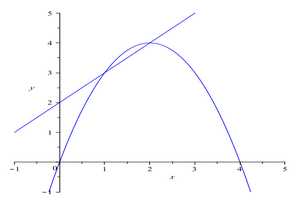
15.(a) Nullclines:



(a) $\alpha = 3$



(b) $\alpha = 9/4$



(c) $\alpha = 2$

(b) The critical points are solutions of the algebraic system

$$-4x + y + x^2 = 0$$
$$-\alpha - x + y = 0$$

which are

$$x_0 = \frac{3+\sqrt{9-4\alpha}}{2} \;\; , \; y_0 = \alpha + \frac{3+\sqrt{9-4\alpha}}{2}$$

and

$$x_0 = \frac{3-\sqrt{9-4\alpha}}{2} \;\; , \; y_0 = \alpha + \frac{3-\sqrt{9-4\alpha}}{2} \; .$$

These critical points exist for $\alpha \leq 9/4$.

(c) For $\alpha = 2$, the critical points are at $(1,3)$ and $(2,4)$. The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 2x-4 & 1 \\ -1 & 1 \end{pmatrix}.$$

At the critical point $(1\,,3)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(1\,,3) = \begin{pmatrix} -2 & 1 \\ -1 & 1 \end{pmatrix}.$$

The eigenvalues of the Jacobian are

$$r = \frac{-1 \pm \sqrt{5}}{2}\,,$$

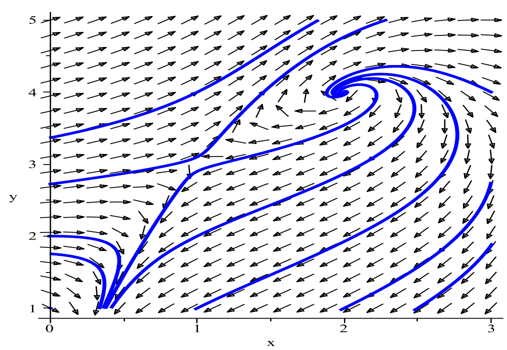
hence the critical point is a saddle point.
At the critical point $(2\,,4)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(2\,,4) = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

The eigenvalues of the Jacobian are

$$r = \frac{1 \pm i\,\sqrt{3}}{2}\,,$$

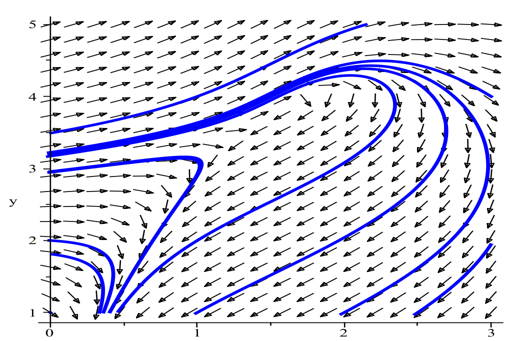
hence the critical point is an unstable spiral.



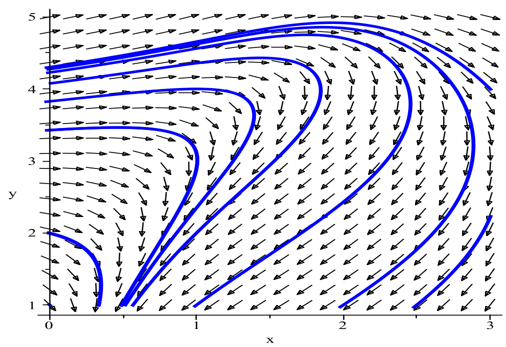
(d) A bifurcation occurs for $\alpha_0 = 9/4$. The coincident critical points are at $(3/2, 15/4)$.
The coefficient matrix of the linearized system is

$$\mathbf{J}(3/2\,,15/4) = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix},$$

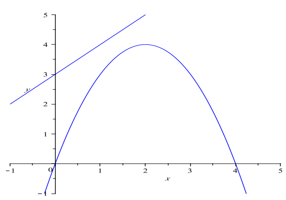
with eigenvalues both equal to zero.



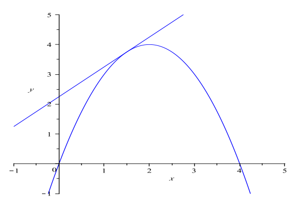
(e)



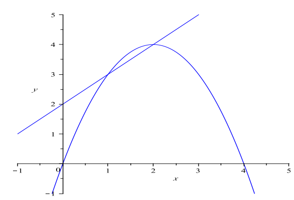
16.(a) Nullclines:



(a) $\alpha = 3$



(b) $\alpha = 9/4$



(c) $\alpha = 2$

(b) The critical points are solutions of the algebraic system

$$-\alpha - x + y = 0$$
$$-4x + y + x^2 = 0$$

which are

$$x_0 = \frac{3 + \sqrt{9 - 4\alpha}}{2} \ , \ y_0 = \alpha + \frac{3 + \sqrt{9 - 4\alpha}}{2}$$

and

$$x_0 = \frac{3 - \sqrt{9 - 4\alpha}}{2} \ , \ y_0 = \alpha + \frac{3 - \sqrt{9 - 4\alpha}}{2} \ .$$

These critical points exist for $\alpha \leq 9/4$.

(c) For $\alpha = 2$, the critical points are at $(1,3)$ and $(2,4)$. The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} -1 & 1 \\ 2x - 4 & 1 \end{pmatrix}.$$

At the critical point $(1\,,3)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(1\,,3) = \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}.$$

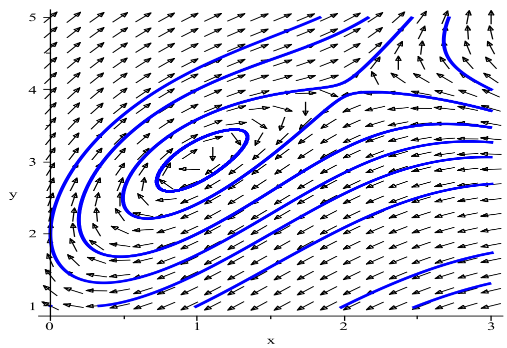
The eigenvalues of the Jacobian are

$$r = \pm\, i\,,$$

hence the critical point is a center.
At the critical point $(2\,,4)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(2\,,4) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}.$$

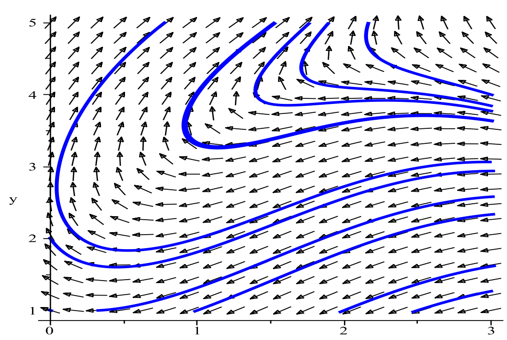
The eigenvalues of the Jacobian, $r = -1$ and 1 , are real and opposite in sign; hence the critical point is a saddle point.



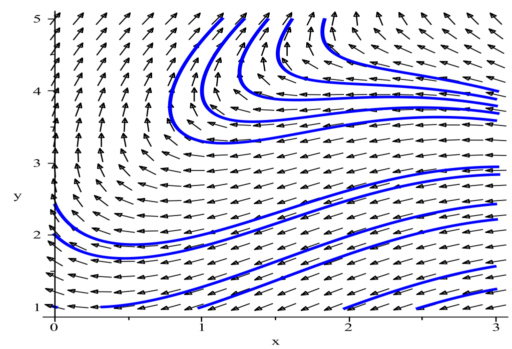
(d) The bifurcation value is $\alpha_0 = 8/3$. The coincident critical point is at $(3/2, 14/4)$. The coefficient matrix of the linearized system is

$$\mathbf{J}(3/2\,,15/4) = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}.$$

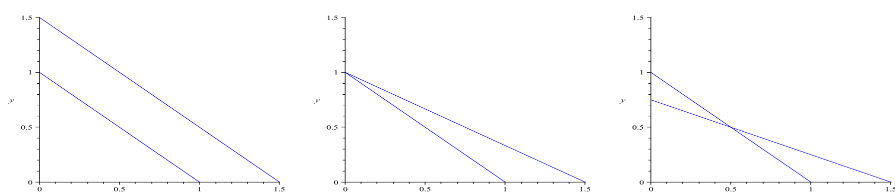
The eigenvalues are both equal to zero.



(e)



18.(a) Nullclines:



(a) $\alpha = 1/2$ (b) $\alpha = 3/4$ (c) $\alpha = 1$

(b) The critical points are $(0,0)$, $(1,0)$, $((4\alpha-3)/(4\alpha-2), 1/(4\alpha-2))$, and $(0, 3/(4\alpha))$. The third critical point is in the first quadrant as long as $\alpha \geq 3/4$.

(c) The third and fourth critical points will coincide (see part (b)) when $\alpha = 3/4$.

(d,e) The Jacobian is

$$\mathbf{J} = \begin{pmatrix} 1-2x-y & -x \\ -y/2 & 3/4-2\alpha y - x/2 \end{pmatrix}.$$

This means that at the origin

$$\mathbf{J}(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 3/4 \end{pmatrix},$$

so the origin is an unstable node. (The eigenvalues are clearly 1 and 3/4.)
At the critical point $(1,0)$ the Jacobian is

$$\mathbf{J}(1,0) = \begin{pmatrix} -1 & -1 \\ 0 & 1/4 \end{pmatrix},$$

which means that this critical point is a saddle.

At the critical point $(0, 3/(4\alpha))$ the Jacobian is

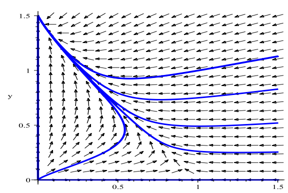
$$\mathbf{J}(0, 3/(4\alpha)) = \begin{pmatrix} 1 - 3/(4\alpha) & 0 \\ -3/(8\alpha) & -3/4 \end{pmatrix},$$

which implies that this critical point is a saddle when $\alpha > 3/4$ and an asymptotically stable node when $0 < \alpha < 3/4$.
At the critical point $(\frac{4\alpha-3}{4\alpha-2}, \frac{1}{4\alpha-2})$ the Jacobian is given by

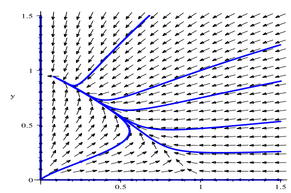
$$\mathbf{J}(\frac{4\alpha - 3}{4\alpha - 2}, \frac{1}{4\alpha - 2}) = \begin{pmatrix} \frac{3-4\alpha}{4\alpha-2} & \frac{3-4\alpha}{4\alpha-2} \\ \frac{-1/2}{4\alpha-2} & \frac{-\alpha}{4\alpha-2} \end{pmatrix}.$$

It can be shown that this is an asymptotically stable node when $\alpha > 3/4$.

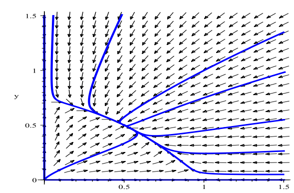
(f) Phase portraits:



(a) $\alpha = 1/2$

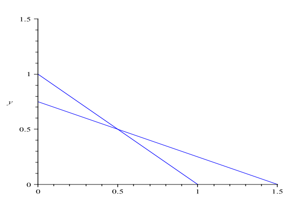


(b) $\alpha = 3/4$

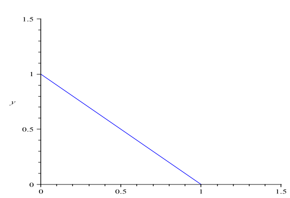


(c) $\alpha = 1$

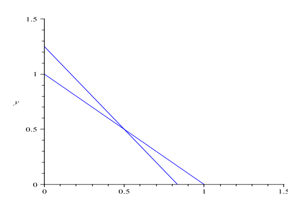
19.(a) Nullclines:



(a) $\alpha = 3/4$



(b) $\alpha = 1$



(c) $\alpha = 5/4$

(b) The critical points are $(0, 0)$, $(1, 0)$, $(0, \alpha)$ and $(1/2, 1/2)$. Also, when $\alpha = 1$, then all points on the line $x + y = 1$ are critical points.

(c) Clearly, $\alpha = 1$ is the bifurcation value.

