# *Exam 2 Solutions*

Multiple Choice Questions

1. Consider the series $\sum_{n=1}^{\infty} \frac{e^n}{n!}$. If the ratio test is applied to the series, which of the following inequalities results, implying that the series converges?

   A. $\lim_{n\to\infty} \frac{e}{n!} < 1$

   B. $\lim_{n\to\infty} \frac{n!}{e} < 1$

   C. $\lim_{n\to\infty} \frac{n+1}{e} < 1$

   **D.** $\lim_{n\to\infty} \frac{e}{n+1} < 1$

   E. $\lim_{n\to\infty} \frac{e}{(n+1)!} < 1$

2. The interval of convergence of the power series $\sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$ is

   A. $[0]$

   B. $\left(-\frac{1}{3}, \frac{1}{3}\right)$

   C. $(-3, 3]$

   **D.** $(-3, 3)$

   E. $(-\infty, +\infty)$

3. The sum of the infinite geometric series $1 + \frac{2}{5} + \frac{4}{25} + \frac{8}{125} + \frac{16}{625} + \cdots$ is

A. $\frac{3}{5}$

B. $\frac{2}{3}$

C. $\frac{5}{3}$

D. $\frac{3}{2}$

E. $\frac{5}{2}$

4. Which of the following sequences converge?

I. $\left\{ \frac{5n}{2n-1} \right\}$

II. $\left\{ \frac{e^n}{n} \right\}$

III. $\left\{ \frac{e^n}{1+e^n} \right\}$

A. I only

B. II only

C. I and II only

**D. I and III only**

E. I, II, and III

5. If $\lim_{M\to\infty} \int_1^M \frac{dx}{x^p}$ converges, then which of the following must be true?

**A. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.**

B. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

C. $\sum_{n=1}^{\infty} \frac{1}{n^{p-2}}$ converges.

D. $\sum_{n=1}^{\infty} \frac{1}{n^{p-1}}$ converges.

E. $\sum_{n=1}^{\infty} \frac{1}{n^{p+1}}$ diverges.

6. A series $\sum a_n$ is convergent if and only if

A. the limit $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ is greater than 1.

B. its sequence of terms $\{a_n\}$ converges to 0.

**C. its sequence of partial sums $\{S_n\}$ converges to some real number.**

D. its sequence of terms $\{a_n\}$ is alternating.

E. its sequence of partial sums $\{S_n\}$ is bounded.

7. Which of the following statements is true? (There is only one.)

   A. If $0 \leq b_n \leq a_n$ and $\sum b_n$ converges then $\sum a_n$ converges.

   B. If $\lim_{n\to\infty} a_n = 0$ then the series $\sum a_n$ is convergent.

   C. The series $\sum_{n=1}^{\infty} n^{-\sin 1}$ is convergent.

   **D. If $\sum a_n$ is convergent for $a_n > 0$ then $\sum(-1)^n a_n$ is also convergent.**

   E. The ratio test can be used to show that $\sum \frac{1}{n^{10}}$ converges.

8. Let $S_N$ be the $N$-th partial sum of the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{2n-1}.$$

Thus, $S_1 = 1$, $S_2 = \frac{2}{3}$. Compute $S_{50} - S_{49}$.

   **A.** $-\frac{1}{99}$.

   B. $-\frac{1}{50}$

   C. 1

   D. $\frac{2}{9603}$

   E. 0

9. Consider the series $\sum_{n=1}^{\infty} \frac{3}{4^n + 6n - 4}$. Applying the comparison test with the series $\sum_{n=1}^{\infty} \frac{3}{4^n}$ leads to the following conclusion.

   A. The test is inconclusive.

   **B. The series converges absolutely.**

   C. The series converges conditionally.

   D. The series diverges.

   E. The test cannot be applied to $a_n = \frac{3}{4^n + 6n - 4}$ and $b_n = \frac{3}{4^n}$.

10. The radius of convergence for the series $\sum_{n=0}^{\infty} \frac{n^2 x^n}{10^n}$ is

   A. 1

   B. $1/10$

   C. 10

   D. $n/10$

   E. $\infty$

11. The series $\sum_{n=0}^{\infty} \frac{n^2+1}{n^4+1}$

A. converges by the Ratio Test.

B. diverges by the Integral Test.

**C. converges by the Limit Comparison Test with the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.**

D. diverges by the Limit Comparison Test with the series $\sum_{n=1}^{\infty} \frac{1}{n}$.

E. diverges because it does not alternate in sign.

12. The series $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n^2}$ is

**A. converges absolutely.**

B. converges conditionally.

C. diverges.

D. eventually oscillates between $-1$ and $1$, but never converges.

E. none of the above.

Free Response Questions

13. Find the first four (4) terms of each of the following sequences.

(a) (6 points) $a_n = \dfrac{1}{(n+1)!}$

**Solution:**

$$1, \frac{1}{2}, \frac{1}{3!}, \frac{1}{4!}$$

(b) (6 points) $a_1 = 2$ and $a_{n+1} = \dfrac{1}{3 - a_n}$

**Solution:**

$$2, 1, \frac{1}{2}, \frac{2}{5}, \frac{5}{13}$$

14. Determine if the sequence is convergent or divergent. If convergent give its limit.

(a) (4 points) $a_n = \dfrac{n+1}{3n-1}$

**Solution:** The sequence converges.

$$\lim_{n\to\infty} \frac{n+1}{3n-1} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{3-\frac{1}{n}} = \frac{1}{3}.$$

(b) (4 points) $a_n = n^2 e^{-n}$

**Solution:** The sequence converges.

$$\lim_{n\to\infty} \frac{n^2}{e^n} = 0.$$

(c) (4 points) $a_n = \dfrac{3^n}{2^n}$

**Solution:** The sequence diverges.

$$\lim_{n\to\infty} \frac{3^n}{2^n} = \lim_{n\to\infty} \left(\frac{3}{2}\right)^n = +\infty.$$

15. Determine the convergence or divergence of each of the following series. State clearly what test you used and show your work.

