MA 123 Elementary Calculus and its Applications
Old Exam 2 Solutions
July 2, 2008

2004 Exam

1. (a) Quotient Rule:

$$f'(x) = \frac{7x^2 - 10x + 5)(4) - (4x - 1)(7 \cdot 2x - 10)}{(7x^2 - 10x + 5)^2}.$$

   (b) Chain Rule:

$$f'(x) = 5 \cdot 7 \cdot (1 + 3x - 5x^4)^6 (3 - 5 \cdot 4x^3)$$

   (c) Chain Rule:

$$f(x) = (3x^4 + 10)^{1/2}$$
$$f'(x) = (1/2)(3x^4 + 10)(3 \cdot 4x^3)$$

2. (a) Critical numbers are the values c for which $f'(c) = 0$ or $f'(c)$ does not exist. So we solve the equation $f'(x) = x^2 - x - 2 = (x+1)(x-2) = 0$ to get critical points -1 and 2.

   (b) The relative maximum must occur at a critical point. We can tell a point is a local max if the derivate is positive to the left and negative to the right. That corresponds to the function increasing to the left and decreasing to the right. The critical point -1 has this property. A quick way to see that is the graph the derivative.

   (c) IGNORE.

3. IGNROE

4. (a)

$$\begin{aligned} f'(x) &= 2A(3x^2 - 4x - 2)(6x - 4) \\ f'(2) = 16 = &= 2A(3(4)^2 - 4(4) - 2)(6(4) - 4) \\ 2A(600) &= 16 \\ A &= 75 \end{aligned}$$

   (b) IGNORE

5. (a) Critical numbera are value where the derivative is zero. So we solve $f'(x) = 3x^2 - 6x - 24 = 0$ for x. Divide both sides by 3 to simplfy the numbers. and you get $f'(x) = x^2 - 2x - 8 = (x + 2)(x - 4) = 0$. Critical points are -2 and 4.

(b) $f(x)$ increases when $f'(c) > 0$. So solve $f'(x) > 0$. Easy approach: graph $y = f'(x)$ and look to see where the graph is above the x-axis. You already have found the exact value of the x-intercepts so you don't need to worry about calculator accuracy. $f(x)$ increases on $(-\infty, -2)$ and $(4, \infty)$.

(c) Local max occurs when the function goes from increasing to decreasing. This happens at $x = -2$.

(d) IGNORE

6. Recall the Extreme Value Theorem: The absolute extrema of a continuous function on a closed bounded interval must obtained at the endpoints, or at a point where the derivate zero or undefined. So find the critical points. (That is, WHERE the derivate is zero. It is a polynomial and thus defined everywhere.) Then find the value of f for the endpoints and the critical points. The largest of those value is the maximum value of f on $[1, 4]$. Note that the problem asks for the maximum VALUE of the function - meaning a y value - not an x value.

   $f'(x) = 4x - 12 = 0$ So $x = 3$. $f(3) = 2(3)^2 - 12(3) + 13 = 112$. $f(1) = 2(1)^2 - 12(1) + 13 = 3$. $f(4) = 2(4)^2 - 12(4) + 13 = -3$ So the maximum value of $f(x)$ on $[1, 4]$ is 112.

7. Done in class.

8. Done in class.

   (a) $(A, D), (F, H)$

   (b) IGNORE: $(B, C), (E, G), (I, J)$

   (c) D, F, H

   (d) IGNORE: B, C, E, G, I (Inflection points are local extrema of the derivative)

   (e) F

9. Your graph should be increasing concave up from 1 to 2, increacing concave down from 2 to 4, decreasing concave down from 4 to 8. It should have a local maximum at 4.

Unlabeled Exam

1. (a) Product Rule: $f'(x) = (3x^2 + 2)(3 + x - x^4) + (x^3 + 2x - 7)(1 - 4x^3)$

   (b) Chain Rule: $f'(x) = \frac{1}{2}(5x^4 + 3x^2 + 2x + 1)^{1/2-1}(4 \cdot 5x^3 + 2 \cdot 3x^{+}2)$

   (c) Quotient and Chain Rules:

$$\frac{(1 - 2x^3)^4 \left(2(3x^4 + 1)^1(12x^3)\right) - (3x^4 + 1)^2 \left(4(1 - 2x^3)^3(-6x^2)\right)}{\left((1 - 2x^3)^4\right)^2}$$

2. IGNORE: $y = f(2) + f'(2)(x - 2)$

3. Give definitions for (a) and (b).

(a) $\ldots f(x_1) \leq f(x_2)$ whenever $x_1 \leq x_2$.