(d,e) The Jacobian is

$$\mathbf{J} = \begin{pmatrix} 1-2x-y & -x \\ -(2\alpha-1)y & \alpha-2y-(2\alpha-1)x \end{pmatrix}.$$

This means that at the origin

$$\mathbf{J}(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix},$$

so the origin is an unstable node when $\alpha > 0$.
At the critical point $(1,0)$ the Jacobian is

$$\mathbf{J}(1,0) = \begin{pmatrix} -1 & -1 \\ 0 & 1-\alpha \end{pmatrix},$$

which means that this critical point is a saddle when $0 < \alpha < 1$ and an asymptotically stable node when $\alpha > 1$.
At the critical point $(0,\alpha)$ the Jacobian is

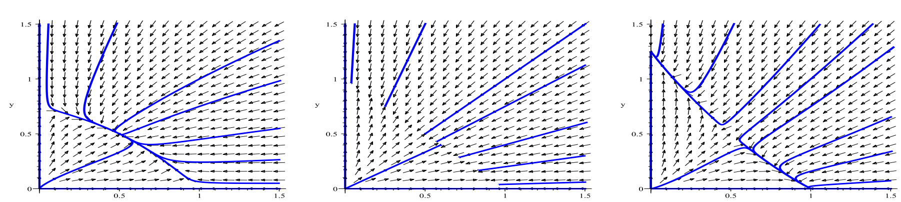
$$\mathbf{J}(0,\alpha)) = \begin{pmatrix} 1-\alpha & 0 \\ -\alpha(2\alpha-1) & -\alpha \end{pmatrix},$$

which implies that this critical point is a saddle when $0 < \alpha < 1$ and an asymptotically stable node when $\alpha > 1$.
At the critical point $(1/2, 1/2)$ the Jacobian is given by

$$\mathbf{J}(1/2,1/2) = \begin{pmatrix} -1/2 & -1/2 \\ 1/2-\alpha & -1/2 \end{pmatrix}.$$

The eigenvalues of $\mathbf{J}(1/2,1/2)$ are $(-1 \pm \sqrt{-1+2\alpha})/2$. Thus the critical point is an asymptotically stable spiral for $0 < \alpha < 1/2$, an asymptotically stable node for $1/2 \le \alpha < 1$, and a saddle for $\alpha > 1$.

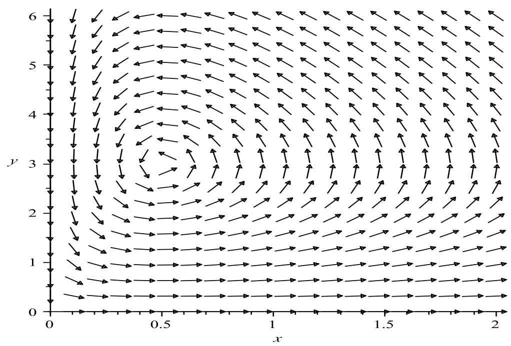
(f) Phase portraits:



(a) $\alpha = 3/4$ (b) $\alpha = 1$ (c) $\alpha = 5/4$

9.5

1.(a)



(b) The critical points are solutions of the system of equations

$$x(1.5 - 0.5\,y) = 0$$
$$y(-0.5 + x) = 0\,.$$

The two critical points are $(0\,,0)$ and $(0.5\,,3)$.

(c) The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 3/2 - y/2 & -x/2 \\ y & -1/2 + x \end{pmatrix}.$$

At the critical point $(0\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0) = \begin{pmatrix} 3/2 & 0 \\ 0 & -1/2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 3/2\;,\;\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\;;\; r_2 = -1/2\;,\;\boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the origin is a saddle, which is unstable. At the critical point $(0.5\,,3)$, the coefficient matrix of the linearized system is

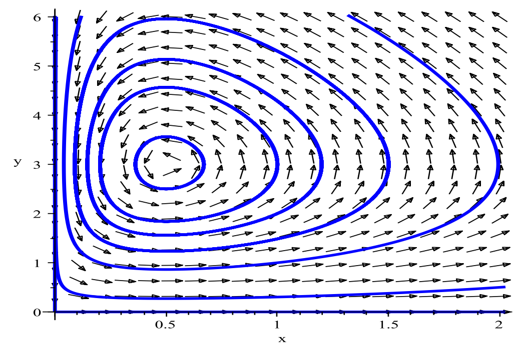
$$\mathbf{J}(0.5\,,3) = \begin{pmatrix} 0 & -1/4 \\ 3 & 0 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = i\,\frac{\sqrt{3}}{2}\;,\;\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -2\,i\sqrt{3} \end{pmatrix}\;;\; r_2 = -i\,\frac{\sqrt{3}}{2}\;,\;\boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 2\,i\sqrt{3} \end{pmatrix}.$$

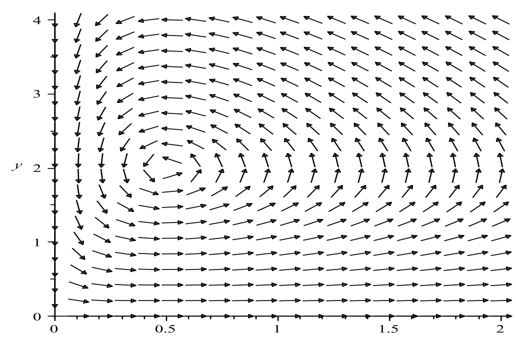
The eigenvalues are purely imaginary. Hence the critical point is a center, which is stable.

(d,e)



(f) Except for solutions along the coordinate axes, almost all trajectories are closed curves about the critical point $(0.5\,,3)$.

2.(a)



(b) The critical points are the solution set of the system of equations

$$x(1-0.5\,y)=0$$
$$y(-0.25+0.5\,x)=0\,.$$

The two critical points are $(0\,,0)$ and $(0.5\,,2)$.

(c) The Jacobian matrix of the vector field is

$$\mathbf{J}=\begin{pmatrix}1-y/2 & -x/2\\ y/2 & -1/4+x/2\end{pmatrix}.$$

At the critical point $(0\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0)=\begin{pmatrix}1 & 0\\ 0 & -1/4\end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1=1\ ,\ \boldsymbol{\xi}^{(1)}=\begin{pmatrix}1\\0\end{pmatrix}\ ;\ r_2=-1/4\ ,\ \boldsymbol{\xi}^{(2)}=\begin{pmatrix}0\\1\end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the origin is a saddle, which is unstable. At the critical point $(0.5\,,2)$, the coefficient matrix of the linearized system is

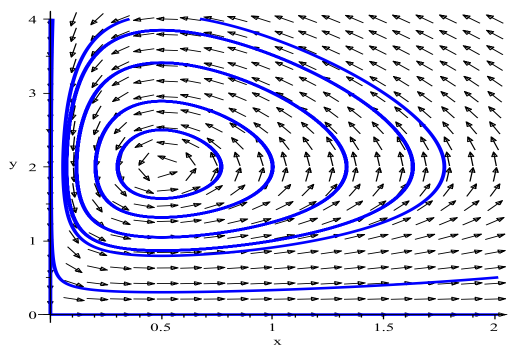
$$\mathbf{J}(0.5\,,2) = \begin{pmatrix} 0 & -1/4 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = i/2\ ,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -2\,i \end{pmatrix}\ ;\ r_2 = -i/2\ ,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 2\,i \end{pmatrix}.$$

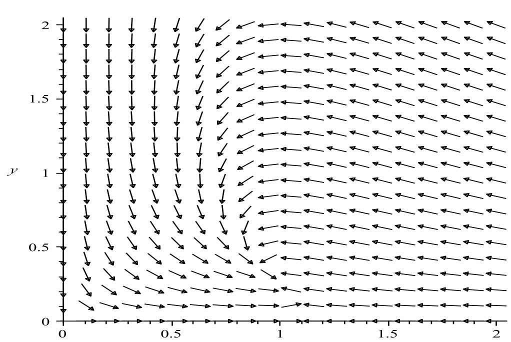
The eigenvalues are purely imaginary. Hence the critical point is a center, which is stable.

(d,e)



(f) Except for solutions along the coordinate axes, almost all trajectories are closed curves about the critical point $(0.5\,,2)$.

4.(a)



(b) The critical points are the solution set of the system of equations

$$x(9/8 - x - y/2) = 0$$
$$y(-1 + x) = 0\,.$$

The three critical points are $(0\,,0)$, $(9/8\,,0)$ and $(1\,,1/4)$.

(c) The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 9/8 - 2x - y/2 & -x/2 \\ y & -1 + x \end{pmatrix}.$$

At the critical point $(0\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0) = \begin{pmatrix} 9/8 & 0 \\ 0 & -1 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 9/8\ ,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\ ;\ r_2 = -1\ ,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the origin is a saddle, which is unstable. At the critical point $(9/8\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(9/8\,,0) = \begin{pmatrix} -9/8 & -9/16 \\ 0 & 1/8 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = -\frac{9}{8}\ ,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\ ;\ r_2 = \frac{1}{8}\ ,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 9 \\ -20 \end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the critical point $(9/8\,,0)$ is a saddle, which is unstable.
At the critical point $(1\,,1/4)$, the coefficient matrix of the linearized system is

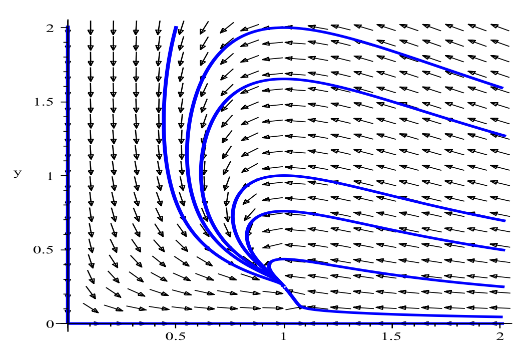
$$\mathbf{J}(1\,,1/4) = \begin{pmatrix} -1 & -1/2 \\ 1/4 & 0 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = \frac{-2+\sqrt{2}}{4}\ ,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} -2+\sqrt{2} \\ 1 \end{pmatrix}\ ;\ r_2 = \frac{-2-\sqrt{2}}{4}\ ,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} -2-\sqrt{2} \\ 1 \end{pmatrix}.$$

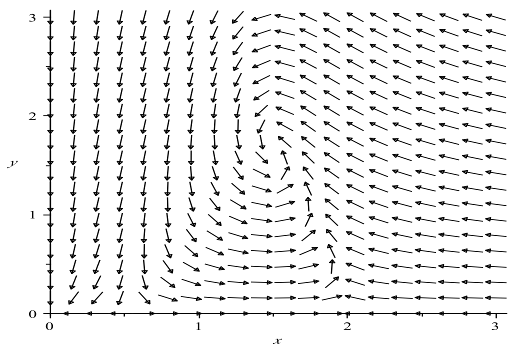
The eigenvalues are both negative. Hence the critical point is a stable node, which is asymptotically stable.

(d,e)



(f) Except for solutions along the coordinate axes, all solutions converge to the critical point $(1\,,1/4)$.

5.(a)



(b) The critical points are solutions of the system of equations

$$x(-1+2.5\,x-0.3\,y-x^2)=0$$
$$y(-1.5+\ x)=0\,.$$

The four critical points are $(0\,,0)$, $(1/2\,,0)$, $(2\,,0)$ and $(3/2\,,5/3)$.

(c) The Jacobian matrix of the vector field is

$$\mathbf{J}=\begin{pmatrix}-1+5x-3x^2-3y/10 & -3x/10\\ y & -3/2+x\end{pmatrix}.$$

At the critical point $(0\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0)=\begin{pmatrix}-1 & 0\\ 0 & -3/2\end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1=-1\ ,\ \boldsymbol{\xi}^{(1)}=\begin{pmatrix}1\\0\end{pmatrix}\ ;\ r_2=-3/2\ ,\ \boldsymbol{\xi}^{(2)}=\begin{pmatrix}0\\1\end{pmatrix}.$$

The eigenvalues are both negative, hence the critical point $(0\,,0)$ is a stable node, which is asymptotically stable.
At the critical point $(1/2\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(1/2\,,0)=\begin{pmatrix}3/4 & -3/20\\ 0 & -1\end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1=\frac{3}{4}\ ,\ \boldsymbol{\xi}^{(1)}=\begin{pmatrix}1\\0\end{pmatrix}\ ;\ r_2=-1\ ,\ \boldsymbol{\xi}^{(2)}=\begin{pmatrix}3\\35\end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the critical point $(1/2\,,0)$ is a saddle, which is unstable.

