MA114 Calculus II

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Problem. *If the nth partial sum of a series* $\sum_{n=1}^{\infty} a_n$ *is*

$$s_n = \frac{n-1}{n+1},$$

find a_n *and* $\sum_{n=1}^{\infty} a_n$.

Proof. For $n = 1$, we get $s_1 = 0$ so $a_1 = 0$. For $n > 1$, we know that $s_n = a_1 + a_2 + \ldots + a_n$. Therefore,

$$\begin{aligned}
a_n &= s_n - (a_1 + a_2 + a_3 + \ldots + a_{n-1}) \\
&= s_n - s_{n-1} \\
&= \frac{n-1}{n+1} - \frac{(n-1)-1}{(n-1)+1} \\
&= \frac{n(n-1)}{n(n+1)} - \frac{(n-2)(n+1)}{n(n+1)} \\
&= \frac{n^2 - n - (n^2 - n - 2)}{n^2 + n} \\
&= \frac{2}{n^2 + n}.
\end{aligned}$$

We now know what a_n is. To find $\sum_{n=1}^{\infty} a_n$, we simply need to take the limit of the sequence of partial sums. Therefore, we get

$$\begin{aligned}
\lim_{n\to\infty} s_n &= \lim_{n\to\infty} \frac{n-1}{n+1} \\
&= \lim_{n\to\infty} \frac{\frac{n}{n} - \frac{1}{n}}{\frac{n}{n} + \frac{1}{n}} \\
&= 1.
\end{aligned}$$

Thus,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{2}{n^2 + n} = 1$$

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