MA322 Matrix Algebra

Answers to assigned even problems

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Section 1.1:

10. (-3, -5, 6, -3)
14. (2,-1,1)
24. a. True. See the blue shaded box on page 8.
b. False. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ have the same number of rows but are not row equivalent.
c. False. By definition, inconsistent means no solution.
d. True. See definition of equivalent on page 3.
26. Many answers. For example, $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, (R1+R3) $\begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, (R2+R3) $\begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Section 1.2

2. a. Reduced echelon b. echelon c. not echelon d. echelon.
10. $x_1 = -4 + 2x_2$, x_2 is free, $x_3 = -7$.
14. $x_1 = -9 - 7x_3$, $x_2 = 2 + 6x_3 + 3x_4$, x_3 is free, x_4 is free, $x_5 = 0$.
16. a. Unique solution. b. Infinitly many solutions.
20. a. Inconsistent when $h = 9$ and $k \neq 6$.
b. Unique solution when $h \neq 9$.
c. Many solutions when $h = 9$ and $k = 6$.
24. If the fifth column is a pivot column, then the reduced echelon form of the augmented matrix will have a row like [0 0 0 0 1]. Since this is the augmented matrix, the system is inconsistent (See theorem 2 page 24).
30. Many answers. For example:

$$\begin{aligned} x_1 + x_2 + x_3 &= 4 \\ 2x_1 + 2x_2 + 2x_3 &= 5. \end{aligned}$$

Section 1.3

4. Just see me if you have questions about this one.
6.

$$\begin{aligned} -2x_1 + 8x_2 + x_3 &= 0 \\ 3x_1 + 5x_2 - 6x_3 &= 0. \end{aligned}$$

10. $x_1 \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -7 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 15 \end{bmatrix}$.
12. No. $\mathbf{b}$ is not a linear combination of $\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$.
16. Many answers, for example: $1\mathbf{v_1} + 1\mathbf{v_2} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$.
20. The span is the xz-plane.

Section 1.4

2. The product is not defined because the number of columns of the first matrix does not equal the number of rows of the vector.

12. $\mathbf{x}= \begin{bmatrix} 3/5 \\ -4/5 \\ 1 \end{bmatrix}$

18. The reduced echelon form of B has row of zeros (i.e. there is a row without a pivot position). Thus, the columns of B do not span $\mathbb{R}^4$ (see theorem 4 on page 43).

32. Consider the matrix that has n columns (vectors) and m rows with $n < m$. There may be at most n pivot positions (since there can be only one per column). Since $n < m$, this implies that not all rows have a pivot position. Therefore, the columns of this matrix do not span all of $\mathbb{R}^m$, again by theorem 4 page 43.

34. If the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, then the corresponding linear system does not have any free variables. That means, that there is a pivot in every column of A. Since A is 3×3, this also implies that A has a pivot in every row. Therefore, the columns of A must span $\mathbb{R}^3$ by theorem 4.

Section 1.5

6. $\mathbf{x}=x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$

16. $\mathbf{x}= \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} +x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix} = p + x_3 q$. The solution set is a line through p parallel to q.

24. a. False. A non-trivial solution just means that not all entries are zero. Some of the entries may be zero.
b. True. See example 2, page 51.
c. True. If $\mathbf{0}$ is a solution, then is must be true that $\mathbf{b} = \mathbf{0}$ since $A \cdot \mathbf{0} = \mathbf{0}$.
d. True. See the paragraph after example 3, page 53.
e. False. This is true only if $A\mathbf{x} = \mathbf{b}$ is consistent. Therefore, in general, it is false.

26. Let $\mathbf{p}$ be a possible solution to $A\mathbf{x} = \mathbf{b}$. By theorem 6, we know that the solution set of this equation is $\mathbf{p} + \mathbf{v}_h$ where $\mathbf{v}_h$ is any solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$. Therefore, if $\mathbf{p}$ is the only solution, then $\mathbf{v}_h$ must only be zero.

30. a. Since there are only two pivot positions, there must be a free variable because the pivot positions correspond to basic variables (i.e. non-free variables). So there must be non-trivial solutions.
b. Since there are only two pivot positions, there is not a pivot in every row. Therefore, there is not a solution for every $\mathbf{b} \in \mathbb{R}^3$ (again, by theorem 4, page 43).

Section 1.7

2. Linearly independent
10. a. No h.
b. All h.
16. Linearly dependent
18. Linearly dependent
38. True. If $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ had a non-trivial solution, then so would the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + 0\mathbf{v}_4 = \mathbf{0}$. However, we know that this is not the case since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly independent (meaning the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 = \mathbf{0}$ has only the trivial solution. Thus, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are also linearly independent.