At the critical point $(2\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(2\,,0) = \begin{pmatrix} -3 & -3/5 \\ 0 & 1/2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = -3\ ,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\ ;\ r_2 = 1/2\ ,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 6 \\ -35 \end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the critical point $(2\,,0)$ is a saddle, which is unstable.
At the critical point $(3/2\,,5/3)$, the coefficient matrix of the linearized system is

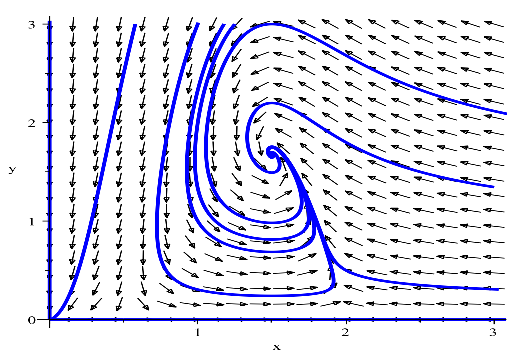
$$\mathbf{J}(3/2\,,5/3) = \begin{pmatrix} -3/4 & -9/20 \\ 5/3 & 0 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = \frac{-3 + i\sqrt{39}}{8}\ ,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} \frac{-9+i\,3\sqrt{39}}{40} \\ 1 \end{pmatrix}\ ;\ r_2 = \frac{-3 - i\sqrt{39}}{8}\ ,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} \frac{-9-i\,3\sqrt{39}}{40} \\ 1 \end{pmatrix}.$$

The eigenvalues are complex conjugates. Hence the critical point $(3/2\,,5/3)$ is a stable spiral, which is asymptotically stable.

(d,e)



(f) The single solution curve that converges to the node at $(1/2\,,0)$ is a separatrix. Except for initial conditions on the coordinate axes, trajectories on either side of the separatrix converge to the node at $(0\,,0)$ or the stable spiral at $(3/2\,,5/3)$.

6. Given that t is measured from the time that x is a maximum, we have

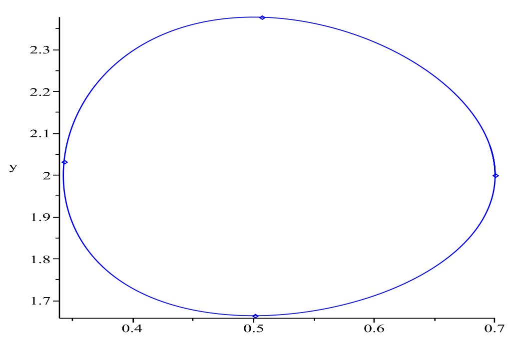
$$x = \frac{c}{\gamma} + \frac{cK}{\gamma}\cos(\sqrt{ac}\,t)$$
$$y = \frac{a}{\alpha} + K\frac{a}{\alpha}\sqrt{\frac{c}{\alpha}}\,\sin(\sqrt{ac}\,t)\,.$$

(a) Since $\sin\theta$ reaches a maximum at $\theta = \pi/2$, y reaches a maximum at $\sqrt{ac}\,t = \pi/2$, or $t = \pi/(2\sqrt{ac}) = T/4$, where $T = 2\pi/\sqrt{ac}$ is the period of oscillation.

(b) Note that
$$\frac{dx}{dt} = -\frac{cK\sqrt{ac}}{\gamma}\sin(\sqrt{ac}\,t).$$
Since $\sin\theta$ reaches a minimum of -1 at $\theta = 3\pi/2, 7\pi/2, \ldots$, x is increasing most rapidly at $\sqrt{ac}\,t = 3\pi/2, 7\pi/2, \ldots$, or $t = 3T/4, 7T/4, \ldots$. Since $\sin\theta$ reaches a maximum of 1 at $\theta = \pi/2, 5\pi/2, \ldots$, x is decreasing most rapidly at $\sqrt{ac}\,t = \pi/2, 5\pi/2, \ldots$, or $t = T/4, 5T/4, \ldots$. Since $\cos\theta$ reaches a minimum at $\theta = \pi, 3\pi, \ldots$, x reaches a minimum at $\sqrt{ac}\,t = \pi, 3\pi, \ldots$, or $t = T/2, 3T/2, \ldots$.

(c) Note that
$$\frac{dy}{dt} = Kc\left(\frac{a}{\alpha}\right)^{3/2}\cos(\sqrt{ac}\,t).$$
Since $\cos\theta$ reaches a minimum of -1 at $\theta = \pi, 3\pi, \ldots$, y is decreasing most rapidly at $\sqrt{ac}\,t = \pi, 3\pi, \ldots$, or $t = T/2, 3T/2, \ldots$. Since $\cos\theta$ reaches a maximum of 1 at $\theta = 0, 2\pi, \ldots$, y is increasing most rapidly at $\sqrt{ac}\,t = 0, 2\pi, \ldots$, or $t = 0, T, \ldots$. Since $\sin\theta$ reaches a minimum at $\theta = 3\pi/2, 7\pi/2, \ldots$, y reaches a minimum at $\sqrt{ac}\,t = 3\pi/2, 7\pi/2, \ldots$, or $t = 3T/4, 7T/4, \ldots$.

(d) In the following example, the system in Problem 2 is solved numerically with the initial conditions $x(0) = 0.7$ and $y(0) = 2$. The critical point of interest is at $(0.5\,, 2)$. Since $a = 1$ and $c = 1/4$, it follows that the period of oscillation is $T = 4\pi$.



8.(a) The period of oscillation for the linear system is $T = 2\pi/\sqrt{ac}$. In system (2), $a = 1$ and $c = 0.75$. Hence the period is estimated as $T = 2\pi/\sqrt{0.75} \approx 7.2552$.

(b) The estimated period appears to agree with the graphic in Figure 9.5.3.

(c) The critical point of interest is at $(3\,, 2)$. The system is solved numerically, with $y(0) = 2$ and $x(0) = 3.5\,, 4.0\,, 4.5\,, 5.0$. The resulting periods are shown in the table:

	$x(0) = 3.5$	$x(0) = 4.0$	$x(0) = 4.5$	$x(0) = 5.0$
T	7.26	7.29	7.34	7.42

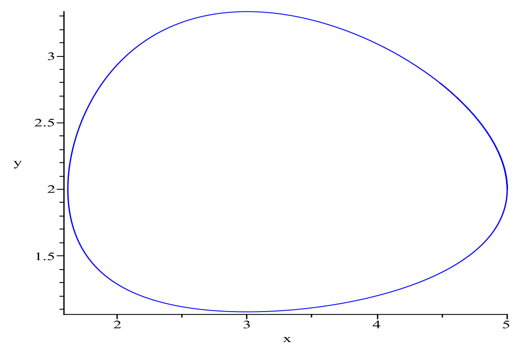
The actual period steadily increases as $x(0)$ increases.

9. The system

$$\frac{dx}{dt} = a\,x(1 - \frac{y}{2})$$
$$\frac{dy}{dt} = b\,y(-1 + \frac{x}{3})$$

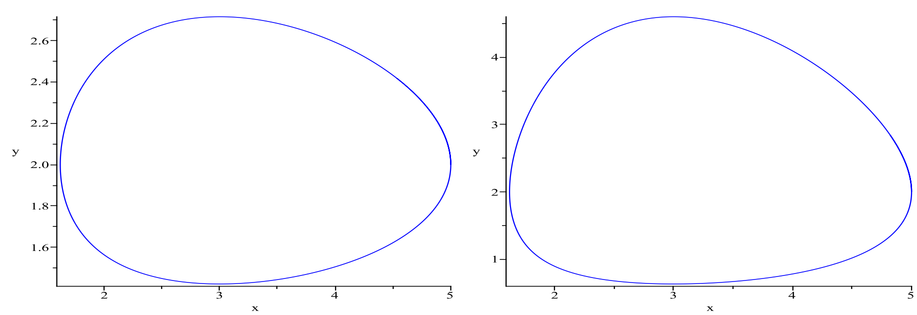
is solved numerically for various values of the parameters. The initial conditions are $x(0) = 5$, $y(0) = 2$.

(a) $a = 1$ and $b = 1$:



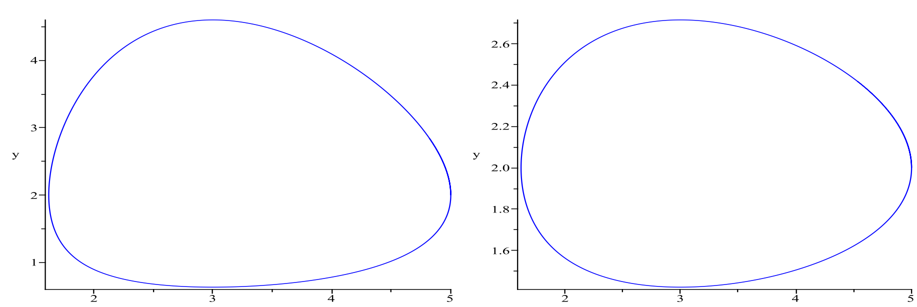
The period is estimated by observing when the trajectory becomes a closed curve. In this case, $T \approx 6.45$.

(b) $a = 3$ and $a = 1/3$, with $b = 1$:



For $a = 3$, $T \approx 3.69$. For $a = 1/3$, $T \approx 11.44$.

(c) $b = 3$ and $b = 1/3$, with $a = 1$:



For $b = 3$, $T \approx 3.82$. For $b = 1/3$, $T \approx 11.06$.

(d) It appears that if one of the parameters is fixed, the period varies inversely with the other parameter. Hence one might postulate the relation

$$T = \frac{k}{f(a\,,b)}\,.$$

10.(a) Since $T = 2\pi/\sqrt{ac}$, we first note that

$$\int_A^{A+T} \cos(\sqrt{ac}\,t + \phi)dt = \int_A^{A+T} \sin(\sqrt{ac}\,t + \phi)dt = 0\,.$$

Hence

$$\overline{x} = \frac{1}{T}\int_A^{A+T} \frac{c}{\gamma}\,dt = \frac{c}{\gamma} \quad \text{and} \quad \overline{y} = \frac{1}{T}\int_A^{A+T} \frac{a}{\alpha}\,dt = \frac{a}{\alpha}\,.$$

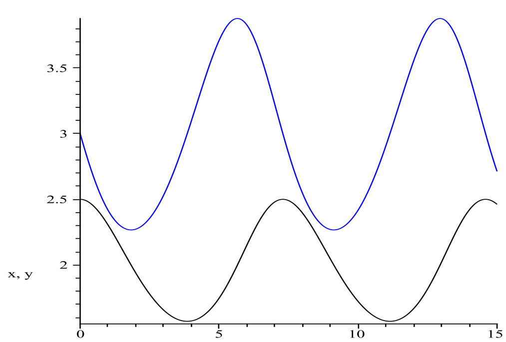
(b) One way to estimate the mean values is to find a horizontal line such that the area above the line is approximately equal to the area under the line. From Figure 9.5.3, it appears that $\overline{x} \approx 3.25$ and $\overline{y} \approx 2.0$. In Example 1, $a = 1$, $c = 0.75$, $\alpha = 0.5$ and $\gamma = 0.25$. Using the result in part (a), $\overline{x} = 3$ and $\overline{y} = 2$.

(c) The system

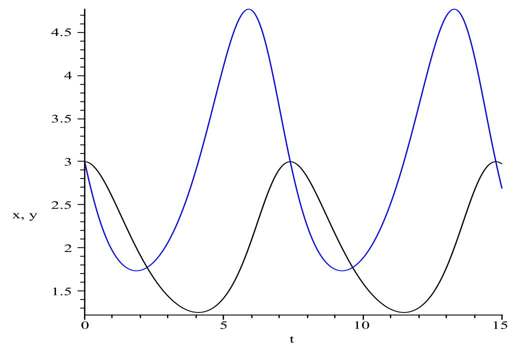
$$\frac{dx}{dt} = x(1 - \frac{y}{2})$$
$$\frac{dy}{dt} = y(-\frac{3}{4} + \frac{x}{4})$$

is solved numerically for various initial conditions.

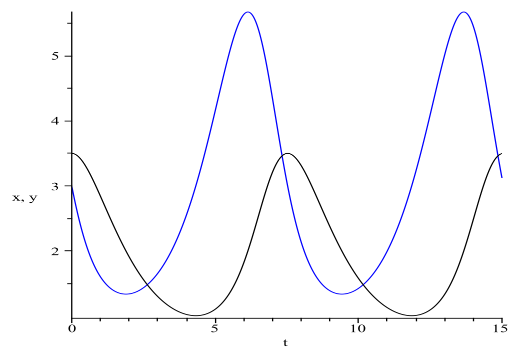
$x(0) = 3$ and $y(0) = 2.5$:



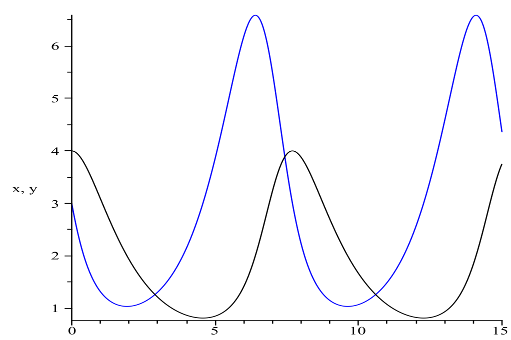
$x(0) = 3$ and $y(0) = 3.0$:



$x(0) = 3$ and $y(0) = 3.5$:



$x(0) = 3$ and $y(0) = 4.0$:



It is evident that the mean values increase as the amplitude increases. That is, the mean values increase as the initial conditions move farther from the critical point.

