MA201 Test 2 Key - 11/3/09

1. (a) (4 pts.) State George Polya's four principles for problem solving.
   Understand the problem, devise a plan, carry out the plan, and look back.

   (b) (4 pts.) Using one or two sentences for each, explain what these principles mean.
   U.t.P: Before being able to solve a problem, you must have a clear understanding of what the problem is asking. This includes determining the type of answer the problem is asking for and making sure you understand all of the words used in the problem.
   D.a.P: Try to determine the best way to attack the problem. C.o.t.P: Attempt to solve the problem using the plan from the previous principle. In particular, if that plan does not work, go back to part two and try again.
   L.B: Look back at the problem and solution together in order to ascertain what the key was in the solution. This will help you remember how to solve similar problems in the future.

2. Let a and b be mathematical statements.

   (a) (5 pts.) Suppose that the statement "If a then b" is true. Suppose additionally that b is NOT true. What, if anything, can you conclude? Illustrate this with an example.
   You can conclude that a is NOT true. Consider, for example, the sentence, "If I jump in a pool, then I get wet." This is true. If I tell you also, "I did not get wet," you can conclude that I did not jump in a pool.

   (b) (5 pts.) Suppose that the statement "If a then b" is true. Suppose additionally that a is NOT true. What, if anything, can you conclude? Illustrate this with an example.
   You cannot conclude anything. Using the same example as above, if I tell you additionally that, "I did not jump in a pool," you CANNOT conclude that I did not get wet. I could have, for instance, gotten caught in the rain!

3. (5 pts.) Use Gauss's trick to find the sum $1+2+3+4+\cdots+(n-1)+n$. You must show your work.
   Consider

$$\begin{array}{ccccccccccc} s & = & 1 & + & 2 & + & \dots & + & n-1 & + & n \\ s & = & n & + & n-1 & + & \dots & + & 2 & + & 1 \\ \hline 2s & = & (n+1) & + & (n+1) & + & \dots & + & (n+1) & + & (n+1) \end{array}$$

   Thus, $2s = n\cdot(n+1)$. Therefore,

$$s = \frac{n\cdot(n+1)}{2}.$$

4. Consider the sequence $1, 5, 9, 13, \dots$

   (a) (4 pts.) List the next 4 terms in this sequence.
   17,21,25,29

   (b) (4 pts.) What is the 10th term?
   1+9(4)

   (c) (4 pts.) What is the 100th term?
   1+99(4)

   (d) (4 pts.) What is the nth term?
   $1+(n-1)\cdot 4$

5. (a) (5 pts.) Explain the pigeonhole principle.
   The pigeonhole principle states that if you have n pigeons and m pigeonholes with $n > m$ then there is at least one pigeonhole with more that one pigeon.

(b) (5 pts.) You have a classroom with 15 boys and 15 girls. When forming a group, how many students must you select at random to ensure that you EITHER have at least two boys OR at least two girls? Explain.

3. In the worst case, our first two picks could be a boy and a girl. However, the next person picked will give us either two girls or two boys. In the case, the sexes "girl" and "boy" are the pigeonholes and the students are the pigeons.

(c) (5 pts.) How many people do you need in a room to ensure that at least two people were born on the same day of the week? Explain.

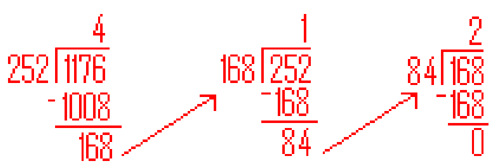
8. In the worst case, we could have 7 people who were each born on a different day of the week. However, the 8th person would have to double up a day somewhere. In this case, the seven days of the week are the pigeonholes and people are the pigeons.

6. (5 pts.) Sue can work a puzzle in 2 hours. Patty can work the same puzzle in 3 hours. How long would it take the two to work the puzzle together? Make sure you show your work.