Section 1.8

4. $\mathbf{x} = \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}$, which is the unique solution.
8. 5 rows and 4 columns.
20. $\begin{bmatrix} -2 & 7 \\ 5 & -3 \end{bmatrix}$

24.

$$\begin{aligned} T(\mathbf{x}) &= T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p) \text{ since the v's span } \mathbb{R}^n \\ &= c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \ldots + c_pT(\mathbf{v}_p) \text{ since } T \text{ is linear} \\ &= c_1\mathbf{0} + \ldots + c_p\mathbf{0} \\ &= \mathbf{0}. \end{aligned}$$

32. Pick any vector with $x_2 \neq 0$ and let c be any negative number. Use this to check and see if $T(c\mathbf{x}) = cT(\mathbf{x})$.

Section 1.9

4. $\begin{bmatrix} 1/\sqrt(2) & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt(2) \end{bmatrix}$

8. $\begin{bmatrix} 0 & \text{-1} \\ 1 & 0 \end{bmatrix}$

12. Look at the rotation matrix on page 84. Note that if we let $\phi = \pi/2$, we get the matrix from number 4. So the transformation from 4 is the same as a rotation counter-clockwise by $\pi/2$.

18. $\begin{bmatrix} \text{-3} & 2 \\ 1 & \text{-4} \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$

Section 2.1

$$4. A - 5I_3 = \begin{bmatrix} 4 & -1 & 3 \\ -8 & 2 & -6 \\ -4 & 1 & 3 \end{bmatrix}.$$

$(5I_3)A = \begin{bmatrix} 45 & -5 & 15 \\ -40 & 35 & 30 \\ -20 & 5 & 40 \end{bmatrix}$

10. $AB = AC = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$.

12. One example: $\begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$.

20. The second column of AB is also all zeros because $A\mathbf{b}_2 = A\mathbf{0} = \mathbf{0}$.

22. If the columns of B are linearly dependent, then there exists a nonzero vector $\mathbf{x}$ such that $B\mathbf{x} = \mathbf{0}$. From this, $A(B\mathbf{x}) = (AB)\mathbf{x} = \mathbf{0}$. Since $\mathbf{x}$ is nonzero, the columns of AB must be linearly dependent.

28. $\mathbf{u}^T\mathbf{v} = \mathbf{v}^T\mathbf{u}$ and $\mathbf{u}\mathbf{v}^T = \mathbf{v}\mathbf{u}^T$ by theorem 3.

Section 2.2

8. Since $AD = I$, $A^{-1}AD = A^{-1}I$ which implies $ID = A^{-1}$ or $D = A^{-1}$.

24. If the equation $A\mathbf{x} = \mathbf{b}$ has as solution for each $\mathbf{b}$ in $\mathbb{R}^n$, then A has a pivot position in each row. Since A is square, the pivots must be on the diagonal of A so A is row equivalent to I_n. Thus, A is invertible.

26. Take the formula for A^{-1} and actually verify that $AA^{-1} = A^{-1}A = I$.

32. Not invertible.

Section 2.3

2. Not invertible because the determinant is zero. Or you could note that the second column is $-3/2$ times the first column.

4. The columns of the matrix are linearly dependent because one of the columns is zero, so the matrix is not invertible.

22. Statment (g) of the Invertible Matrix theorem is false for H, so statement (d) is false, too.

Section 3.1

2. 2
4. 20
10. -6

16. 2
20. The determinant of the first is $ad - bc$ and the determinant of the second is $k(ad - bc)$. So scaling a row by k multiplies the determinant by k.
Section 3.2
6. A constant may be factored out of a row.
12. 114
16. 21
24. linearly independent
34.

$$\begin{aligned} \det(PAP^{-1}) &= \det(P)\det(A)\det(P^{-1}) \\ &= \det(P)\det(P^{-1})\det(A) \\ &= \det(PP^{-1})\det(A) \\ &= \det(I)\det(A) \\ &= \det(A) \end{aligned}$$

