Math 322 – 008 April 19, 2016 SOLUTIONS

1. Define a transformation $T: M_{2\times2} \to \mathbb{P}_3$ by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ax^3 + (b+c)x + d$.

a. Find the kernel of T, a basis for $\ker T$, and $\dim \ker T$.

ker T = matrices which map to the zero polynomial.

This requires $a=0$, $d=0$, and $b+c=0 \Rightarrow b=-c$.

$$\ker T = \left\{ \begin{bmatrix} 0 & -c \\ c & 0 \end{bmatrix} \,\middle|\, c \text{ any real number} \right\}$$

Basis for kernel: $\left\{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$ ← or any nonzero multiple of this

$\dim \ker T = 1$

b. Find a basis for the image of T, and its dimension.

image = all polynomials in $\mathbb{P}_3$ which can be mapped to.

Using appropriate choices for a, b, c and d, we can achieve any polynomial except those with x^2 terms.

our image will be the set $\{ex^3 + fx + g\}$,

and can have basis $\{x^3, x, 1\}$.

The dimension of the image is 3.

2. $\mathcal{B} = \{2t^2+t, t+1, 3t+2\}$ is a basis for $\mathbb{P}_2$. Let $p = 4t^2+5t+6$. Find $[p]_{\mathcal{B}}$. $= \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

$$c_1(2t^2+t) + c_2(t+1) + c_3(3t+2) = 4t^2+5t+6.$$

Squares: $2c_1t^2 = 4t^2$ $\quad\boxed{c_1 = 2}$

Linear terms: $c_1t + c_2t + 3c_3t = 5t$

$2 + c_2 + 3c_3 = 5$

$c_2 + 3c_3 = 3$

Constant terms: $c_2 + 2c_3 = 6 \longmapsto$ $-c_2 - 2c_3 = -6$

$\boxed{c_3 = -3}$

$c_2 - 6 = 6$

$\boxed{c_2 = 12}$

thus $[p]_B = \begin{bmatrix} 2 \\ 12 \\ -3 \end{bmatrix}$