SOLUTIONS

1. Let $\mathscr{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\mathfrak{D} = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ be bases of a vector space V. Suppose that $\mathbf{a}_1 = 4\mathbf{d}_1 - \mathbf{d}_2$, $\mathbf{a}_2 = -\mathbf{d}_1 + \mathbf{d}_2 + \mathbf{d}_3$ and $\mathbf{a}_3 = \mathbf{d}_2 - 2\mathbf{d}_3$.

   a. Find the change of coordinate matrix from $\mathscr{A}$ to $\mathfrak{D}$.

$$\underset{\mathfrak{D} \leftarrow \mathscr{A}}{P} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

   b. Find $[\mathbf{x}]_{\mathfrak{D}}$ for $\mathbf{x} = 3\mathbf{a}_1 + 4\mathbf{a}_2 + \mathbf{a}_3$.

$$[X]_{\mathfrak{D}} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 12-4 \\ -3+4+1 \\ 0+4-2 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 2 \end{bmatrix}$$

2. In $\mathbb{P}_2$, find the change of coordinate matrix from the basis $\mathfrak{B} = \{1-3x^2,\ 2+x-5x^2,\ 1+2x\}$ to the standard basis $\{1, x, x^2\}$. Then write x^2 as a linear combination of the polynomials in $\mathfrak{B}$.

$$\underset{\mathfrak{S} \leftarrow \mathfrak{B}}{P} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix}$$

write X^2 in terms of standard basis

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

solve the system:
(or, could find inverse of matrix and multiply.)

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ -3 & -5 & 0 & 1 \end{array}\right] \xrightarrow[R_3+3R_1]{R_1-2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 3 & 1 \end{array}\right]$$

$$\xrightarrow[R_3-R_2]{} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

thus

$$X^2 = 3(1-3x^2) - 2(2+x-5x^2) + 1(1+2x)$$