(b) ... there is some small open interval containing x_1 such that $f(x_1) \geq f(x_2)$ for points x_2 in that small open interval.

(c) IGNORE: $\ldots f''(x) > 0$ f has a minimum is $f' = 0$ and the graph is concave up there (smile). Graph is "smiling" if the second derivate is positive.

4. (a) Find when $f'(x) = 0$ and when $f'(x)$ DNE. f is a polynomial and thus is differentiable everywhere, so $f'(x)$ exits everwhere. Solve $f'(x) = -6x^2 + 6x + 12 = -x^2 + x + 2 = (2 - x)(x + 1)0$ Critical points are 2 and -1.

   (b) f is increasing when f' is positive, decreasing when f' is negative. f' can only change signs at -1 and 2. Since the leading coefficient of $f'(x)$ is negative, the parabola opens down. So $f'(x)$ is negative outside $(-1, 2)$ and positive on $(-1, 2)$. Thus $f(x)$ decreases on $(-\infty, -1)$ and $(2, \infty)$ and increases on $(-1, 2)$.

   (c) IGNORE.

   (d) We already worked out that f goes from decreasing to increasing at -1 and from increasing to decreasing at 2. Thus there is a local minimum at -1 and a local maximum at 2.

   (e) I don't feel like creating a graph right now.

5. (a) A function $f(x)$ is continuous at $x = c$ if

      i. $f(c)$ is defined.

      ii. $\lim_{x\to c} f(x)$ exists.

      iii. $f(c) = \lim_{x\to c} f(x)$

   (b)

$$\ldots \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} \text{ exists.}$$

6. Your graph must be increasing to the left of -2 and from -2 to 3. It must be decreasing to the right of 3. The graph must have horizontal tangents at -2 and 3. There will be a point of inflection at -2 and an absolute maximum at 3.

7. (a) $f'(2)$ does not exist. The graph is "pointy" there. Specifically, it appears from the graph that the slope of tangent lines just to the left of 2 have negative slope and those the the right have positive slope, but they slopes are not approaching zero.

   (b) f is continuous at 2 since the limits of $f(x)$ from both the left and the right are equal to the $f(2)$.

   (c) f is not continuous at -1. The limit exists, but is not equal to the value of the function.

2007 Multiple Choice Exam

1. (e) The limit is the definition of the derivative of f a 1. $f'(1) = 4(1)^3 = 4$.

2. (e) $f'(.7) = \lim_{h\to 0} \frac{2^{.7+h} - 2^{.7}}{h}$ Approximate the limit by evaluating the expression for a non-zero value of h very close to 0. If using the table at the end of the exam, choose a value of h so that $.7 + h$ is on the table. $h = .05$ will work.

$$f'(.7) \approx \frac{2^{.7+.05} - 2^{.7}}{.05} \approx 1.1457607559.$$

   So the best approximation given is 1.13. The actual derivative is $2^{.7}\ln(2)$.

3. (e) again. Use the Quotient Rule. The numerator will simplify to just -2.

4. (b) The slope of the line tangent to the graph at $t = 1$ is the derivative of the function at $t = 1$. Write $F(t)$ with a negative exponent instead of a fraction and use the Power and Chain Rules (derivative of the inside is just 1). $F'(t) = -2(t+1)^{-2}$, so $F'(t) = -2(1+1)^{-2} = -2/4 = -1/2$.

5. (d) Find the derivative (which is the slope of the desired line) $w'(8) = 1/6$. Only one answer has slope 1/6.

6. (d) The graph of $y = 7 - |x|$ is the same as the graph of $y = |x|$ reflected over the x-axis (flipped upside-down) and then shifted up seven units. So it is decreasing everywhere to the right of zero.

