MA322 021 Midterm 2 - 7/16/07

Name:______________________________

Directions:

Please print your name clearly. This is 60 minute test and is worth 20% of your final grade. There are 90 points possible. **Calculators are not permitted on this exam.** You must show all of your work. **Answers without work or explanation will receive little or no credit.**

Problem	**Max. score**	**Assigned score**
1	15	
2	12	
3	18	
4	10	
5	10	
6	5	
7	10	
8	10	
Total	90	

1. (15 pts)

(a) (5 pts) Find det $\begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ 2 & 4 & 0 \end{bmatrix}$ by cofactor expansion.

Expand about the third column. det $= -2$.

(b) (5 pts) Find det $\begin{bmatrix} 1 & 1 & 4 \\ -1 & 2 & 12 \\ 2 & 4 & 14 \end{bmatrix}$ by row reduction.

You can use the row operations R1-R2, R3-2R1, R3-2/3R2 to find that det $= -14$.

(c) (5 pts) Let $A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ 2 & 4 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 12 \\ 14 \end{bmatrix}$. Using part (a), part (b), and Cramer's Rule, find the value of x_3 in the matrix equation $A\mathbf{x} = \mathbf{b}$.

$x_3 = \frac{-14}{-2} = 7$

2. (12 pts) Determine whether or not the set $H = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} : a, b \in \mathbb{R} \right\}$ is a subspace of $M_{2\times 2}$.

H is a subspace. We need to check the three criteria.

(1) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in H.$

(2) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ -(b+d) & a+c \end{bmatrix} \in H.$

(3)$k \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} ka & kb \\ -(kb) & ka \end{bmatrix} \in H.$

3. (18 pts) Let $A = \begin{bmatrix} 1 & 0 & 2 \\ -2 & 1 & -1 \\ 5 & -1 & 7 \end{bmatrix}$. Find a basis for the following:

A is row equivalent to the matrix $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

(a) (5 pts) col(A).

From the reduced form, we see that the pivot columns are columns 1 and 2, so a basis for the column space is $\left\{ \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

(b) (5 pts) null(A)

Using the reduced form, we can find a solution for the corresponding homogeneous equation and write it in parametric vector form. This gives us that a basis for the null space is $\left\{ \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}, \right\}$.

(c) (5 pts) row(A)

From the reduced form, we know that a basis for the row space is given by the non-zero rows. Therefore, a basis for the row space is

$$\{(1, 0, 2), (0, 1, 3)\}$$

(d) (3 pts) col(A^T)

Since col(A^T) =row(A), we get that a basis for the columns space of A^T is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}$

4. (10 pts)

(a) (5 pts) What is the dimension of $\mathbb{P}_2$, the vector space of polynomials with degree ≤ 2. Explain.

We know that $\{1, x, x^2\}$ *is a basis for* $\mathbb{P}_2$, *so the dimension of* $\mathbb{P}_2$ *is* 3.

(b) (5 pts) Determine whether or not $\{1, 1 + x, 1 + x + x^2\}$ a basis for $\mathbb{P}_2$?

Since there are 3 elements in this set and the dimension of the space is 3, we only need to show that this set is linearly indepedent. Consider

$$\begin{aligned} c_1(1) + c_2(1 + x) + c_3(1 + x + x^2) &= 0 \\ \Rightarrow (c_1 + c_2 + c_3)(1) + (c_2 + c_3)(x) + c_3 x^2 &= 0 \end{aligned}$$

This implies that $c_3 = 0$, $c_2 + c_3 = 0$, *and* $c_1 + c_2 + c_3 = 0$ *(because for polynomials to be equal, each coeffiecient must be equal). But this implies that* $c_1 = c_2 = c_3 = 0$. *Thus, this set is linearly independent by definition.*

5. (10 pts) If A is a 4×9 matrix,

(a) (5 pts) What is the largest possible value for rank(A)? Explain.

Since the columns of A are from $\mathbb{R}^4$, they can only span at most $\mathbb{R}^4$. Thus, the largest possible value for the rank is 4.

(b) (5 pts) What is the smallest possible value for nullity(A)? Explain.

We know that rank + nullity = number of columns and since the largest the rank can be is 4, the smallest possible value of the nullity is $9 - 4 = 5$.

6. (5 pts) Let $\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$. Find the change of basis matrix $P_{\mathcal{C} \leftarrow \mathcal{B}}$.

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}.$$

7. (10 pts) Is it possible that all solutions of a homogeneous system of ten linear equations in twelve variables are multiples of one fixed nonzero solution? Explain. *HINT: In such a case, how many free variables would there be?*

No. If all solutions of a homogeneous system can be written as multiples of one fixed nonzero solution, then there must be only one free varible. This means that the nullity of the coefficient matrix is 1. Therefore, by the rank theorem, the rank of the coefficient matrix must be 11. However, this is not possible since the columns of the coefficient matrix are only in $\mathbb{R}^{10}$.

8. (10 pts) Let H be the set of triples of numbers (x_1, x_2, x_3) with the operations

$$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$$
$$k(x_1, x_2, x_3) = (0, 0, 0)$$

Why is this set NOT a vector space?

Axiom 10 is not met. For example,

$$\begin{aligned} 1 \cdot (1, 2, 3) &= (0, 0, 0) \\ &\neq (1, 2, 3) \end{aligned}$$