12.(a) The critical points are the solutions of the system

$$x(a - \sigma x - \alpha y) = 0$$
$$y(-c + \gamma x) = 0.$$

If $x = 0$, then $y = 0$. If $y = 0$, then $x = a/\sigma$. The third solution is found by substituting $x = c/\gamma$ into the first equation. This implies that $y = a/\alpha - \sigma c/(\gamma\alpha)$.

So the critical points are $(0,0)$, $(\frac{a}{\sigma},0)$ and $(\frac{c}{\gamma},\frac{a}{\alpha}-\frac{\sigma c}{\gamma\alpha})$. When σ is increasing, the critical point $(\frac{a}{\sigma},0)$ moves to the left and the critical point $(\frac{c}{\gamma},\frac{a}{\alpha}-\frac{\sigma c}{\gamma\alpha})$ moves down. The assumption $a>\frac{\sigma c}{\gamma}$ is necessary for the third critical point to be in the first quadrant. (When $a=\frac{\sigma c}{\gamma}$, then the two nonzero critical points coincide.)

(b,c) The Jacobian of the system is

$$\mathbf{J}=\begin{pmatrix} a-2\sigma x-\alpha y & -\alpha x \\ \gamma y & -c+\gamma x\end{pmatrix}.$$

This implies that at the origin

$$\mathbf{J}(0,0)=\begin{pmatrix} a & 0 \\ 0 & -c\end{pmatrix},$$

which implies that the origin is a saddle point. ($a>0$ and $c>0$ by our assumption.) At the critical point $(\frac{a}{\sigma},0)$

$$\mathbf{J}(\frac{a}{\sigma},0)=\begin{pmatrix} -a & -\alpha a/\sigma \\ 0 & -c+\gamma a/\sigma\end{pmatrix},$$

which implies that this critical point is also a saddle as long as our assumption $a>\frac{\sigma c}{\gamma}$ is valid.
At the critical point $(\frac{c}{\gamma},\frac{a}{\alpha}-\frac{\sigma c}{\gamma\alpha})$

$$\mathbf{J}(\frac{c}{\gamma},\frac{a}{\alpha}-\frac{\sigma c}{\gamma\alpha})=\begin{pmatrix} -\sigma c/\gamma & -\alpha c/\gamma \\ \gamma a/\alpha-\sigma c/\alpha & 0\end{pmatrix}.$$

The eigenvalues of the matrix are

$$\frac{-c\sigma\pm\sqrt{c^2\sigma^2+4c^2\gamma\sigma-4ac\gamma^2}}{2\gamma}.$$

We set the discriminant equal to zero and find that the greater solution is

$$\sigma_1=-2\gamma+\frac{2\gamma\sqrt{ac+c^2}}{c}.$$

First note that $\sigma_1>0$, since $\sqrt{ac+c^2}>c$. Next we note that $\sigma_1<a\gamma/c$. Since

$$\sqrt{ac+c^2}<\sqrt{\frac{a^2}{4}+ac+c^2}=\frac{a}{2}+c,$$

we see that

$$\sigma_1=-2\gamma+\frac{2\gamma\sqrt{ac+c^2}}{c}<-2\gamma+\frac{2\gamma}{c}\left(\frac{a}{2}+c\right)=-2\gamma+\frac{a\gamma}{c}+2\gamma=\frac{a\gamma}{c}.$$

For $0<\sigma<\sigma_1$, the eigenvalues will be complex conjugates with negative real part, so the critical point will be an asymptotically stable spiral point. For $\sigma=\sigma_1$, the eigenvalues will be repeated and negative, so the critical point will be an asymptotically stable spiral point or node. For $\sigma_1<\sigma<ac/\gamma$, the eigenvalues will be distinct and negative, so the critical point will be an asymptotically stable node.

(d) Since the third critical point is asymptotically stable for $0 < \sigma < ac/\gamma$, and the other critical points are saddle points, the populations will coexist for all such values of σ.

13.(a) The critical points are the solutions of the system

$$x(1 - \frac{x}{5} - \frac{2y}{x+6}) = 0$$
$$y(-\frac{1}{4} + \frac{x}{x+6}) = 0.$$

If $x = 0$, then $y = 0$. If $y = 0$, then $x = 5$. The third critical point can be found by setting $1/4 = x/(x+6)$, which gives $x = 2$ and then $y = 2.4$. So the critical points are $(0,0)$, $(5,0)$ and $(2, 2.4)$.

(b) The Jacobian of the system is

$$\mathbf{J} = \begin{pmatrix} 1 - \frac{2x}{5} - \frac{12y}{(x+6)^2} & -\frac{2x}{x+6} \\ \frac{6y}{(x+6)^2} & -\frac{1}{4} + \frac{x}{x+6} \end{pmatrix}.$$

This implies that at the origin

$$\mathbf{J}(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -1/4 \end{pmatrix},$$

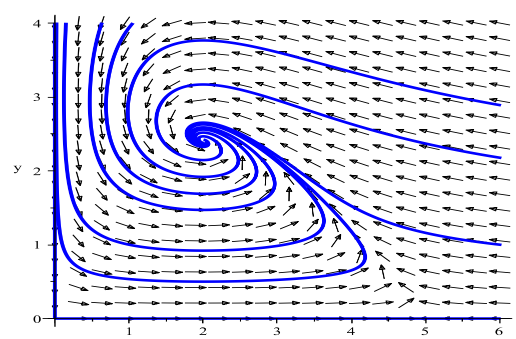
which implies that the origin is a saddle point.
At the critical point $(5,0)$

$$\mathbf{J}(5,0) = \begin{pmatrix} -1 & -10/11 \\ 0 & 9/44 \end{pmatrix},$$

which implies that this critical point is also a saddle point.
At the critical point $(2, 2.4)$

$$\mathbf{J}(2,2.4) = \begin{pmatrix} -1/4 & -1/2 \\ 9/40 & 0 \end{pmatrix},$$

whose eigenvalues are complex with negative real part, which implies that this critical point is an asymptotically stable spiral.



15.(a) Solving for the equilibrium of interest we obtain

$$x = \frac{E_2 + c}{\gamma} \qquad y = \frac{a}{\alpha} - \frac{\sigma}{\alpha} \cdot \frac{E_2 + c}{\gamma} - \frac{E_1}{\alpha}.$$

So if $E_1 > 0$ and $E_2 = 0$, then we have the same amount of prey and fewer predators.

(b) If $E_1 = 0$ and $E_2 > 0$, then we have more prey and fewer predators.

(c) If $E_1 > 0$ and $E_2 > 0$, then we have more prey and even fewer predators.

16.(b) The equilibrium solutions are given by the solutions of the system

$$x(1 - \frac{y}{2}) = H_1$$
$$y(-\frac{3}{4} + \frac{x}{4}) = H_2.$$

Now if $H_2 = 0$, then $x = 3$ and the first equation gives $y = 2 - 2H_1/3$. This means we have the same amount of prey and fewer predators.

(c) If $H_1 = 0$, then $y = 2$ and the second equation gives $x = 3 + 2H_2$. This means we have the same amount of predators and more prey.

(d) If $H_1 > 0$ and $H_2 > 0$, then the second equation gives $(x-3)y = 4H_2$ and using this we obtain from the first equation that $x(1 - \frac{2H_2}{x-3}) = H_1$. This gives the quadratic equation $x^2 - (3 + 2H_2 + H_1)x + 3H_1 = 0$. Now at the old value $x = 3$ this expression is $-6H_2$, so there is a root which is bigger than $x = 3$. The other root of the quadratic equation is closer to 0, so the equilibrium increases here: we have more prey. (Check this with e.g. $H_1 = H_2 = 1$: the roots are $3 \pm \sqrt{6}$, so both of the original roots get bigger.) A similar analysis shows that we will have fewer predators in this case.

9.6

2. We consider the function $V(x,y) = a\,x^2 + c\,y^2$. The rate of change of V along any trajectory is

$$\dot{V} = V_x \frac{dx}{dt} + V_y \frac{dy}{dt} = 2ax(-\frac{1}{2}x^3 + 2xy^2) + 2cy(-y^3) = -ax^4 + 4ax^2y^2 - 2cy^4.$$

Let us complete the square now the following way:

$$\dot{V} = -ax^4 + 4ax^2y^2 - 2cy^4 = -a(x^4 - 4x^2y^2) - 2cy^4 =$$
$$= -a(x^2 - 2y^2)^2 + 4ay^4 - 2cy^4 = -a(x^2 - 2y^2)^2 + (4a - 2c)y^4.$$

If $a > 0$ and $c > 0$, then $V(x,y)$ is positive definite. Clearly, if $4a - 2c < 0$, i.e. when $c > 2a$, then $\dot{V}(x,y)$ is negative definite. One such example is $V(x,y) =$

$x^2 + 3\,y^2$. It follows from Theorem 9.6.1 that the origin is an asymptotically stable critical point.

4. Given $V(x,y) = a\,x^2 + c\,y^2$, the rate of change of V along any trajectory is

$$\dot{V} = V_x \frac{dx}{dt} + V_y \frac{dy}{dt} = 2ax(x^3 - y^3) + 2cy(2xy^2 + 4x^2y + 2y^3) =$$
$$= 2a\,x^4 + (4c - 2a)xy^3 + 8c\,x^2y^2 + 4c\,y^4 .$$

Setting $a = 2c$,

$$\dot{V} = 4c\,x^4 + 8c\,x^2y^2 + 4c\,y^4 \geq 4c\,x^4 + 4c\,y^4 .$$

As long as $a = 2c > 0$, the function $V(x,y)$ is positive definite and $\dot{V}(x,y)$ is also positive definite. It follows from Theorem 9.6.2 that $(0\,,0)$ is an unstable critical point.

5. Given $V(x,y) = c(x^2 + y^2)$, the rate of change of V along any trajectory is

$$\dot{V} = V_x \frac{dx}{dt} + V_y \frac{dy}{dt} = 2c\,x\,[y - xf(x,y)] + 2cy\,[-x - yf(x,y)] = -2c(x^2 + y^2)f(x,y)\,.$$

If $c > 0$, then $V(x,y)$ is positive definite. Furthermore, if $f(x,y)$ is positive in some neighborhood of the origin, then $\dot{V}(x,y)$ is negative definite. Theorem 9.6.1 asserts that the origin is an asymptotically stable critical point. On the other hand, if $f(x,y)$ is negative in some neighborhood of the origin, then $V(x,y)$ and $\dot{V}(x,y)$ are both positive definite. It follows from Theorem 9.6.2 that the origin is an unstable critical point.

9.(a) Letting $x = u$ and $y = u'$, we obtain the system of equations

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -g(x) - y\,.$$

Since $g(0) = 0$, it is evident that $(0\,,0)$ is a critical point of the system. Consider the function

$$V(x,y) = \frac{1}{2}y^2 + \int_0^x g(s)ds\,.$$

It is clear that $V(0,0) = 0$. Since $g(u)$ is an odd function in a neighborhood of $u = 0$,

$$\int_0^x g(s)ds > 0 \text{ for } x > 0\,,$$

and

$$\int_0^x g(s)ds = -\int_x^0 g(s)ds > 0 \text{ for } x < 0\,.$$

Therefore $V(x,y)$ is positive definite. The rate of change of V along any trajectory is

$$\dot{V} = V_x \frac{dx}{dt} + V_y \frac{dy}{dt} = g(x)\cdot(y) + y\,[-g(x) - y] = -y^2\,.$$

It follows that $\dot{V}(x,y)$ is only negative semidefinite. Hence the origin is a stable critical point.

