Exam 3 Solutions

Multiple Choice Questions

1. The average value of the function $f(x) = x + \sin(x)$ on the interval $[0, 2\pi]$ is:

   A. $\dfrac{2\pi^2 - 1}{2\pi}$

   B. π

   C. $\dfrac{2\pi^2 + 1}{2\pi}$

   D. $\dfrac{4\pi^2 - 1}{2\pi}$

   E. $\dfrac{4\pi^2 + 1}{2\pi}$

2. Let $f(x)$ and $g(x)$ be two functions such that $f(x) \geq g(x)$ in the interval $[a, b]$. If we want to find the volume of the solid of revolution obtained by rotating around the y-axis the region R between $f(x)$ and $g(x)$ and $a \leq x \leq b$, the right integral to compute is:

   A. $\displaystyle\int_a^b 2\pi x\sqrt{1 + (f(x) - g(x))'}dx$

   B. $\displaystyle\int_a^b \pi(f(x))^2 - \pi(g(x))^2 dx$

   C. $\displaystyle\int_a^b 2\pi x(f(x) - g(x))dx$

   D. $\displaystyle\int_a^b \sqrt{1 + (f(x) - g(x))'}dx$

   E. $\displaystyle\int_a^b f(x) - g(x)dx$

3. Given that $f(x) = 1 - 3x^2$ find all of the values of x that satisfy the Mean Value Theorem for Integrals on the interval $[-2, 4]$.

   A. $\pm\sqrt{\frac{67}{3}}$
   B. ± 2
   C. $\pm\sqrt{6}$
   D. $\sqrt{\frac{19}{3}}$
   E. There is no such value of x.

4. The base of the solid S is the region enclosed by the parabola $y = 1 - x^2$ and the x-axis. The cross-sections perpendicular to the x-axis are squares. The volume of this solid is

   A. $\frac{22}{15}$
   B. $\frac{2}{3}$
   C. $\frac{16\pi}{15}$
   D. $\frac{28}{15}$
   E. $\frac{16}{15}$

5. Consider the region in the first quadrant bounded by the graph of $f(x) = \ln(x)$ and the line $x = 8$. Which of the following integrals represents the volume obtained by rotating the region about the line $x = 10$?

   A. $2\pi \int_1^8 (10 - x) \ln(x)\, dx$**.**

   B. $2\pi \int_0^8 (10 - x) \ln(x)\, dx$.

   C. $2\pi \int_1^8 x \ln(x)\, dx$.

   D. $2\pi \int_1^8 (10 + x) \ln(x)\, dx$.

   E. $\pi \int_1^{10} \ln(x)^2\, dx$.

6. Which of the following integrals represents the arc length of the graph of $f(x) = \ln(x)$ over the interval $[1, 8]$?

   A. $\int_1^8 \frac{1}{x}\sqrt{1 + x^2}\, dx$**.**

   B. $\int_1^8 \ln(x)\, dx$.

   C. $\int_1^8 \frac{1}{x^2}\sqrt{1 + x^2}\, dx$.

   D. $\int_1^8 \sqrt{1 + \ln(x)^2}\, dx$.

   E. $\int_1^8 \sqrt{1 + x^2}\, dx$.

7. Which of the following represents the integral for the surface area obtained by rotating the graph of $f(x) = 4\cos(x^3)$ over $[1, 2]$ about the x-axis?

   A. $2\pi \int_1^2 4\cos(x^3)\sqrt{1 + 12x^2 \sin(x^3)}\, dx.$

   B. $2\pi \int_1^2 4\cos(x^3)\sqrt{1 + 144x^4 \sin^2(x^3)}\, dx.$

   C. $2\pi \int_1^2 4\cos(x^3)\sqrt{1 + 144x^4 \cos^2(x^3)}\, dx.$

   D. $2\pi \int_1^2 4\cos(x^3)\sqrt{1 + 144x^4 \sin(x^6)}\, dx.$

   E. $2\pi \int_1^2 \sqrt{1 + 144x^4 \sin^2(x^3)}\, dx.$

8. Which of the following integrals represents the y-moment M_y of a thin plate of constant density $\rho = 3$ covering the region enclosed by the graphs of $f(x) = x^2 - 4x + 6$ and $g(x) = x + 2$?

   A. $M_y = \int_1^4 3x(x^2 - 5x + 4)\, dx.$

   B. $M_y = \frac{3}{2} \int_1^4 \left((2 + x)^2 - (x^2 - 4x + 6)^2\right) dx.$

   C. $M_y = \int_1^4 3x(-x^2 + 5x - 4)\, dx.$

   D. $M_y = \int_1^4 3(-x^2 + 5x - 4)\, dx.$

   E. $M_y = \int_1^4 (-x^2 + 5x - 4)\, dx.$

9. If $x = e^{2t}$ and $y = \sin(2t)$, then $\frac{dy}{dx} =$

   A. $4e^{2t}\cos(2t)$

   B. $\frac{e^{2t}}{\cos(2t)}$

   C. $\frac{\sin(2t)}{2e^{2t}}$

   D. $\frac{\cos(2t)}{2e^{2t}}$

   E. $\frac{\cos(2t)}{e^{2t}}$

10. For what values of t does the curve given by the parametric equations $x = t^3 - t^2 - 1$ and $y = t^4 + 2t^2 - 8t$ have a vertical tangent line?