7. (e) $F'(x) = u'(x^2)\cdot(2x) + 2v(x)\cdot v'(x)$ So $F'(1) = u'(1^2)\cdot(2) + 2v(1)\cdot v'(1) = 9\cdot 2 + 2\cdot 3\cdot 7 = 60$

8. (b) $h'(x) = 1\cdot g(x) + x\cdot g'(x)$, so $h'(2) = \cdot g(2) + 2\cdot g'(2)$. The equation of the tangent line tells us that $g(2) = -5$ and $g'(2) = 4$. Thus $h'(2) = -5 + 2\cdot 4 = 3$

9. (a) $|s-1|$ is not differentiable at $s - 1$. The derivative to the left of 1 is alwas -1, and to the right the derivative is always -1.

10. (e) To apply the Extreme Value Theorem, find the value of g at the endpoints, at the points in the interval with zero derivative and the points in the interval where the function is defined but the derivative is not. $g'(s) = \frac{-1}{(s+1)^2}$. Notice that g' is never zero and is undefined only at $s = -1$ where the g is also undefined. So we need only to check the endpoints. $g(-2) = -1, g(0) = 1$. BUT WAIT! We don't know that 1 is a maximum. Because g is not continuous on the interval the Extreme Value Theorem does not apply. The limit of $g(s)$ as s approaches -1 from the right is ∞, so there is no maximum value on the interval.

11. (e) If $-1 \le x \le 1$, then $|x| < 1$ and thus $h'(x) = 1 - |x| > 0$. Therefore $h(x)$ is increasing on the whole interval and so the maximum value is at the right endpoint.

12. (b) f is continuous on the interval, so the Extreme Value Theorem applies, we need only check the endpoint, the zeroes of the derivative, and the points where the derivative is undefined. The point where the two formulas are glued together ($t = 4$) is a likely place for the derivative to not exists. Rather than checking to see if it does exist, just include it in the list of points to check just to be sure.

    On the left half of hte interval, $f'(t) = -1/2(4-t)^{-1/2}$ and on the right half $f'(t) = 1/2(t-4)^{-1/2}$. When is $f'(t) = 0$? Never. The numerators of both formulae are constants. When is the derivative undefined? only when $t = 4$, the point we were going to check anyway. So we check: $f(0) = 2, f(4) = 0, f(6) = \sqrt{(}2)$ and the maximum is... 2.

13. (e) Recall that the Mean Value Theorem says that for a continuous differentiable function f defined on the interval $[a, b]$ there is a point c between a and b so that the average rate of f with respect to x between $x = a$ and $x = b$ is equal to the instantaneous rate of change of f with respect to x at $x = c$. What does this mean to us here? Only that the answer exists. (The fact that there si no "does not exist" options tells you that too.) So how do we find A? Write down the equation implied by the problem and solve for A.

$$\begin{aligned} \frac{g(A) - g(0)}{A - 0} &= g'(2) \\ \frac{A^3 - 0}{A} &= 3(2)^2 \\ A^2 &= 12 \\ |A| = \sqrt{(}12) & \end{aligned}$$

    So is $A = \sqrt{(}12)$ or $-\sqrt{(}12)$? (Well, only one is an option...) The problem mentions the interval $[0, A]$ so we know that $A > 0$.

14. (c) The problem is: $\frac{dC}{dt} = C'(t) = 5$ and $C(t) = (5/9)(F(t) - 32)$, what is $\frac{dF}{dt} = F'(t)$? Well, $5 = C'(t) = (5/9)F'(t)$ so $F'(t) = 9$.

15. (d) $A(q) = C(q)/q = 1000q^{-1} + 10 + .1q$ What is the minimum of $A(q)$ on (q, ∞)? Well, the minimum must occur where the derivative is zero or undefined. The derivative $(-1000q^{-2} + .1)$ is undefined only at 0. $.1 - 1000t^{-2} = 0$ when $t^2 = 10000$ which occurs when $t = 100$ (remember that $t > 0$). So the minimum is $A(100) = 1000/100 + 10 + .1 \cdot 100 = 30$