(b) Given

$$V(x,y)=\frac{1}{2}y^2+\frac{1}{2}y\,\sin(x)+\int_0^x \sin(s)ds\,,$$

It is easy to see that $V(0\,,0)=0$. The rate of change of V along any trajectory is

$$\dot{V}=V_x\frac{dx}{dt}+V_y\frac{dy}{dt}=\left[\sin\,x+\frac{y}{2}\cos\,x\right](y)+\left[y+\frac{1}{2}\sin\,x\right][-\sin\,x-y]=$$

$$=\frac{1}{2}y^2\,\cos\,x-\frac{1}{2}\sin^2 x-\frac{y}{2}\sin\,x-y^2\,.$$

For $-\pi/2<x<\pi/2$, we can write $\sin\,x=x-\alpha\,x^3/6$ and $\cos\,x=1-\beta\,x^2/2$, in which $\alpha=\alpha(x)$, $\beta=\beta(x)$. Note that $0<\alpha\,,\beta<1$. Then

$$\dot{V}(x\,,y)=\frac{y^2}{2}\,(1-\frac{\beta\,x^2}{2})-\frac{1}{2}(x-\frac{\alpha\,x^3}{6})^2-\frac{y}{2}(x-\frac{\alpha\,x^3}{6})-y^2\,.$$

Using polar coordinates,

$$\dot{V}(r\,,\theta)=-\frac{r^2}{2}\,[1+\sin\,\theta\,\cos\,\theta+h(r\,,\theta)]=-\frac{r^2}{2}\,\left[1+\frac{1}{2}\sin\,2\theta+h(r\,,\theta)\right]\,.$$

It is easy to show that

$$|h(r\,,\theta)|\leq\frac{1}{2}r^2+\frac{1}{72}r^4.$$

So if r is sufficiently small, then $|h(r\,,\theta)|<1/2$ and $\left|\frac{1}{2}\sin\,2\theta+h(r\,,\theta)\right|<1$. Hence $\dot{V}(x\,,y)$ is negative definite. Now we show that $V(x,y)$ is positive definite. Since $g(u)=\sin\,u$,

$$V(x,y)=\frac{1}{2}y^2+\frac{1}{2}y\,\sin(x)+1-\cos\,x\,.$$

This time we set

$$\cos\,x=1-\frac{x^2}{2}+\gamma\,\frac{x^4}{24}\,.$$

Note that $0<\gamma<1$ for $-\pi/2<x<\pi/2$. Converting to polar coordinates,

$$\begin{aligned}V(r,\theta)\;&=\frac{r^2}{2}\left[1+\sin\,\theta\,\cos\,\theta-\frac{r^2}{12}\sin\,\theta\,\cos^3\theta-\gamma\,\frac{r^2}{24}\cos^4\theta\right]\\&=\frac{r^2}{2}\left[1+\frac{1}{2}\sin\,2\theta-\frac{r^2}{12}\sin\,\theta\,\cos^3\theta-\gamma\,\frac{r^2}{24}\cos^4\theta\right].\end{aligned}$$

Now

$$-\frac{r^2}{12}\sin\,\theta\,\cos^3\theta-\gamma\,\frac{r^2}{24}\cos^4\theta>-\frac{1}{8}\text{ for }r<1\,.$$

It follows that when $r>0$,

$$V(r,\theta)>\frac{r^2}{2}\left[\frac{7}{8}+\frac{1}{2}\sin\,2\theta\right]\geq\frac{3\,r^2}{16}>0\,.$$

Therefore $V(x, y)$ is indeed positive definite, and by Theorem 9.6.1 , the origin is an asymptotically stable critical point.

12.(a) We consider the linear system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Let $V(x, y) = Ax^2 + Bxy + Cy^2$, in which

$$\begin{aligned} A &= -\frac{a_{21}^2 + a_{22}^2 + (a_{11}a_{22} - a_{12}a_{21})}{2\Delta} \\ B &= \frac{a_{12}a_{22} + a_{11}a_{21}}{\Delta} \\ C &= -\frac{a_{11}^2 + a_{12}^2 + (a_{11}a_{22} - a_{12}a_{21})}{2\Delta}, \end{aligned}$$

and $\Delta = (a_{11} + a_{22})(a_{11}a_{22} - a_{12}a_{21})$. Based on the hypothesis, the coefficients A and B are negative. Therefore, except for the origin, $V(x, y)$ is negative on each of the coordinate axes. Along each trajectory,

$$\dot{V} = (2Ax + By)(a_{11}\,x + a_{12}\,y) + (2Cy + Bx)(a_{21}\,x + a_{22}\,y) = -x^2 - y^2.$$

Hence $\dot{V}(x, y)$ is negative definite. Theorem 9.6.2 asserts that the origin is an unstable critical point.

(b) We now consider the system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} F_1(x\,, y) \\ G_1(x\,, y) \end{pmatrix},$$

in which $F_1(x\,, y)/r \to 0$ and $G_1(x\,, y)/r \to 0$ as $r \to 0$. Let

$$V(x, y) = Ax^2 + Bxy + Cy^2,$$

in which

$$\begin{aligned} A &= \frac{a_{21}^2 + a_{22}^2 + (a_{11}a_{22} - a_{12}a_{21})}{2\Delta} \\ B &= -\frac{a_{12}a_{22} + a_{11}a_{21}}{\Delta} \\ C &= \frac{a_{11}^2 + a_{12}^2 + (a_{11}a_{22} - a_{12}a_{21})}{2\Delta}, \end{aligned}$$

and $\Delta = (a_{11} + a_{22})(a_{11}a_{22} - a_{12}a_{21})$. Based on the hypothesis, $A\,, B > 0$. Except for the origin, $V(x, y)$ is positive on each of the coordinate axes. Along each trajectory,

$$\dot{V} = x^2 + y^2 + (2Ax + By)F_1(x\,, y) + (2Cy + Bx)G_1(x\,, y)\,.$$

Converting to polar coordinates, for $r \neq 0$,

$$\begin{aligned} \dot{V} = &= r^2 + r(2A\cos\,\theta + B\sin\,\theta)\,F_1 + r(2C\sin\,\theta + B\cos\,\theta)\,G_1 \\ &= r^2 + r^2\left[(2A\cos\,\theta + B\sin\,\theta)\,\frac{F_1}{r} + (2C\sin\,\theta + B\cos\,\theta)\,\frac{G_1}{r}\right]. \end{aligned}$$

Since the system is almost linear, there is an R such that

$$\left|(2A\cos\theta + B\sin\theta)\,\frac{F_1}{r} + (2C\sin\theta + B\cos\theta)\,\frac{G_1}{r}\right| < \frac{1}{2}\,,$$

and hence

$$(2A\cos\theta + B\sin\theta)\,\frac{F_1}{r} + (2C\sin\theta + B\cos\theta)\,\frac{G_1}{r} > -\frac{1}{2}$$

for $r < R$. It follows that

$$\dot{V} > \frac{1}{2}r^2$$

as long as $0 < r < R$. Hence $\dot{V}$ is positive definite on the domain

$$D = \left\{(x\,,y)\,|\,x^2 + y^2 < R^2\right\}.$$

By Theorem 9.6.2, the origin is an unstable critical point.

9.7

3. The critical points of the ODE

$$\frac{dr}{dt} = r(r-1)(r-3)$$

are given by $r_1 = 0$, $r_2 = 1$ and $r_3 = 3$. Note that

$$\frac{dr}{dt} > 0 \text{ for } 0 < r < 1 \text{ and } r > 3\ ;\ \frac{dr}{dt} < 0 \text{ for } 1 < r < 3\,.$$

$r = 0$ corresponds to an unstable critical point. The critical point $r_2 = 1$ is asymptotically stable, whereas the critical point $r_3 = 3$ is unstable. Since the critical values are isolated, a limit cycle is given by

$$r = 1\,,\ \theta = t + t_0$$

which is asymptotically stable. Another periodic solution is found to be

$$r = 3\,,\ \theta = t + t_0$$

which is unstable.

5. The critical points of the ODE

$$\frac{dr}{dt} = \sin \pi r$$

are given by $r = n$, $n = 0\,,1\,,2\,,\ldots$. Based on the sign of r' in the neighborhood of each critical value, the critical points $r = 2k$, $k = 1\,,2\,,\ldots$ correspond to unstable periodic solutions, with $\theta = t + t_0$. The critical points $r = 2k+1$, $k = 0\,,1\,,2\,,\ldots$ correspond to stable limit cycles, with $\theta = t + t_0$. The solution $r = 0$ represents an unstable critical point.

6. The critical points of the ODE

$$\frac{dr}{dt} = r|r-2|(r-3)$$

are given by $r_1 = 0$, $r_2 = 2$ and $r_3 = 3$. Note that

$$\frac{dr}{dt} < 0 \text{ for } 0 < r < 3\,; \; \frac{dr}{dt} > 0 \text{ for } r > 3\,.$$

$r = 0$ corresponds to an asymptotically stable critical point. The critical points $r_2 = 2$ is semistable, whereas the critical point $r_3 = 3$ is unstable. Since the critical values are isolated, a semistable limit cycle is given by

$$r = 2\,,\; \theta = -t + t_0.$$

Another periodic solution is found to be

$$r = 3\,,\; \theta = -t + t_0$$

which is unstable.

10. Given $F(x\,,y) = a_{11}\,x + a_{12}\,y$ and $G(x\,,y) = a_{21}\,x + a_{22}\,y$, it follows that

$$F_x + G_y = a_{11} + a_{22}\,.$$

Based on the hypothesis, $F_x + G_y$ is either positive or negative on the entire plane. By Theorem 9.7.2, the system cannot have a nontrivial periodic solution.

12. Given that $F(x\,,y) = -2x - 3y - xy^2$ and $G(x\,,y) = y + x^3 - x^2 y$,

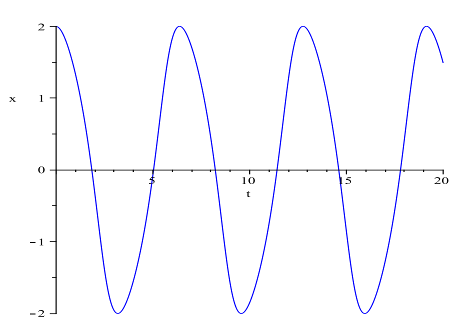
$$F_x + G_y = -1 - x^2 - y^2.$$

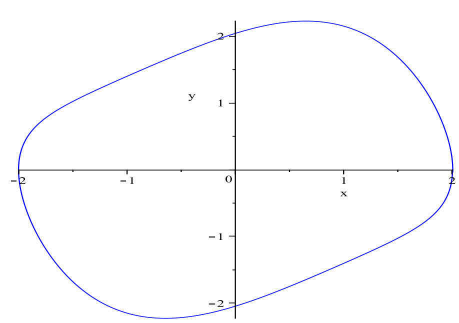
Since $F_x + G_y < 0$ on the entire plane, Theorem 9.7.2 asserts that the system cannot have a nontrivial periodic solution.

14.(a) Based on the given graphs, the following table shows the estimated values:

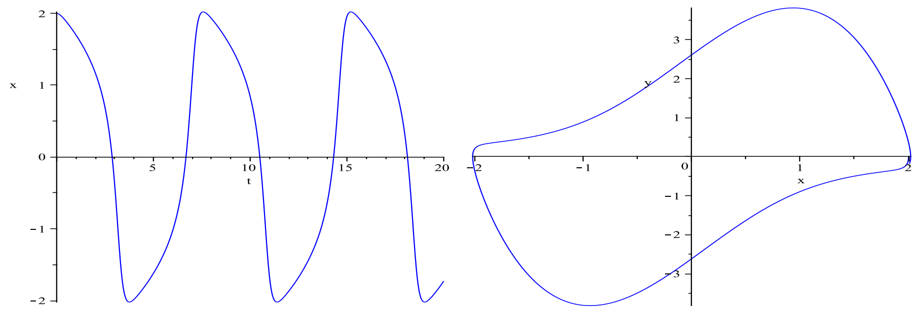
$\mu = 0.2$	$T \approx 6.29$
$\mu = 1.0$	$T \approx 6.66$
$\mu = 5.0$	$T \approx 11.60$

(b) The initial conditions were chosen as $x(0) = 2$, $y(0) = 0$.
For $\mu = 0.5$, $T \approx 6.38$:

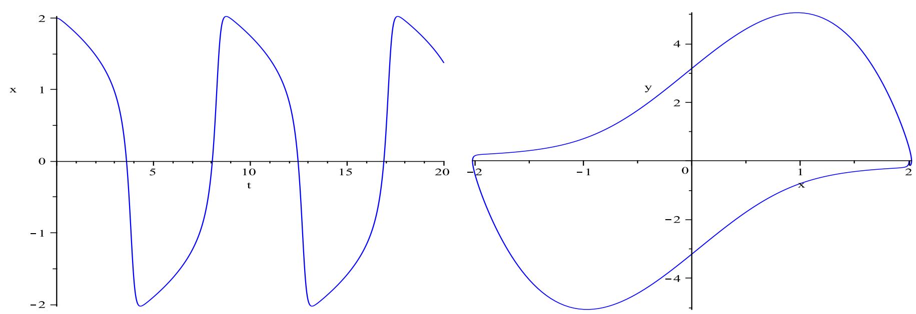




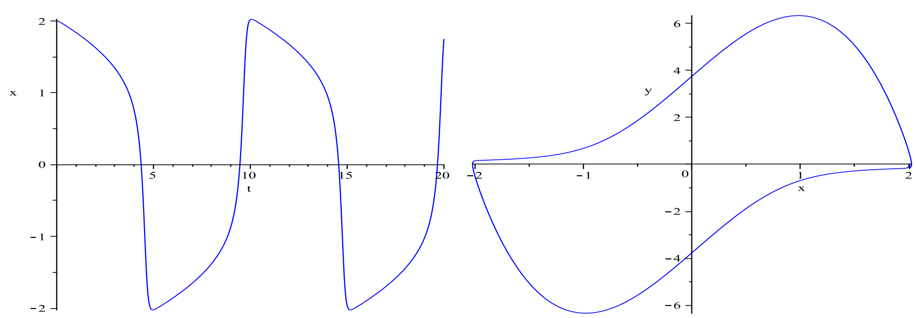
For $\mu = 2$, $T \approx 7.65$:



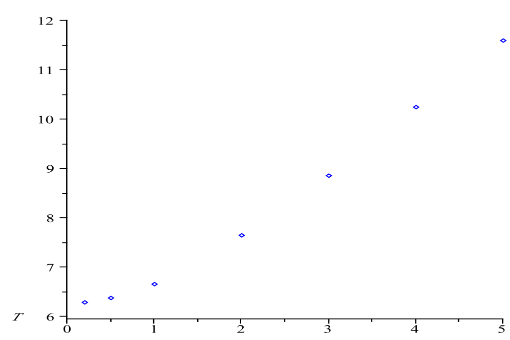
For $\mu = 3$, $T \approx 8.86$:



For $\mu = 5$, $T \approx 10.25$:



(c) The period, T, appears to be a quadratic function of μ.



