Math 322 – 008 Review Problems for Final

SOLUTIONS

1. The matrix $A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}$ has characteristic polynomial $-(\lambda-3)^2(\lambda-8)$.

a. Find a basis for the eigenspace corresponding to the eigenvalue $\lambda = 3$.

$$A - \lambda I = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 4 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

x_3 free
x_2 free
$x_1 = -2x_2 - 3x_3$

$$\boxed{\text{Basis for eigenspace} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

x_2 x_3

b. Is A diagonalizable? Explain briefly.

Yes: A has three linearly independent eigenvectors. We found two for the eigenvalue $\lambda = 3$ (which matches its multiplicity), and know we could find one for $\lambda = 8$.

2. Given the system

$$x_1 + 3x_2 = 2$$
$$3x_1 + hx_2 = k$$

for what values of h and k will the system have a unique solution?

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 3 & h & k \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & h-9 & k-6 \end{array}\right]$$

To have a unique solution, both columns must have a pivot. Thus we require $h-9 \neq 0$, so $\boxed{h \neq 9}$.
There are no restrictions on k: $\boxed{k \text{ can be any real \#}}$

3. Let $T:\mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T(x_1,x_2,x_3) = (x_1 - x_2 - 2x_3,\ \ -x_1 + 2x_2 + 3x_3)$.

a. What properties would we have to check to show that T a linear transformation? (If you finish the rest of the worksheet, come back and show them.)

Check that $T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$ (preserves addition)

and that $T(c\vec{v}) = cT(\vec{v})$ (preserves scalar multiplication).

To begin, write $\vec{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$.

b. Find $\ker T$, a basis for $\ker T$, and the dimension of $\ker T$.

$\ker T$ = all vectors $\vec{v}$ in $\mathbb{R}^3$ so that $T(\vec{v}) = \vec{0}$.

$$\Rightarrow \left.\begin{array}{l} x_1 - x_2 - 2x_3 = 0 \\ -x_1 + 2x_2 + 3x_3 = 0 \end{array}\right] \text{ Solve this system.}$$

$$\begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

x_3 is free,

$x_2 = -x_3$

$x_1 = x_3$

$$\boxed{\begin{array}{l} \ker T = \left\{ \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} \middle| \ x_3 \text{ any real} \right\} \\ \text{Basis for } \ker T = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}, \qquad \dim \ker T = 1 \end{array}}$$

c. Is T onto? (Justify.)

Yes. In the echelon form of the system, we can check that there is a pivot in every row.

4. Let A be a 42×35 matrix, and let $T:\mathbb{R}^{35}\to\mathbb{R}^{42}$ be the transformation $T(\mathbf{x})=A\mathbf{x}$.

   a. Suppose A has 30 pivot columns. Find $\operatorname{rank} A$ and $\dim \operatorname{Nul} A$.

   $\operatorname{rank} A = 30 \qquad \dim \operatorname{Nul} A = 35-30 = 5$

   b. If $\operatorname{Col} A$ is a subspace of $\mathbb{R}^j$ and $\operatorname{Nul} A$ is a subspace of $\mathbb{R}^k$, find j and k.

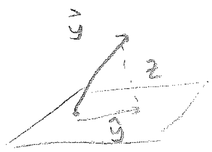
   $\operatorname{Col} A$ = image of T, so $j=42$. $\operatorname{Nul} A = \ker T$ so $k=35$

5. In $\mathbb{R}^3$, let $W=\operatorname{span}\left\{ \begin{array}{cc} \vec{u} & \vec{v} \\ \begin{bmatrix}2\\1\\4\end{bmatrix}, & \begin{bmatrix}-2\\0\\1\end{bmatrix} \end{array} \right\}$. Let $\mathbf{y}=\begin{bmatrix}1\\2\\3\end{bmatrix}$. Find the closest point to $\mathbf{y}$ in W and the distance between $\mathbf{y}$ and W. (No need to simplify the ugly fractions.) Don't forget the first step.

Option 1: Compute the projection $\hat{y}$ of y onto W.

Step 1: Verify the basis vectors are orthogonal.

$\vec{u}\cdot\vec{v} = -4+0+4=0.$ ✓



Step 2: $\hat{y} = \dfrac{\vec{u}\cdot\vec{y}}{\vec{u}\cdot\vec{u}}\cdot\vec{u} + \dfrac{\vec{v}\cdot\vec{y}}{\vec{v}\cdot\vec{v}}\vec{v} = \dfrac{2+2+12}{4+1+16}\begin{bmatrix}2\\1\\4\end{bmatrix} + \dfrac{-2+0+3}{4+0+1}\begin{bmatrix}-2\\0\\1\end{bmatrix}$

$= \dfrac{16}{21}\begin{bmatrix}2\\1\\4\end{bmatrix} + \dfrac{1}{5}\begin{bmatrix}-2\\0\\1\end{bmatrix} = \begin{bmatrix}\frac{32}{21}-\frac{2}{5}\\ \frac{16}{21}\\ \frac{64}{21}+\frac{1}{5}\end{bmatrix} \approx \begin{bmatrix}1.123\\.761\\3.248\end{bmatrix}$

Closest point to $\vec{y}$ in W: exact; for comparison

For the distance, first find

$\vec{z} = \vec{y}-\hat{y} = \begin{bmatrix}1\\2\\3\end{bmatrix} - \begin{bmatrix}\text{that}\end{bmatrix}$, and then take the norm. (Approximately 3.52)

Option 2: Find $\hat{y}$ using least-squares

6. Let $\mathcal{B}=\left\{\begin{bmatrix}7\\-2\end{bmatrix},\begin{bmatrix}2\\-1\end{bmatrix}\right\}$ and $\mathcal{C}=\left\{\begin{bmatrix}4\\1\end{bmatrix},\begin{bmatrix}5\\2\end{bmatrix}\right\}$ be bases of $\mathbb{R}^2$. Find the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$ and the change of basis matrix from $\mathcal{C}$ to $\mathcal{B}$.

