MA322 021 Midterm 1 - 6/28/07

Name:SOLUTIONS

Directions:

Please print your name clearly. This is 60 minute test and is worth 20% of your final grade. There are 90 points possible. **Calculators are not permitted on this exam.** You must show all of your work. **Answers without work or explanation will receive little or no credit.**

Problem	**Max. score**	**Assigned score**
1	10	
2	10	
3	13	
4	12	
5	10	
6	10	
7	15	
8	10	
Total	90	

1. (10 pts) Let $A = \begin{bmatrix} -1 & 1 \\ -4 & 2 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -3 \\ 10 \end{bmatrix}$.

a.(5pts) What is A^{-1}?

SOLUTION:

For a 2×2, we know that $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. So $A^{-1} = \begin{bmatrix} 1 & -1/2 \\ 2 & -1/2 \end{bmatrix}$.

b.(5pts) Use part a. to solve $A\mathbf{x} = \mathbf{b}$.

SOLUTION:

$A^{-1}\mathbf{b} = \begin{bmatrix} -8 \\ -11 \end{bmatrix}$.

2. (10 pts) Let

$$A = \begin{bmatrix} -5 & 3 & 4 \\ 10 & -8 & -9 \\ 15 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & -5 & 1 \end{bmatrix} \begin{bmatrix} -5 & 3 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & 9 \end{bmatrix}$$

Use this LU decomposition to solve $A\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ -17 \end{bmatrix}$.

SOLUTION:

$A\mathbf{x} = \mathbf{b}$ implies $LU\mathbf{x} = \mathbf{b}$. Let $\mathbf{y} = U\mathbf{x}$ and first solve $L\mathbf{y} = \mathbf{b}$ by forward substitution. We get $\mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ -9 \end{bmatrix}$. We next solve $U\mathbf{x} = \mathbf{y}$ by back substitution to get $\mathbf{x} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$.

3. (13 pts) Consider the linear system

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 5x_3 & = & 8 \\ 3x_1 & + & 4x_2 & + & 9x_3 & = & 18 \\ -2x_1 & + & 2x_2 & + & 8x_3 & = & 2 \end{array}$$

a. (2 pts) What is the corresponding augmented matrix?

SOLUTION:

$$\left[\begin{array}{ccc|c} 1 & 2 & 5 & 8 \\ 3 & 4 & 9 & 18 \\ -2 & 2 & 8 & 2 \end{array}\right]$$

b. (5 pts) What is the reduced echelon form of this matrix? Make sure you show which row operations you use.

SOLUTION:

Use the following row operations: $R_2 - 3R_1, R_3 + 2R_1, R_3 + 3R_2, R_1 + R_2, \frac{-1}{2}R_2$. This leaves you with the matrix $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

c. (3 pts) Write the solution set for this system in parametric vector form.

SOLUTION:

From part b. we see that $x_1 = 2 - x_3, x_2 = 3 - 3x_3$ and x_3 is free. Therefore, we get that $\mathbf{x} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$.

d. (3 pts) Using your answer from part c.(do NOT directly solve this system), what is the solution set for the corresponding homogeneous linear system (i.e. the system whose right side is all zeros)?

SOLUTION:

We know from part c and the theorem on page 53 that $\mathbf{x}$ is a solution to the corresponding homogeneous system if $\mathbf{x} = k \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$ for any constant k.