15.(a) Setting $x = u$ and $y = u'$, we obtain the system of equations

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -x + \mu(1 - \frac{1}{3}y^2)y\,.$$

(b) Evidently, $y = 0$. It follows that $x = 0$. Hence the only critical point of the system is at $(0\,,0)$. The components of the vector field are infinitely differentiable everywhere. Therefore the system is almost linear. The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & \mu - \mu y^2 \end{pmatrix}.$$

At the critical point $(0\,,0)$, the coefficient matrix of the linearized system is

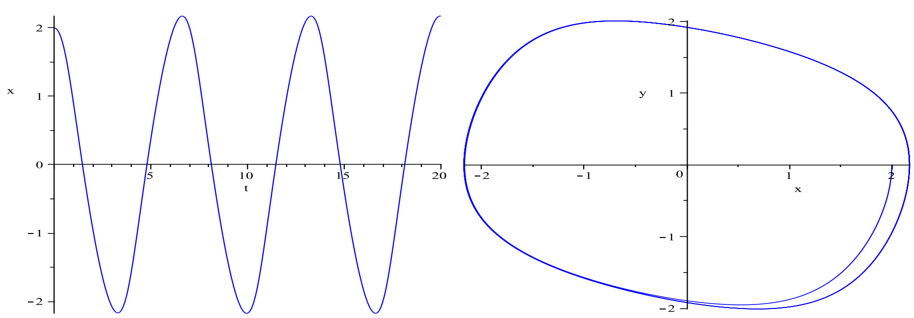
$$\mathbf{J}(0\,,0) = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix},$$

with eigenvalues

$$r_{1,2} = \frac{\mu}{2} \pm \frac{1}{2}\sqrt{\mu^2 - 4}\,.$$

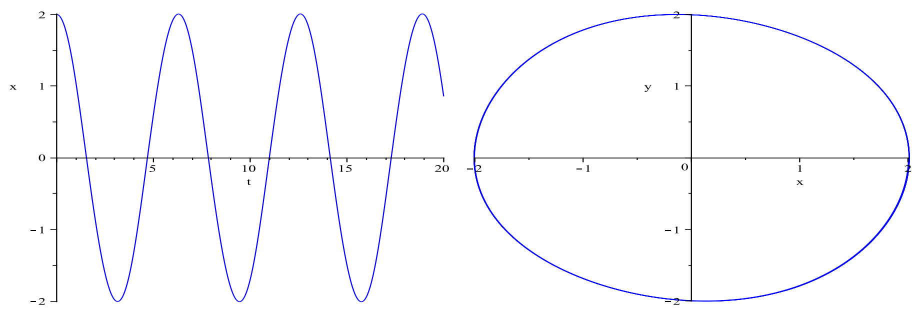
If $\mu = 0$, the equation reduces to the ODE for a simple harmonic oscillator. For the case $0 < \mu < 2$, the eigenvalues are complex, and the critical point is an unstable spiral. For $\mu \geq 2$, the eigenvalues are real, and the origin is an unstable node.

(c) The initial conditions were chosen as $x(0) = 2$, $y(0) = 0$.

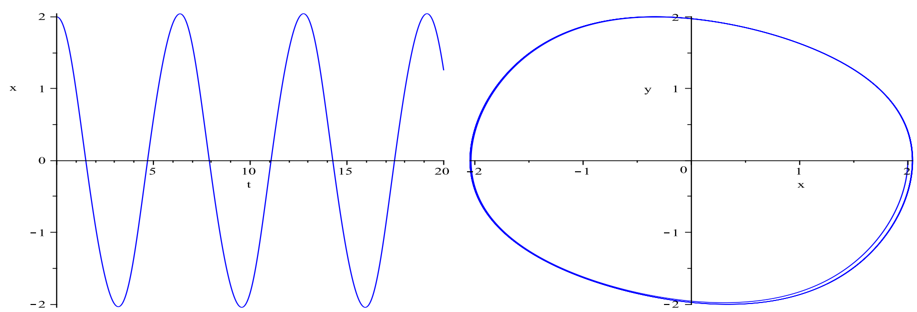


$\mu = 1$: $A \approx 2.16$ and $T \approx 6.65$.

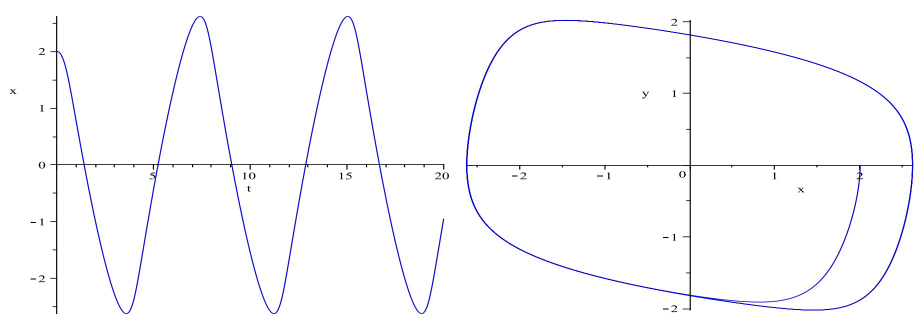
(d)



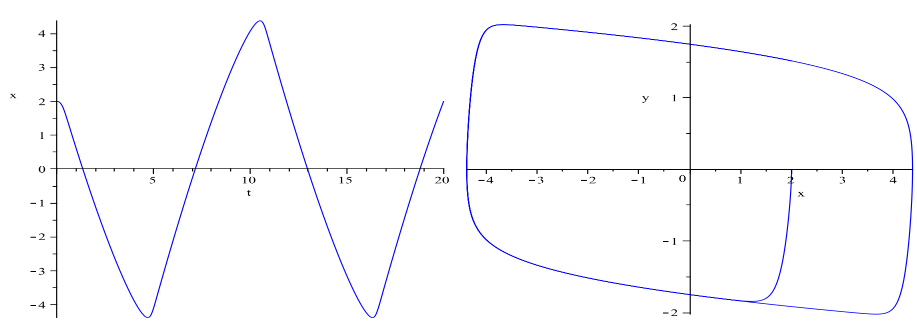
$\mu = 0.2$: $A \approx 2.00$ and $T \approx 6.30$.



$\mu = 0.5$: $A \approx 2.04$ and $T \approx 6.38$.



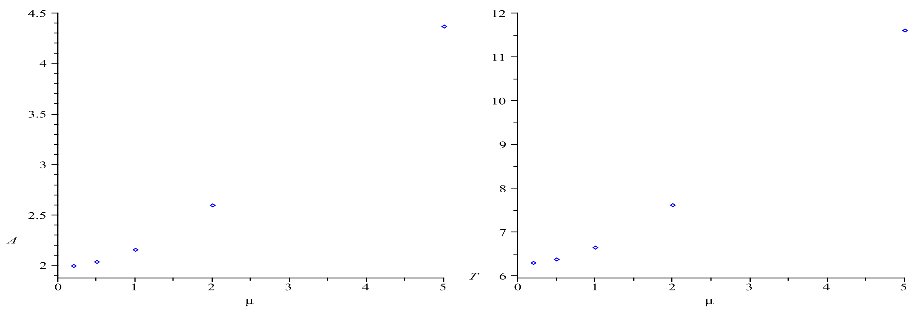
$\mu = 2$: $A \approx 2.6$ and $T \approx 7.62$.



$\mu = 5$: $A \approx 4.37$ and $T \approx 11.61$.

(e)

	A	T
$\mu = 0.2$	2.00	6.30
$\mu = 0.5$	2.04	6.38
$\mu = 1.0$	2.16	6.65
$\mu = 2.0$	2.6	7.62
$\mu = 5.0$	4.37	11.61



16.(a) The critical points are solutions of the algebraic system

$$\mu\, x + y - x(x^2 + y^2) = 0$$
$$-x + \mu\, y - y(x^2 + y^2) = 0\,.$$

Multiply the first equation by y and the second equation by x to obtain

$$\mu\, xy + y^2 - xy(x^2 + y^2) = 0$$
$$-x^2 + \mu\, xy - xy(x^2 + y^2) = 0\,.$$

Subtraction of the two equations results in

$$x^2 + y^2 = 0\,,$$

which is satisfied only for $x = y = 0$.

(b) The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} \mu - 3x^2 - y^2 & 1 - 2xy \\ -1 - 2xy & \mu - x^2 - 3y^2 \end{pmatrix}.$$

At the critical point $(0\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0) = \begin{pmatrix} \mu & 1 \\ -1 & \mu \end{pmatrix},$$

resulting in the linear system

$$x' = \mu\, x + y$$
$$y' = -x + \mu\, y\,.$$

The characteristic equation for the coefficient matrix is $\lambda^2 - 2\mu\,\lambda + \mu^2 + 1 = 0$, with solution

$$\lambda = \mu \pm i\,.$$

For $\mu < 0$, the origin is a stable spiral. When $\mu = 0$, the origin is a center. For $\mu > 0$, the origin is an unstable spiral.

(c) Introduce polar coordinates r and θ, so that $x = r\cos\theta$ and $y = r\sin\theta$ for $r \geq 0$. Multiply the first of Eqns (i) by x and the second equation by y to obtain

$$\begin{aligned} x\,x' &= \mu\,x^2 + xy - x^2(x^2+y^2) \\ y\,y' &= -xy + \mu\,y^2 - y^2(x^2+y^2)\,. \end{aligned}$$

Addition of the two equations results in

$$x\,x' + y\,y' = \mu(x^2+y^2) - (x^2+y^2)^2.$$

Since $r^2 = x^2 + y^2$ and $r\,r' = x\,x' + y\,y'$, it follows that $r\,r' = \mu\,r^2 - r^4$ and

$$\frac{dr}{dt} = \mu\,r - r^3$$

for $r > 0$. Multiply the first of Eqns (i) by y and the second equation by x, the difference of the two equations results in

$$y\,x' - x\,y' = x^2 + y^2.$$

Since $y\,x' - x\,y' = -r^2\,\theta'$, the above equation reduces to

$$\frac{d\theta}{dt} = -1\,.$$

(d) From $r' = r(\mu - r^2)$ and $\theta' = -1$, it follows that one solution of the system is given by

$$r = \sqrt{\mu} \quad \text{and } \theta = -t + t_0$$

valid for $\mu > 0$. This corresponds to a periodic solution with a circular trajectory. Since $r \geq 0$, observe that

$$\begin{aligned} \frac{dr}{dt} &= r(\mu - r^2) < 0 \ \text{ for } \ r > \sqrt{\mu} \\ &= r(\mu - r^2) > 0 \ \text{ for } \ \sqrt{\mu} > r > 0\,. \end{aligned}$$

Hence solutions with initial condition $r(0) \neq \sqrt{\mu}$ are attracted to the limit cycle.