Let T represent the number of hours it takes Sue and Patty to complete the puzzle together. In T hours, Sue can complete $\frac{T}{2}$ of the puzzle while Patty can complete $\frac{T}{3}$. Thus, together they complete $\frac{T}{2}+\frac{T}{3}$ of the puzzle in T hours. This is all of the puzzle though. So we get

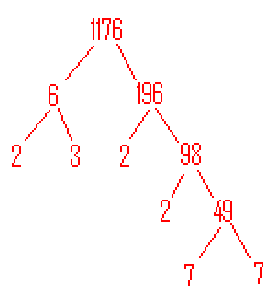
$$\begin{aligned} \frac{T}{2}+\frac{T}{3} &= 1 \\ \Rightarrow \frac{3T}{6}+\frac{2T}{6} &= 1 \\ \Rightarrow \frac{5T}{6} &= 1 \\ \Rightarrow T = \frac{5}{6}. \end{aligned}$$

7. (5 pts.) Use the Euclidean Algorithm to find GCD(1176,252).



GCD(1176,252) = GCD(252,168) = GCD(168,84) = GCD(84, 0) = 84

8. (a) (5 pts.) Find the prime factorization of 1176 by using a factor tree.



Therefore, $1176 = 2^3 \cdot 3 \cdot 7^2$.

(b) (5 pts.) Suppose $252 = 2^2 \cdot 3^2 \cdot 7$. Use the prime factorizations to calculate GCD(1176, 252).
GCD(1176,252)$= 2^2 \cdot 3 \cdot 7 = 84$.

(c) (5 pts.) Use the prime factorizations to calculate LCM(1176, 252).
LCM(1176,252) $= 2^3 \cdot 3^2 \cdot 7^2$.

9. (5 pts.) Let $23a, 5a7, a13$ be a 9 digit number where the a represents a few missing digits. Explain why this number is divisible by three, regardless of the choice of a.
Note that the sum of the digits is

$$2 + 3 + a + 5 + a + 7 + a + 1 + 3 = 21 + 3a.$$

Since both 21 and $3a$ are divisible by 3, then so is their sum. Therefore, the sum of the digits is divisible by three. This means that the number itself is also divisible by three.

10. (6 pts.) List all factors of the number $n = 23 \cdot 31 \cdot 59^2$ (Note that this is a prime factorization).

$23^0 \cdot 31^0 \cdot 59^0$	$23^0 \cdot 31^1 \cdot 59^1$	$23^1 \cdot 31^0 \cdot 59^2$
$23^0 \cdot 31^0 \cdot 59^1$	$23^0 \cdot 31^1 \cdot 59^2$	$23^1 \cdot 31^1 \cdot 59^0$
$23^0 \cdot 31^0 \cdot 59^2$	$23^1 \cdot 31^0 \cdot 59^0$	$23^1 \cdot 31^1 \cdot 59^1$
$23^0 \cdot 31^1 \cdot 59^0$	$23^1 \cdot 31^0 \cdot 59^1$	$23^1 \cdot 31^1 \cdot 59^2$

11. Are the following statements true or false?

(a) (1 pt.) 1 is prime.
False

(b) (1 pt.) 0 is neither even nor odd.
False

12. (8 pts.) Which of the following divide the number 22355080? 2,3,4,5,6,9,10,11. Explain why or why not for each.

Number	Divisible?	Reason
2	Yes	The last digit is a 0 which is divisible by 2
3	No	2+2+3+5+5+0+8+0=25 which is not divisible by 3.
4	Yes	The number created by the last two digits, 80, is divisible by 4.
5	Yes	The last digit is a 0 which is divisible by 5.
6	No	The number is divisible by 2 but not by 3.
9	No	2+2+3+5+5+0+8+0=25 which is not divisible by 9.
10	Yes	The last digit is a 0 which is divisible by 10.
11	Yes	0-8+0-5+5-3+2-2=-11 which is divisibly by 11.

Bonus:

13. (5 pts.) Prove why the divisibility test for 3 works.

14. (5 pts.) Explain why there are infinitely many primes.