   A. 0 only

   B. 1 only

   C. 0 and $\frac{2}{3}$ only

   D. 0, $\frac{2}{3}$, and 1

   E. No value

11. A curve C is defined by the parametric equations $x = t^2 - 4t + 1$ and $y = t^3$. Which of the following is an equation of the line tangent to the graph of C at the point $(-3, 8)$?

   A. $x = -3$

   B. $x = 2$

   C. $y = 8$

   D. $y = -\frac{27}{10}(x + 3) + 8$

   E. $y = 12(x + 3) + 8$

12. The length of the path described by the parametric equations $x = \frac{1}{3}t^3$ and $y = \frac{1}{2}t^2$, where $0 \leq t \leq 1$, is given by

   A. $\int_0^1 \sqrt{t^2 + 1}dt$

   B. $\int_0^1 \sqrt{t^2 + t}dt$

   C. $\int_0^1 \sqrt{t^4 + t^2}dt$

   D. $\frac{1}{2} \int_0^1 \sqrt{4 + t^2}dt$

   E. $\frac{1}{6} \int_0^1 t^2\sqrt{4t^2 + 9}dt$

Free Response Questions

13. Setup (and do not compute) the integral that needs to be calculated if we want to find the volume of the solid of revolution obtained by rotating the region R enclosed by $y = e^x + \sin(x) + 1$, $y \geq 0$ and $0 \leq x \leq 2$ using the most suitable method, when R is rotated:

   (a) (5 points) around the x-axis

   Solution:
   $$\int_0^2 \pi \left(e^x + \sin x + 1\right)^2 dx.$$

   (b) (5 points) around the y-axis

   Solution:
   $$2\pi \int_0^2 x \left(e^x + \sin x + 1\right) dx.$$

14. (10 points) Compute the arc length of the graph of $f(x) = 2x^{\frac{3}{2}} + 4$ over the interval $[0, 7]$. **Give the exact answer.**

Solution: $f'(x) = 3x^{1/2}$ so the arc length is given by

$$\begin{aligned} L &= \int_0^7 \sqrt{1 + [f'(x)]^2} dx \\ &= \int_0^7 \sqrt{1 + 9x} dx \\ &= \frac{2}{27} (1 + 9x)^{3/2} \Big|_0^7 \\ &= \frac{1022}{27} \end{aligned}$$

15. Consider the functions $f(x) = 6\sqrt{x}$ and $g(x) = 3x$.

(a) (4 points) Compute the intersection points of $f(x)$ and $g(x)$.

Solution:

$$6\sqrt{x} = 3x$$
$$4x = x^2$$
$$x = 0, 4$$

(b) (2 points) Give a sketch of the region enclosed by the graphs of $f(x)$ and $g(x)$.

Solution:

(c) (12 points) Compute the centroid of the region.

Solution:

$$M = \int_0^4 (6\sqrt{x} - 3x)dx = 8$$
$$M_y = \int_0^4 x(6\sqrt{x} - 3x)dx = \frac{64}{5} = 12.8$$
$$M_x = \int_0^4 \frac{1}{2}\left((6\sqrt{x})^2 - (3x)^2\right)dx = 48$$
$$\overline{x} = \frac{M_y}{M} = \frac{8}{5} = 1.6$$
$$\overline{y} = \frac{M_x}{M} = 6$$

16. The surface S is generated by revolving the parametric curve given by $x = 1 + \cos t$ and $y = 2 - \sin t$ for $0 \leq t \leq 2\pi$ around the x-axis.

(a) (6 points) Write down the integral that is required to compute the surface area of this surface, S.

Solution:

$$\int_0^{2\pi} 2\pi(2 - \sin t)\sqrt{(-\sin t)^2 + (-\cos t)^2} dt.$$

(b) (4 points) Find the **exact** value of this integral.

Solution:

$$\begin{aligned}\int_0^{2\pi} 2\pi(2 - \sin t)\sqrt{\sin^2 t + \cos^2 t} &= \int_0^{2\pi} 2\pi(1 + \sin t) dt \\ &= 2\pi \int_0^{2\pi} 2 - \sin t \, dt \\ &= 2\pi(4\pi + 0) \\ &= 8\pi^2\end{aligned}$$

17. Consider the parametric equations $x = 4\cos(t)$ and $y = 6\cos(2t)$.

(a) (6 points) Find $\frac{dy}{dx}$

Solution:

$$\begin{aligned}\frac{dy}{dt} &= y' = -12\sin(2t) = -24\sin t\cos t\\ \frac{dx}{dt} &= x' = -4\sin t\\ \frac{dy}{dx} &= \frac{y'}{x'} = \frac{-24\sin t\cos t}{-4\sin t} = \frac{3\sin(2t)}{\sin t}\\ &= 6\cos t\end{aligned}$$

(b) (2 points) Find the slope of the tangent line when $t = \pi/3$.

Solution:

$$\left.\frac{dy}{dx}\right|_{t=\pi/3} = 6\cos\frac{\pi}{3} = 3.$$

(c) (3 points) Find the equation of the tangent line when $t = \pi/3$.

Solution: $x(\pi/3) = 2$ and $y(\pi/3) = -3$, thus

$$y + 3 = 3(x-2) \qquad or \qquad y = 3x - 9.$$

(d) (5 points) Find $\frac{d^2y}{dx^2}$.

Solution:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{6\sin t\cos(2t) - 3\sin(2t)\cos t}{\sin^2 t}}{-4\sin t} = \frac{-6\sin t}{-4\sin t} = \frac{3}{2}.$$