17.(a) The critical points are solutions of the algebraic system

$$\begin{aligned} y &= 0 \\ -x + \mu\,(1 - x^2)y &= 0\,. \end{aligned}$$

Clearly, $y = 0$, and this implies that $x = 0$. So the origin is the only critical point. The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 - 2\mu xy & \mu(1 - x^2) \end{pmatrix}.$$

At the critical point $(0\,,0)$, the coefficient matrix of the linearized system is

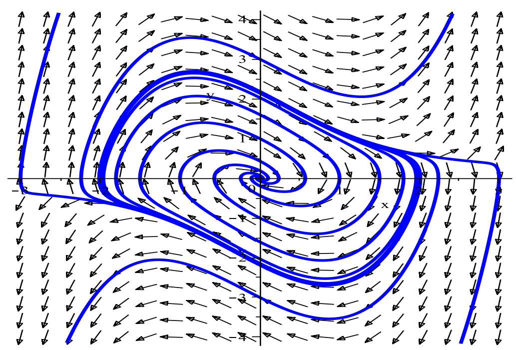
$$\mathbf{J}(0\,,0) = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}.$$

The characteristic equation is $\lambda^2 - \mu\lambda + 1 = 0$, and the roots are

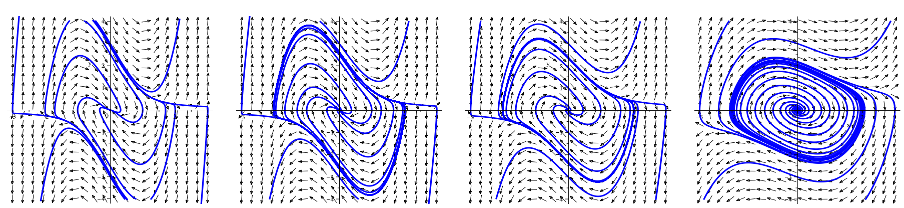
$$\lambda = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2}.$$

This implies that the origin is an asymptotically stable node for $\mu < -2$, an asymptotically stable spiral point for $-2 < \mu < 0$, an unstable spiral point for $0 < \mu < 2$ and an unstable node for $\mu > 2$.

(b)

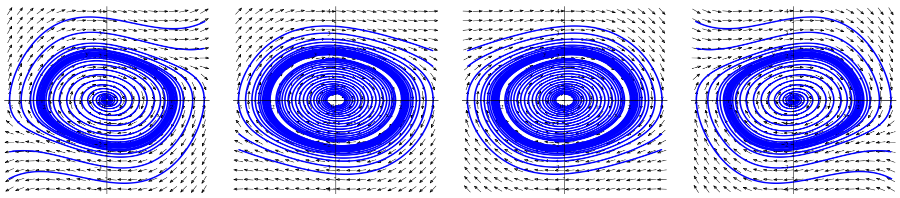


(c)



(a) $\mu = -2.5$ (b) $\mu = -2$ (c) $\mu = -1.5$ (d) $\mu = -0.5$

(d)



(a) $\mu = -0.2$ (b) $\mu = -0.1$ (c) $\mu = 0.1$ (d) $\mu = 0.2$

19.(a) The critical points are solutions of the algebraic system

$$x(a - \frac{x}{5} - \frac{2y}{x+6}) = 0$$
$$y(-\frac{1}{4} + \frac{x}{x+6}) = 0\,.$$

If $x = 0$, then $y = 0$. If $y = 0$, then $x = 5a$ from the first equation. The third critical point comes from $x = 2$, and then the first equation implies that $y = 4a - 8/5$. So the critical points are $(0, 0)$, $(5a, 0)$ and $(2, 4a - 8/5)$.

(b) The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} a - \frac{2x}{5} - \frac{12y}{(x+6)^2} & -\frac{2x}{x+6} \\ \frac{6y}{(x+6)^2} & -\frac{1}{4} + \frac{x}{x+6} \end{pmatrix}.$$

At the critical point $(2\,, 4a - 8/5)$, the coefficient matrix of the linearized system is

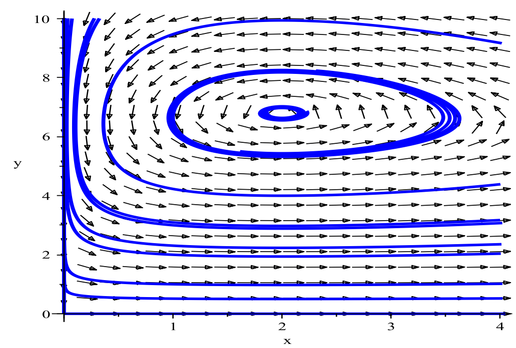
$$\mathbf{J}(2\,, 4a - 8/5) = \begin{pmatrix} a/4 - 1/2 & -1/2 \\ 3a/8 - 3/20 & 0 \end{pmatrix}.$$

The characteristic equation is $\lambda^2 + \lambda(1/2 - a/4) + (3a/8 - 3/20)/2 = 0$, and the roots are

$$\lambda = \frac{a}{8} - \frac{1}{4} \pm \frac{1}{2}\sqrt{\frac{a^2}{16} - a + \frac{11}{20}}.$$

We can conclude that $a_0 = 2$.

(c)



20.(a) The critical points are solutions of the algebraic system

$$1-(b+1)x+x^2y/4=0$$
$$b\,x-x^2y/4=0\,.$$

Add the two equations to obtain $1-x=0$, with solution $x=1$. The second of the above equations yields $y=4\,b$.

(b) The Jacobian matrix of the vector field is

$$\mathbf{J}=\frac{1}{4}\begin{pmatrix}-4(b+1)+2xy & x^2\\ 4b-2xy & -x^2\end{pmatrix}.$$

At the critical point $(1\,,4b)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(1\,,4b)=\begin{pmatrix}b-1 & 1/4\\ -b & -1/4\end{pmatrix},$$

with characteristic equation

$$\lambda^2+(\frac{5}{4}-b)\lambda+\frac{1}{4}=0$$

and eigenvalues

$$\lambda=-\frac{5-4b\,\pm\,\sqrt{9-40\,b+16\,b^2}}{8}=-\frac{(5/4-b)\,\pm\,\sqrt{(5/4-b)^2\,-1}}{2}\,.$$

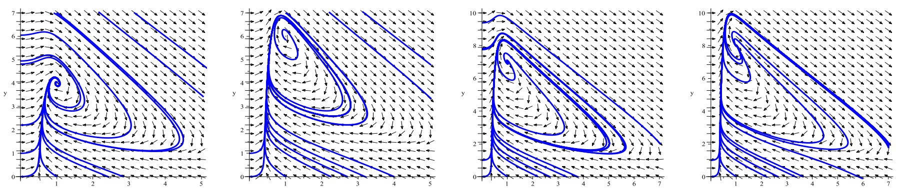
(c) Let $L=5/4-b$. The eigenvalues can be expressed as

$$\lambda=-\frac{L\,\pm\,\sqrt{L^{\,2}\,-1}}{2}\,.$$

We find that the eigenvalues are real if $L^{\,2}\geq 1$ and are complex if $L^{\,2}<1$.
For the complex case, that is, $-1<L<1$, the critical point is a stable spiral if $0<L<1$; it is an unstable spiral if $-1<L<0$. That is, the critical point is an asymptotically stable spiral if $1/4<b<5/4$ and is an unstable spiral if $5/4<b<9/4$. When $L^{\,2}>1$, the critical point is a node. The critical point is a stable node if $L>1$; it is an unstable node if $L<-1$. That is, the critical point is an asymptotically stable node if $0<b<1/4$ and is an unstable node if $b>9/4$.

(d) From part (c), the critical point changes from an asymptotically stable spiral to an unstable spiral when $b_0 = 5/4$.

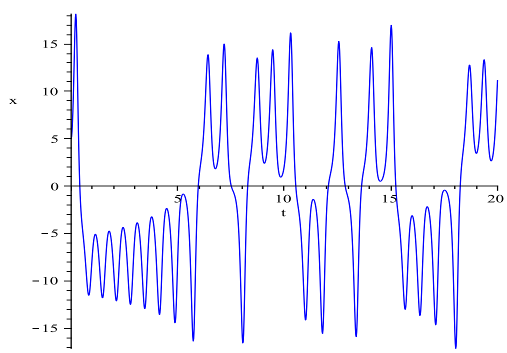
(e,f)



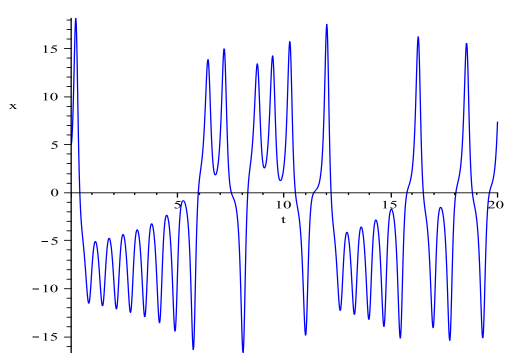
(a) $b = 1$ (b) $b = 1.5$ (c) $b = 1.75$ (d) $b = 2$

9.8

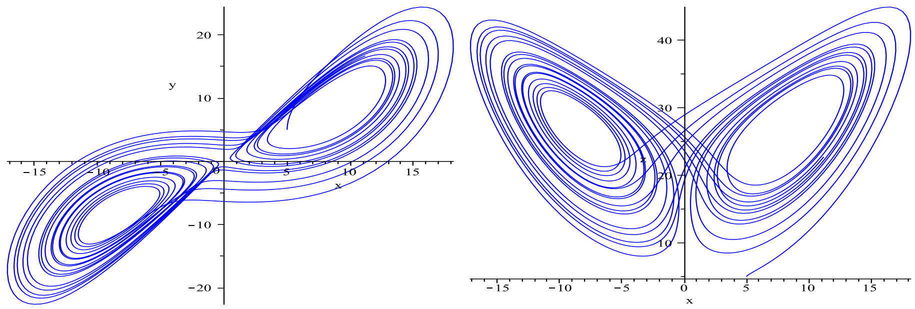
6. $r = 28$, with initial point $(5\,, 5\,, 5)$:



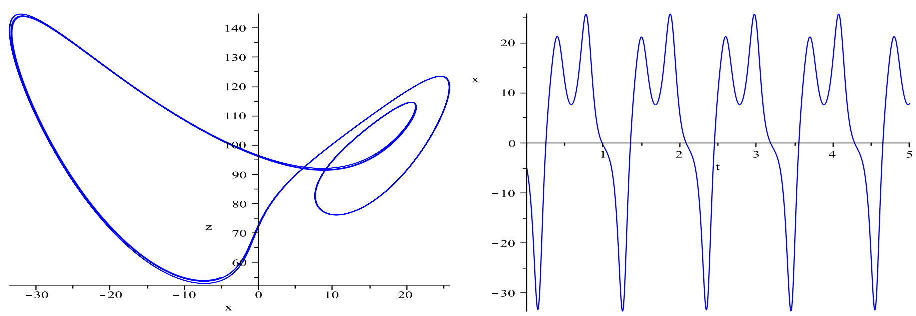
$r = 28$, with initial point $(5.01\,, 5\,, 5)$:



7. $r = 28$:

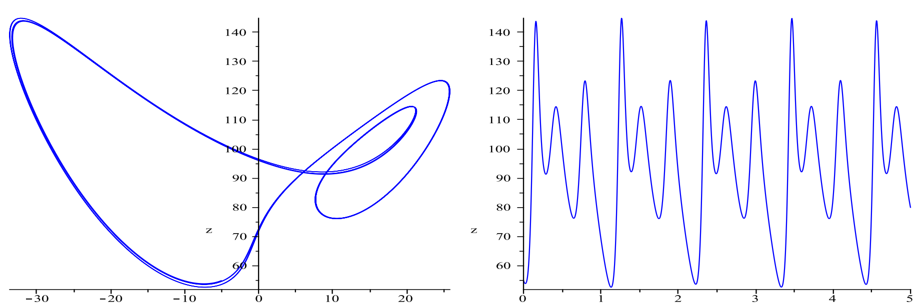


9.(a) $r = 100$, initial point $(-5\,, -13\,, 55\,)$:



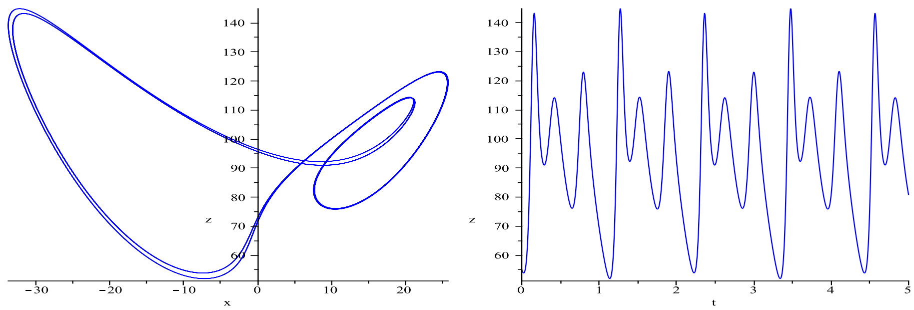
The period appears to be $T \approx 1.12$.