(a) (5 points) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

**Solution:** This series diverges by the $p$-series test with $p = 1/2$.

(b) (5 points) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^3}$

**Solution:**

$$\sin^2(n) \leq 1$$
$$\frac{\sin^2(n)}{n^3} \leq \frac{1}{n^3}$$

The latter series converges by the $p$-series test with $p = 3$, so the given series converges by the Comparison Test with the series $\sum \frac{1}{n^3}$.

(c) (5 points) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

**Solution:** Use the Ratio Test.

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}} \\ &= \lim_{n\to\infty} \left( \frac{n+1}{n} \right)^2 \frac{1}{2} = \frac{1}{2} < 1 \end{aligned}$$

Since the limit is less than 1, the series converges by the Ratio Test.

16. (5 points) Use the integral test to determine whether the series

$$\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$$

converges or diverges. Show your work and clearly state your answer.

**Solution:** Let $f(x) = \frac{1}{x\ln x}$. $f'(x) = -\frac{\ln x + 1}{x^2(\ln x)^2)} < 0$ for $x > 2$ so the function is decreasing. Let $u = \ln x$ then $du = \frac{dx}{x}$ and

$$\begin{aligned}\int_2^{\infty} \frac{1}{x\ln x} dx &= \lim_{M\to\infty} \int_2^M \frac{1}{x\ln x}\,dx \\ &= \lim_{M\to\infty} \int_{\ln 2}^M \frac{1}{u}\,du \\ &= \lim_{M\to\infty} \ln u|_{\ln 2}^M \\ &= \text{diverges}\end{aligned}$$

Since the integral diverges, then the series also diverges.

17. (4 points) Use the comparison test to determine whether the series

$$\sum_{k=1}^{\infty} \frac{\ln k}{k}$$

converges or diverges.

**Solution:** For $k > 3$, $\ln k > 1$, so $\frac{\ln k}{k} > \frac{1}{k}$. $\sum \frac{1}{k}$ is the harmonic series and diverges, so by the Comparison Test, $\sum \frac{\ln k}{k}$ diverges.

18. A function $f$ is defined by

$$f(x) = \frac{1}{3} + \frac{2}{3^2}x + \frac{3}{3^3}x^2 + \frac{4}{3^4}x^3 + \cdots + \frac{n+1}{3^{n+1}}x^n + \cdots = \sum_{n=0}^{\infty} \frac{n+1}{3^{n+1}}x^n.$$

for all $x$ in the interval of convergence for the power series.

(a) (4 points) Find the radius of convergence for the power series. ***Show your work.***

**Solution:** Use the Ratio Test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n+2}{n+1}\frac{|x^{n+1}|}{|x^n|}\frac{3^{n+1}}{3^{n+2}} = \frac{|x|}{3} < 1.$$

To converge we must have $|x| < 3$, so the radius of convergence is 3.

(b) (4 points) Find the interval of convergence for the power series. ***Show your work.***

**Solution:** The radius of convergence is 3, so we need to check the endpoints: $x = 3$ and $x = -3$. At $x = 3$, we have

$$\sum_{n=0}^{\infty}\frac{n+1}{3^{n+1}}3^n = \sum_{0}^{\infty}\frac{n+1}{3}$$

which diverges by the Divergence Test. Likewise, at $x = -3$

$$\sum_{n=0}^{\infty}\frac{n+1}{3^{n+1}}3^n = \sum_{0}^{\infty}(-1)^n\frac{n+1}{3}$$

which also diverges by the Divergence Test.
Thus, the interval of convergence is $-3 < x < 3$ or $(-3, 3)$.

(c) (4 points) Find the power series representation for $f'(x)$ and state its radius of convergence.

**Solution:**

$$f'(x) = \sum_{n=1}^{\infty}\frac{n(n+1)}{3^{n+1}}x^{n-1} = \frac{2}{3^2} + \frac{2\cdot 3}{3^3}x + \frac{3\cdot 4}{3^4}x^2 + \cdots + \frac{n(n+1)}{3^{n+1}}x^{n-1} + \cdots.$$

The radius of convergence does not change and remains at 3.

(d) (4 points) Find the power series representation for $\int f(x)\,dx$.

**Solution:**

$$\int f(x)dx = C + \sum_{n=0}^{\infty}\frac{x^{n+1}}{3^{n+1}} = C + \frac{x}{3} + \frac{x^2}{3^2} + \frac{x^3}{3^3} + \cdots + \frac{x^n}{3^n} + \cdots.$$