4. (12 pts) Determine whether the following sets of vectors are linearly independent or linearly dependent. Briefly justify your answer (HINT: you shouldn't have to solve a system for any of these)

a.

$$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ -13 \end{bmatrix} \right\}$$

SOLUTION:

These are linearly dependent because we've seen numerous times in class and in the homework that n vectors cannot be linearly independent in $\mathbb{R}^m$ if $n > m$.

b.

$$\left\{ \begin{bmatrix} -2 \\ 3 \\ 7 \end{bmatrix}, \begin{bmatrix} -6 \\ 9 \\ 21 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

SOLUTION:

These are linearly dependent because the first two vectors are scalar multiples of each other, making them a dependent set. If you add any vectors to a dependent set, it is still dependent.

c.

$$\left\{ \begin{bmatrix} 2 \\ 9 \end{bmatrix}, \begin{bmatrix} -1 \\ 13 \end{bmatrix} \right\}$$

SOLUTION:

We have two vectors that are not scalar multiples of each other. Therefore, they are independent.

d.

$$\left\{ \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 2 \end{bmatrix} \right\}$$

SOLUTION:

Any collection of vectors containing the zero vector is linearly dependent (because we can write the zero vector as a linear combination of the others).

5. (10 pts) Consider the matrices

$$A^T = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 4 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 2 & 1 \\ 4 & 5 & 7 \end{bmatrix}.$$

a. (5 pts) Circle the expressions that are defined:

$$A + B, \quad \underline{B + A^T}, \quad \underline{(AB)^T}, \quad \underline{AB}, \quad \underline{A^T B^T}$$

The underlined choices are defined

b. (5 pts) For the last of the expressions you circled (going from left to right) calculate the resulting matrix.

SOLUTION:

$$A^T B^T = \begin{bmatrix} 4 & 18 \\ 20 & 74 \end{bmatrix}$$

6. (10 pts) A **basis** for $\mathbb{R}^n$ is a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_p\}$ that is linearly independent AND spans $\mathbb{R}^n$. How many vectors are in a basis for $\mathbb{R}^4$? Explain.

SOLUTION: We have seen numerous times that n vectors cannot span $\mathbb{R}^m$ if $n < m$ and that n vectors cannot be linearly independent in $\mathbb{R}^m$ if $n > m$. Therefore, for a set of vectors to be both linearly independent and span $\mathbb{R}^4$, we must have exactly 4 vectors.

7. (15 pts) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation that maps the vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to $\begin{bmatrix} 2x_1 \\ x_2 \end{bmatrix}$.

a. (5 pts) Find the standard matrix A for this transformation.

SOLUTION:

$A = [T(\mathbf{e}_1)T(\mathbf{e}_2)T(\mathbf{e}_3)] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

b. (5 pts) Is the transformation one-to-one? Justify.

SOLUTION:

To be one-to-one, the system $A\mathbf{x} = \mathbf{b}$ would need to have at most one solution for every $b \in \mathbb{R}^2$ but it is easy to see that $T(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}) = T(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. Therefore, T is not one-to-one.

c. (5 pts) Is the transformation onto? Justify

SOLUTION:

To be an onto function, we need that $A\mathbf{x} = \mathbf{b}$ has a solution for every $b \in \mathbb{R}^2$. We know this is true because A has a pivot position in every row. Thus, T is onto.

8. (10 pts) Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be linearly independent vectors in $\mathbb{R}^5$ and let $\mathbf{v}_4$ be another vector in $\mathbb{R}^5$ that is NOT in the span of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Explain why the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is also linearly independent.

SOLUTION:

To be independent, we need to show that the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 = \mathbf{0}$$

has only the trivial solution (i.e. $x_1 = x_2 = x_3 = x_4 = 0$). There are two cases. If $x_4 = 0$, we are left with $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + 0\mathbf{v}_4 = \mathbf{0}$ which implies that $x_1 = x_2 = x_3 = 0$ also by the linear independence of $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$. If $x_4 \neq 0$, then we can write

$$\mathbf{v}_4 = \frac{x_1}{x_4}\mathbf{v}_1 + \frac{x_2}{x_4}\mathbf{v}_2 + \frac{x_3}{x_4}\mathbf{v}_3,$$

or $\mathbf{v}_4$ can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$. This cannot happen because $\mathbf{v}_4$ is not in their span. Therefore, the only solution is the trivial solution and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a linearly independent set.