(b) $r = 99.94$, initial point $(-5\,, -13\,, 55\,)$:

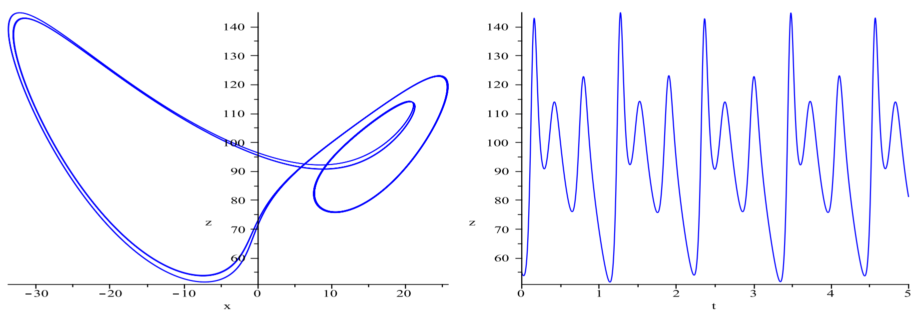


The periodic trajectory appears to have split into two strands, indicative of a period-doubling. Closer examination reveals that the peak values of $z(t)$ are slightly different.

$r = 99.7$, initial point $(-5\,, -13\,, 55\,)$:

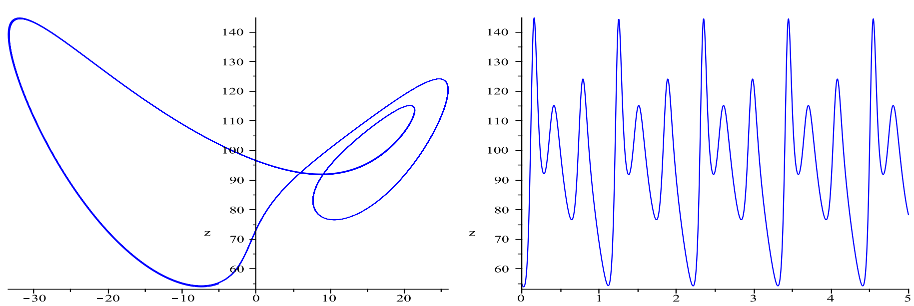


(c) $r = 99.6$, initial point $(-5, -13, 55)$:

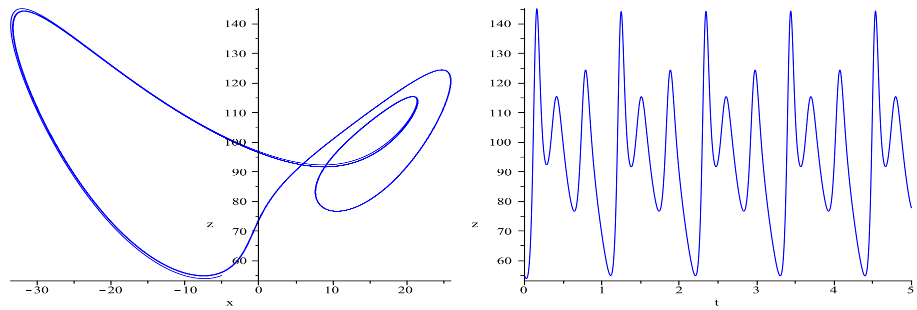


The strands again appear to have split. Closer examination reveals that the peak values of $z(t)$ are different.

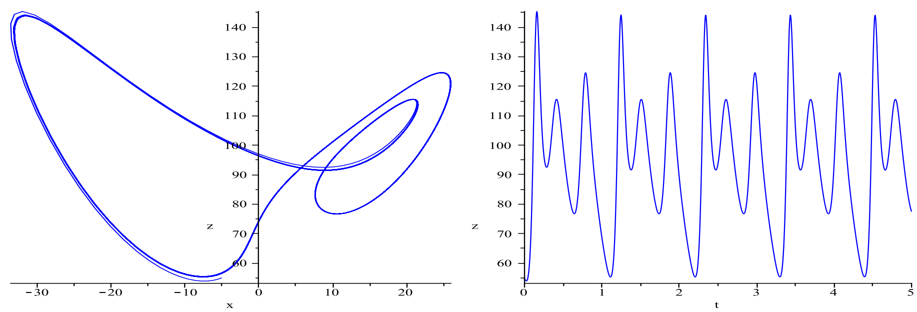
10.(a) $r = 100.5$, initial point $(-5, -13, 55)$:



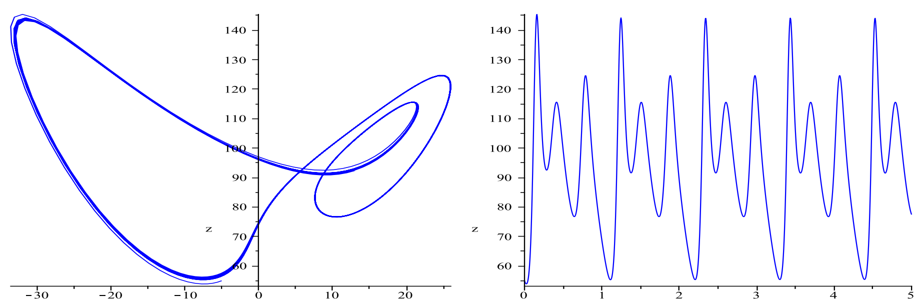
$r = 100.7$, initial point $(-5, -13, 55)$:



(b) $r = 100.8$, initial point $(-5\,, -13\,, 55\,)$:

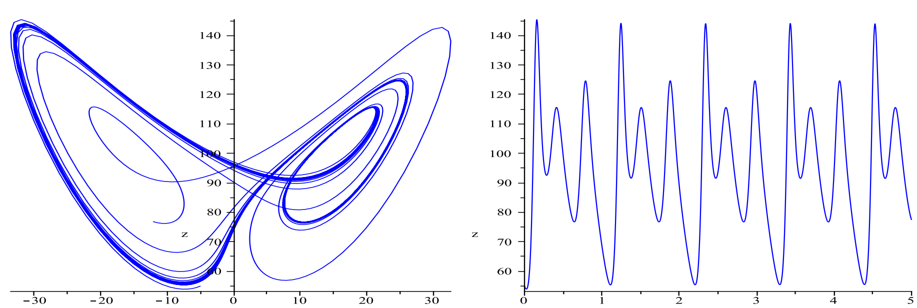


$r = 100.81$, initial point $(-5\,, -13\,, 55\,)$:

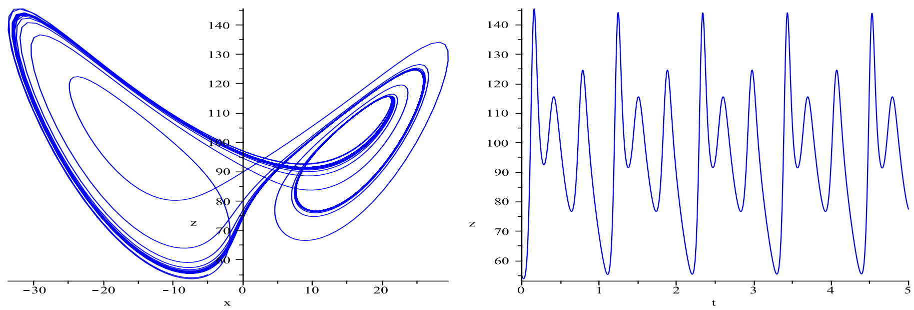


The strands of the periodic trajectory are beginning to split apart.

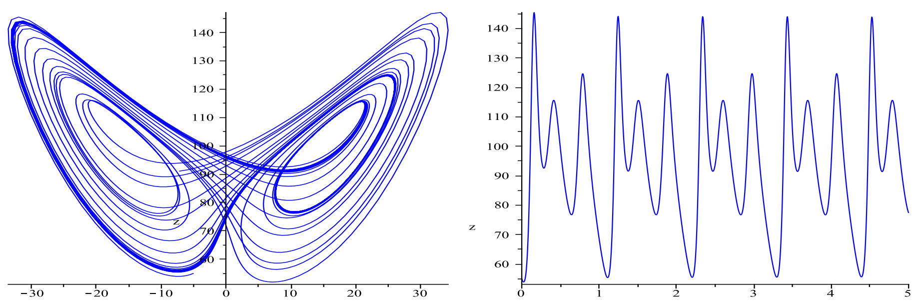
$r = 100.82$, initial point $(-5\,, -13\,, 55\,)$:



$r = 100.83$, initial point $(-5\,,-13\,,55\,)$:



$r = 100.84$, initial point $(-5\,,-13\,,55\,)$:



12. The system is given by

$$
\begin{aligned}
x' &= -y - z \\
y' &= x + y/4 \\
z' &= 1/2 + z(x - c)\,.
\end{aligned}
$$

(a) We obtain that the critical points (when $c^2 > 1/2$) are

$$(\frac{c}{2} + \frac{1}{4}\sqrt{4c^2 - 2}, -2c - \sqrt{4c^2 - 2}, 2c + \sqrt{4c^2 - 2}),$$

$$(\frac{c}{2} - \frac{1}{4}\sqrt{4c^2 - 2}, -2c + \sqrt{4c^2 - 2}, 2c - \sqrt{4c^2 - 2}).$$

The Jacobian of the system is

$$\mathbf{J} = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 1/4 & 0 \\ z & 0 & x - c \end{pmatrix}.$$

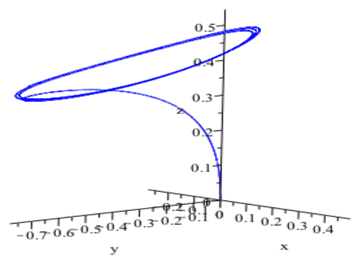
In the following computations, we approximate the values to 4 decimal points. When $c = 1.3$, the two critical points are

$$(1.1954, -4.7817, 4.7817) \qquad \text{and} \qquad (0.1046, -0.4183, 0.4183).$$

The corresponding eigenvalues of the Jacobian are

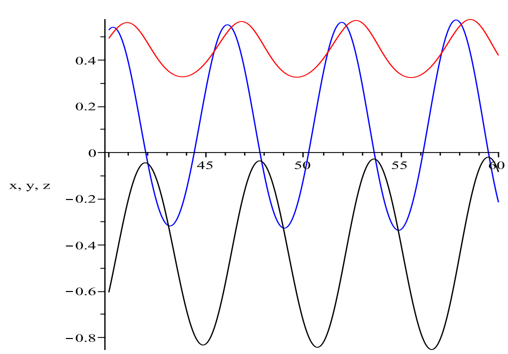
$$0.1893,\ -0.0219 \pm 2.4007i \qquad \text{and } -0.9613,\ 0.0080 \pm 1.0652i.$$

(b)



(c)

(d) $T_1 \approx 5.9$.



13.(b) Using the eigenvalue idea:

c	Critical Point	Complex Eigenvalue Pair
1.2	$(0.1152, -0.4609, 0.4609)$	$-0.0063 \pm 1.0859i$
1.25	$(0.1096, -0.4384, 0.4384)$	$0.0010 \pm 1.0747i$

Clearly, somewhere in between we have a Hopf bifurcation. Similar computations show that the bifurcation value is $c \approx 1.243$.

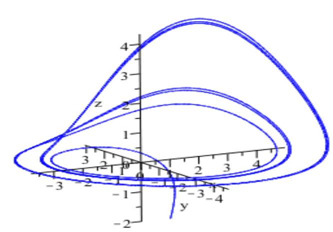
14.(a) When $c = 3$, the two critical points are

$$(2.9577, -11.8310, 11.8310) \qquad \text{and} \qquad (0.0423, -0.1690, 0.1690).$$

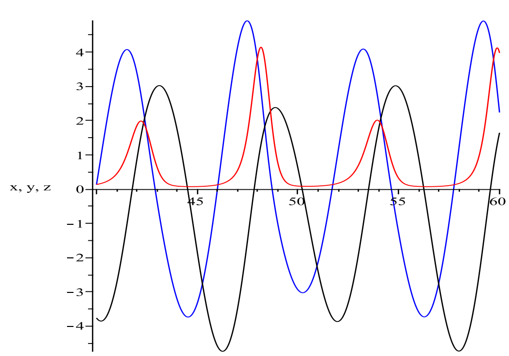
The corresponding eigenvalues of the Jacobian are

$$0.2273,\ -0.0098 \pm 3.5812i \qquad \text{and} \quad -2.9053,\ 0.0988 \pm 0.9969i.$$

(b)



(c) $T_2 \approx 11.8$.



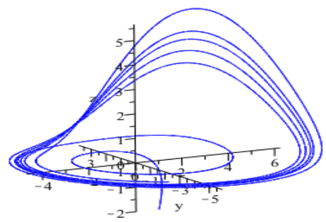
15.(a) When $c = 3.8$, the two critical points are

$$(3.7668, -15.0673, 15.0673) \qquad \text{and} \qquad (0.0332, -0.1327, 0.1327).$$

The corresponding eigenvalues of the Jacobian are

$$0.2324,\ -0.0078 \pm 4.0078i \qquad \text{and} \ -3.7335,\ 0.1083 \pm 0.9941i.$$

(b) $T_4 \approx 23.6$.



(c)