1a) $\left[\begin{array}{cc|c} h & 1 & 3 \\ 1 & 2 & k \end{array}\right]$

$2R_1 - R_2 \to R_2$

$\left[\begin{array}{cc|c} h & 1 & 3 \\ 2h-1 & 0 & 6-k \end{array}\right]$

in order to have inffinite solutions, we need free variables, therefore

$2h-1=0.$ $h=\frac{1}{2}.$

But the equation must also be consistent, so $6-k=0.$

$k=6$

so: $h=\frac{1}{2}$, $k=6$

b) $\left[\begin{array}{cc|c} h & 1 & 3 \\ 2h-1 & 0 & 6-k \end{array}\right]$

in order to have no solutions, we need a row to be in $[0 \; 0 \mid b]$ $b\neq 0$ to cause inconsistency.

therefore, $2h-1=0$, $h=\frac{1}{2}$

and

$6-k\neq 0$, $k\neq 6$

so: $h=\frac{1}{2}$, $k\neq 6$

c) $\left[\begin{array}{cc|c} h & 1 & 3 \\ 2h-1 & 0 & 6-k \end{array}\right]$

in order to have a unique solution, we cannot have free variable, so we can not have a row with either $[0 \; 0 \mid 0]$ or $[0 \; 0 \mid b]$.

therefore $2h-1\neq 0$, $h\neq\frac{1}{2}$.

k can be anything as long as $h\neq\frac{1}{2}$.

so: $h\neq\frac{1}{2}$, $k\in R$

(2) $T(\vec{x}) = \begin{bmatrix} 2x_1 \\ -x_3 + x_4 \\ x_2 + x_3 - 2x_4 \end{bmatrix}$

* We want A to be a 3×4 matrix that when multiplied by x, we get $T(\vec{x})$. So if we look at the first component of $T(x_i)$ we can strip the coefficients to be that of the first row of A. Repeat for other rows/variables.

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & -2 \end{bmatrix}$$

Check to make sure it's true.

$Ax = T(x)$

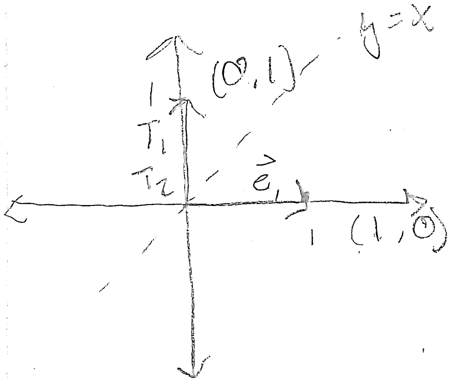
$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ -x_3 + x_4 \\ x_2 + x_3 - 2x_4 \end{bmatrix}$$

3) Find the standard matrix A for the transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ that first reflects points through the line $y = x$ and then reflects through the vertical y-axis.

answer on next page ↓

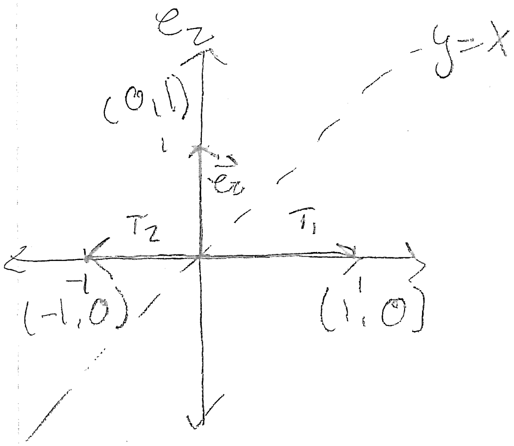
③ $\vec{e}_1$

transformation 1 = through $y=x$
transformation 2 = through y-axis



$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $T_1(\vec{e}_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $T_2(\vec{e}_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\vec{e}_2$



$\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $T_1(\vec{e}_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $T_2(\vec{e}_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

4 No

Reasoning 1:
We know that the matrix is square and spans all of $\mathbb{R}^5$. Therefore we know that $A\vec{x} = b$ has a unique solution for every b. Therefore this is true by the invertible matrix theorem.

Reasoning 2:
We know it is consistent for every b, therefore there is a pivot in every row. In order for this to be true there must be a pivot in every column. If there is a pivot in every column then the matrix spans $\mathbb{R}^5$ therefore there must be a unique solution for every b.

5 a) $A = \begin{bmatrix} 1 & -2 & \varnothing & \varnothing & 3 \\ \varnothing & \varnothing & 1 & \varnothing & \varnothing \\ \varnothing & \varnothing & \varnothing & 1 & 2 \end{bmatrix}$ $T: \mathbb{R}^5 \to \mathbb{R}^3$ given by $T(\vec{x}) = A\vec{x}$

The vector $\begin{bmatrix} 1 \\ 2 \\ \varnothing \\ 1 \\ 3 \end{bmatrix}$ is not an image of T because the transformation goes to $\mathbb{R}^3$

so any image of T will be in $\mathbb{R}^3$. This vector is in $\mathbb{R}^5$ so it cannot be an image of T.

b) Yes, the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is an image of T because as we can see, there is a pivot in every row of A. This means that the columns of A span all of $\mathbb{R}^3$. Since they span all of $\mathbb{R}^3$, the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ must be included as well.

(c), (d) not submitted :(

6. Can a square matrix with identical columns be invertible?
- No because the equation $A\vec{x} = \vec{b}$ does not have a solution. (rule g.) (We would not have linearly ind. columns.)

7. Can a square matrix with two identical rows be invertible?
- No because A wouldn't have n pivots. After reducing, there would be at least one row of all zeros. Also at least one row would be missing a pivot position (rule c.)