Section 3.3
4. $x_1 = -3/2, x_2 = 1/2$
8. All real s; $x_1 = \frac{3s+2}{3(s^2+3)}, x_2 = \frac{2s-9}{5(s^2+3)}$
12. $\text{adj}(A) = \begin{bmatrix} -1 & 3 & 7 \\ 0 & 0 & 5 \\ 2 & -1 & -4 \end{bmatrix}$. So $A^{-1} = \frac{1}{5}\begin{bmatrix} -1 & 3 & 7 \\ 0 & 0 & 5 \\ 2 & -1 & -4 \end{bmatrix}$.
18. Since the cofactors are just sums and products of elements of A, all of the cofactors are integer. Since the $\det(A)$ is 1, we get from the inverse formula that all of the entries of A^{-1} are integer.
Section 4.1
2. a. Given $\begin{bmatrix} x \\ y \end{bmatrix}$ in W and any scalar c, the vector $c\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix}$ is in W because $(cx)(cy) = c^2(xy)$ which is ≥ 0.
b. Let $u = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $v = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$.
8. Yes. The zero polynomial is in the set. If p and q are in the set, then $(p+q)(0) = p(0) + q(0) = 0$ and $(cp)(0) = c(p(0)) = 0$.
22. Yes. The zero matrix is in the set. If B and C are in the set, then $F(B+C) = FB + FC = 0$ and for scalar k, $F(kB) = k(FB) = 0$.
Section 4.2
8. W is not a vector space because the zero vector is not in W.
12. To get the zero vector, we need $d = -1/2$ so that the third entry, $2d + 1$ equals zero. But then it's impossible for the fouth entry to be zero. So this is not a vector space.
28. The two systems have the form $Ax = v$ and $Ax = 5v$. Since we are given that the first system is consistant, we know that v is in the column space of A. Since $\text{col}(A)$ is a vector space, we know that $5v$ is in the column space too. Thus, $Ax = 5v$ must also be consistant.
Section 4.3
4. This is a basis.
6. This is not a basis since it does not span $\mathbb{R}^3$, but they are linearly independent.
24. Let A be the matrix whose columns are the basis elements. Since the columns of A are linearly independent, we have a pivot in every column. Since we have n vectors in $\mathbb{R}^n$, the matrix is square and so there is a pivot in every row as well. Therefore, the columns also span $\mathbb{R}^n$. Therefore, we have a basis.
Section 4.4
2. $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$
6. $\begin{bmatrix} -6 \\ 2 \end{bmatrix}$

Section 4.5

2. $\begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$ is a basis, so the dimension is 2.

16. dim(null(A))=0, dim(col(A))=2.

26. If dim(V)=0, the statement is obvious because the zero vector space is the only subspace of the zero vector space. Otherwise, H has a basis which contains n linearly independent vectors. But these vectors are also linearly independent in V. Since there are n of them and V is n dimensional, these also form a basis for V.

Section 4.6

8. dim(null(A))=2. col(A) cannot be $\mathbb{R}^4$ because the vectors in col(A) have 5 entries.

10. 1

12. 2

20. No. Since there are 2 free variables, the null space of the coefficient matrix A is 2 dimensional. Since there are 8 columns, the rank of A is 6 dimensional and (because there are 6 equations) col(A) is a subspace of $\mathbb{R}^6$. Therefore, col$(A) = \mathbb{R}^6$ so $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b}$.

Section 4.7

2.a. $\begin{bmatrix} -1 & 5 \\ 4 & -3 \end{bmatrix}$.

b. $\begin{bmatrix} 10 \\ 11 \end{bmatrix}$.

4. (i).

Section 5.1

4. Yes. $\lambda = 3 + \sqrt{2}$.

18. 4,0,-3

24. Many solutions. As an example, $\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$ has only 2 as an eigenvalue.

26. We want to know the value of λ if $A\mathbf{x} = \lambda\mathbf{x}$ where $\mathbf{x} \neq \mathbf{0}$. Now

$$\begin{aligned} A^2\mathbf{x} &= A(A\mathbf{x}) \\ &= A(\lambda\mathbf{x}) \\ &= \lambda A\mathbf{x} \\ &= \lambda^2\mathbf{x}. \end{aligned}$$

But $A^2\mathbf{x} = 0$ since $A^2 = 0$, so $\lambda^2\mathbf{x} = \mathbf{0}$ as well. Since $\mathbf{x} \neq \mathbf{0}$, it must be that $\lambda = 0$. Therefore, the only eigenvalue of A is zero.

Section 5.2

6.$\lambda^2 - 11\lambda + 40$. No real eigenvalues.

10.$-\lambda^3 + 14\lambda + 12$

18.$h = 6$

20.

$$\begin{aligned} \det(A^T - \lambda I) &= \det(A^T - (\lambda I)^T) \\ &= \det((A - \lambda I)^T) \\ &= \det(A - \lambda I). \end{aligned}$$

Section 5.3

28. Since A has n linearly independent eigenvectors, A is diagonalizable by the Diagonalization theorem. So A can be written as PDP^{-1}. Therefore,

$$\begin{aligned} A^T &= (PDP^{-1})^T \\ &= (P^{-1})^T D^T (P^T) \\ &= (P^T)^{-1} D (P^T). \end{aligned}$$

Therefore, A^T can be written as QDQ^{-1} where $Q = (P^T)^{-1}$, so A^T is diagonalizable.

Section 5.5

2. $\lambda = 3 + i$, $\begin{bmatrix} 2+i \\ 1 \end{bmatrix}$.

$\lambda = 3 - i$, $\begin{bmatrix} 2-i \\ 1 \end{bmatrix}$.

10.$\lambda = -5 \pm 5i, \phi = 3\pi/4, r = 5\sqrt{2}$

14. $P = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$. $C = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$