For $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$, $\left[\begin{array}{cc|cc}4&5&7&2\\1&2&-2&-1\end{array}\right] \to \left[\begin{array}{cc|cc}1&2&-2&-1\\0&-3&15&6\end{array}\right] \to \left[\begin{array}{cc|cc}1&2&-2&-1\\0&1&-5&-2\end{array}\right]$

$\to \left[\begin{array}{cc|cc}1&0&8&3\\0&1&-5&-2\end{array}\right]$ ← reduced to identity, so $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix}8&3\\-5&-2\end{bmatrix}$

Now for $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ could redo above, or just find the inverse: $\dfrac{1}{16-15}\begin{bmatrix}-2&-3\\5&8\end{bmatrix} = \begin{bmatrix}-2&-3\\5&8\end{bmatrix}$

7. Let $T:\mathbb{P}_2 \to \mathbb{P}_1$ be defined using the derivative, so $T(f) = f'$.

   a. Find the matrix for T relative to the bases $\{t^2+2, t+3, t+1\}$ and $\{t,1\}$.

$T(t^2+2) = 2t \leftarrow$ coord. vector $\begin{bmatrix}2\\0\end{bmatrix}$

$T(t+3) = 1$, coord vector $\begin{bmatrix}0\\1\end{bmatrix}$ so $[T] = \begin{bmatrix}2 & 0 & 0\\ 0 & 1 & 1\end{bmatrix}$

$T(t+1) = 1$ "

   b. Find the kernel of T.

$\ker T$ = polynomials in $\mathbb{P}_2$ with $f' = 0$. all constant polynomials

or $\{a \mid a \text{ any real}\}$ or Span$\{1\}$. (all fine!)

   c. Find the range of T.

range = image = all polynomials in $\mathbb{P}_1$ which are the derivative of something in $\mathbb{P}_2$. Since every $ax+b$ has an antiderivative, the range is all of $\mathbb{P}_1$.

8. Consider the matrix $A = \begin{bmatrix}1 & 5\\ -2 & 3\end{bmatrix}$. Find the eigenvalues, and for one of them (your choice) find a corresponding eigenvector.

$\det(A-\lambda I) = \begin{vmatrix}1-\lambda & 5\\ -2 & 3-\lambda\end{vmatrix} = (1-\lambda)(3-\lambda)+10$ (characteristic polynomial)

$= 3-\lambda-3\lambda+\lambda^2+10 = \lambda^2 - 4\lambda + 13$

Set $=0$: $\lambda = \dfrac{4 \pm \sqrt{16-4(13)}}{2} = \dfrac{4\pm\sqrt{16-52}}{2} = \dfrac{4\pm\sqrt{-36}}{2} = 2 \pm 3i$

I chose $\lambda = 2+3i$:

$\begin{bmatrix}1-(2+3i) & 5\\ -2 & 3-(2+3i)\end{bmatrix} = \begin{bmatrix}-1-3i & 5\\ -2 & 1-3i\end{bmatrix} \leftarrow$ write as equations:

$(-1-3i)x_1 + 5x_2 = 0$

$-2x_1 + (1-3i)x_2 = 0.$

Since λ is an eigenvector, guaranteed a free variable. With only 2 variables, the two equations are equivalent; we can pick one. Using the 1st one, if $x_1 = 1$, then $5x_2 = -1-3i$ so $x_2 = \frac{1}{5}(-1-3i)$

$\begin{bmatrix}1\\ \frac{1}{5}(-1-3i)\end{bmatrix}$ or $\begin{bmatrix}5\\ -1-3i\end{bmatrix}$ is an eigenvector. (or any multiple.)

9. Let $W = \text{span}\left\{\begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}2\\-3\\4\end{bmatrix}\right\}$. Use the Gramm-Schmidt process to find an orthogonal basis for W.

x_1, x_2

Check: the vectors are not orthogonal already.

$\vec{v}_1 = \vec{x}_1$

$\vec{v}_2 = \vec{x}_2 - \text{proj}_{\vec{v}_1}\vec{x}_2 = \begin{bmatrix}2\\-3\\4\end{bmatrix} - \dfrac{0-3+4}{0+1+1}\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}2\\-3\\4\end{bmatrix} - \frac{1}{2}\begin{bmatrix}0\\1\\1\end{bmatrix}$

$= \begin{bmatrix}2\\ -7/2\\ 7/2\end{bmatrix}$, or $\begin{bmatrix}4\\-7\\7\end{bmatrix}$ New basis: $\left\{\begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}4\\-7\\7\end{bmatrix}\right\}$